RG improvement and α_s determination from relativistic sum rules

$$R_{\bar{q}q}(s) \equiv \frac{3s}{4\pi\alpha^2}\sigma\left(e^+e^- \to q\bar{q} + X\right)$$

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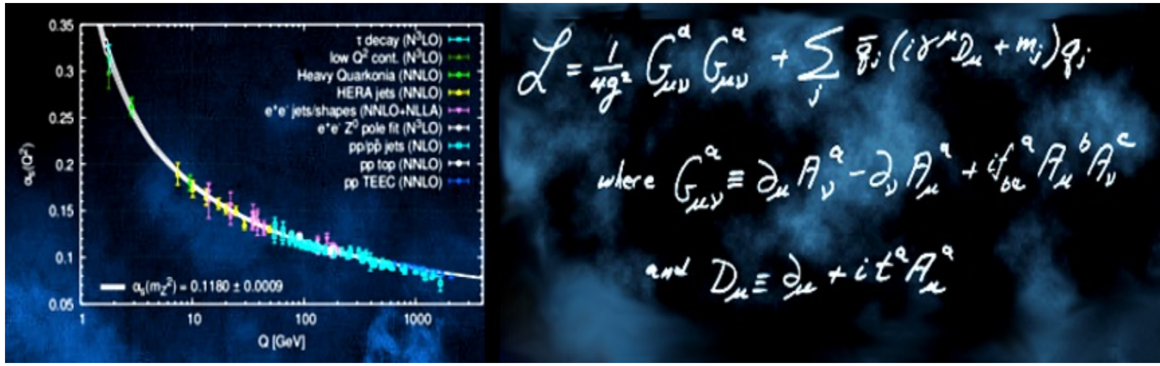
"Renormalization group improved determination of α_s, m_c, and m_b from the low energy moments of heavy quark current correlators" **(Phys.Rev.D 108 (2023) 7, 074029)**



Feb 05, 2024

alphas-2024: Workshop on precision measurements of the QCD coupling constant

Outline

- Motivation
- Definition of various quantities used.
- Prescriptions for the perturbative series.

a. FOPT and RGSPT

b. RG improvements.

- Issues with FOPT using charm moments

a) Charm mass determinations

b) Strong coupling determination

- Results

Motivation

- K. G. Chetyrkin, J. H. Kuhn, A. Maier, P. Maierhofer, P. Marquard, M. Steinhauser and C. Sturm, ``Addendum to Charm and bottom quark masses: An update [arXiv:1710.04249 [hep-ph]].

n	$m_c(3\text{GeV})$	exp	α_s	μ	np LO	total	$m_c\,(m_c)$
1	0.993	0.007	0.004	0.002	0.001	0.008	1.279
2	0.982	0.004	0.007	0.005	0.001	0.010	1.269
3	0.982	0.003	0.008	0.006	0.001	0.010	1.269
4	1.003	0.002	0.005	0.028	0.001	0.029	1.288

Quantities and their definitions

$$R_{\bar{q}q}(s) \equiv \frac{3s}{4\pi\alpha^2}\sigma\left(e^+e^- \to q\bar{q} + X\right) \simeq \frac{\sigma\left(e^+e^- \to q\bar{q}+X\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)}$$

q={c,b}

$$\mathcal{M}_q^{V,n} = \int_0^\infty \frac{ds}{s^{n+1}} R_{q\bar{q}}(s)$$

$$= \int_0^{s_{\text{th.}}} \frac{ds}{s^{n+1}} R_{q\bar{q}}^{\text{exp.}}(s) + \int_{s_{\text{th.}}}^\infty \frac{ds}{s^{n+1}} R_{q\bar{q}}^{\text{cont.}}(s)$$

$$\mathcal{M}_n^{V,\text{th}} = \frac{12\pi^2 Q_q^2}{n!} \frac{d^n}{ds^n} \Pi^V(s)\bigg|_{s=0}$$

$$R_{\bar{q}q}^{\text{cont.}} \equiv 12\pi \,\text{Im}\,\Pi^{\text{V}}(s+i\epsilon) = 12\pi \times \frac{1}{2i}\left(\Pi^{\text{V}}(s+i\epsilon) - \Pi^{\text{V}}(s-i\epsilon)\right)$$

$$\mathcal{M}_n^{P,\text{th}} = \frac{12\pi^2 Q_q^2}{n!} \frac{d^n}{ds^n} P(s)\bigg|_{s=0}$$

$$\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\Pi^V(q^2) = -i\int dx e^{iqx}\langle 0|T\left\{j_\mu(x) j_\nu(0)\right\}|0\rangle$$

$$j_\mu(x) = \bar{q}(x)\gamma^\mu q(x)$$

$$P(s) = \frac{1}{s^2}\left(\Pi^P(s) - \Pi^P(0) - s\left[\frac{d}{ds}\Pi^P(s)\right]_{s=0}\right)$$

$$\Pi^P(q^2) \equiv i\int dx e^{iqx}\langle 0|T\{j_P(x) j_P(0)\}|0\rangle$$

$$j_P = 2 i m_q \bar{q}(x)\gamma^5 q(x)$$

$$\mathcal{M}_n^{X,\text{th.}} = \mathcal{M}_n^{X,\text{pert.}} + \mathcal{M}_n^{X,\text{n.p.}}$$

$$\mathcal{M}_n^{X,\text{pert.}} = m_q^{-2n}\sum_{i,j} T_{i,j}^{X,\text{pert.}} x^i L^j$$

$$\mathcal{M}_n^{X,\text{n.p.}} = \frac{\left\langle x G^2\right\rangle_{\text{RGI}}}{(2m_q)^{4n+4}}\sum_{i,j} T_{i,j}^{X,\text{n.p.}} x^i L^j$$

$$\mathcal{R}_n^X \equiv \frac{\left(\mathcal{M}_n^X\right)^{\frac{1}{n}}}{\left(\mathcal{M}_{n+1}^X\right)^{\frac{1}{n+1}}}$$

$x \equiv \alpha_s(\mu)/\pi$ $\quad L \equiv \log(\mu^2/m_q^2)$

$m_q \equiv m_q(\mu)$

Perturbative prescriptions

- Fixed order perturbation theory (FOPT)
- Renormalization group summed perturbation theory (RGSPT)

N^nLL are summed using Renormalization group

FOPT : $$\mathcal{M}_n^{X,\text{pert}} = m_q^{-2n} \sum_{i,j} T_{i,j}^X x^i L^j$$

$$\beta(x) \equiv \mu^2 \frac{d}{d\mu^2} x(\mu) = -\sum_i \beta_i x^{i+2} \,,$$

$$\gamma_m(x) \equiv \mu^2 \frac{d}{d\mu^2} \log(m_q(\mu)) = -\sum_i \gamma_i\, x^{i+1}$$

RGSPT : $$\mathcal{M}_n^{X,\Sigma} = m_q^{-2\,n} \sum_{i=0} x^i\, S_i(x\,L) \implies S_i\,(x\,L) = \sum_{n=i}^{\infty} T_{n,n-i}^X (x\,L)^{n-i}$$

RGE : $$\mu^2 \frac{d}{d\mu^2} \mathcal{M}_n^X = 0 \implies \left(\beta(x)\partial_x + \gamma_m(x)\partial_m + \partial_L\right) \mathcal{M}_n^X = 0$$

$$\sum_{i=0}^{k} \left[\beta_i(\delta_{i,0} + w - 1) S'_{k-i}(w) + (\beta_i(k-i) - 2n\gamma_i)\, S_{k-i}(w)\right] = 0$$

$$w = 1 - \beta_0\, x\, L$$

$$S_0(w) = w^{\frac{2n\gamma_0}{\beta_0}}, \quad S_1(w) = \left(T_{1,0}^X + \frac{2nL_w\gamma_0\,(\beta_1 - 2\beta_0\gamma_0)}{\beta_0^2} + \frac{2n\,(\beta_1\gamma_0 - \beta_0\gamma_1)}{\beta_0^2}\right) w^{\frac{2n\gamma_0}{\beta_0}-1} + \frac{2n\,(\beta_0\gamma_1 - \beta_1\gamma_0)\, w^{\frac{2n\gamma_0}{\beta_0}}}{\beta_0^2}$$

$$\Omega_{n,a} \equiv \frac{\log^n(w)}{w^a}$$

n: positive integer

$$a \propto \gamma_0/\beta_0$$

Other examples: strong coupling constant

FOPT:

$$x(Q) = x\left(1 + x\beta_0 L + x^2\left(\beta_1 L + \beta_0^2 L^2\right) + x^3\left(\beta_2 L + \frac{5}{2}\beta_1\beta_0 L^2 + \beta_0^3 L^3\right)\right) + \mathcal{O}(x^5)$$

RGSPT:

$$x(Q) = \frac{x}{w} - \frac{x^2\tilde{\beta}_1 L_w}{w^2} + x^3\left(\frac{-\tilde{\beta}_1^2 + \tilde{\beta}_2 + \tilde{\beta}_1^2 L_w^2 - \tilde{\beta}_1^2 L_w}{w^3} + \frac{\tilde{\beta}_1^2 - \tilde{\beta}_2}{w^2}\right) + x^4\left\{\frac{-\frac{\tilde{\beta}_1^3}{2} + \tilde{\beta}_2\tilde{\beta}_1 - \frac{\tilde{\beta}_3}{2}}{w^2}\right.$$
$$\left. + \frac{-\frac{\tilde{\beta}_1^3}{2} + \frac{\tilde{\beta}_3}{2} + \tilde{\beta}_1^3\left(-L_w^3\right) + \frac{5}{2}\tilde{\beta}_1^3 L_w^2 + \left(2\tilde{\beta}_1^3 - 3\tilde{\beta}_1\tilde{\beta}_2\right)L_w}{w^4} + \frac{\tilde{\beta}_1^3 - \tilde{\beta}_2\tilde{\beta}_1 + \left(2\tilde{\beta}_1\tilde{\beta}_2 - 2\tilde{\beta}_1^3\right)L_w}{w^3}\right\} + \mathcal{O}\left(x^5\right)$$

where: $w = 1 - \beta_0 x L,\ L_w \equiv \log(w)$ and $\tilde{\beta}_i = \beta_i/\beta_0$

Not only coupling and moments, even decoupling relations for quark masses and coupling can be summed in closed form!!!

Higher order results

$$\mathcal{M}_n^{X,\mathrm{pert.}} = m_q^{-2n} \sum_{i=0,j=0}^{\infty,i} T_{i,j}^{X,\mathrm{pert.}} x^i L^j$$

$T_{i,0}^{X,\mathrm{pert.}}$ known to four loops $\left(\mathcal{O}\left(\alpha_s^3\right)\right)$

A. Maier, P. Maierhofer, P. Marquard and A. V. Smirnov, Nucl. Phys. B 824(2010), [arXiv:0907.2117 [hep-ph]].

$$\mathcal{M}_n^{X,\mathrm{n.p.}} = \frac{\left\langle x G^2 \right\rangle_{\mathrm{RGI}}}{(2 m_q)^{4n+4}} \sum_{i,j} T_{i,j}^{X,\mathrm{n.p.}} x^i L^j$$

$T_{i,0}^{X,\mathrm{n.p.}}$ is known to two loops $\left(\mathcal{O}\left(\alpha_s\right)\right)$

D. J. Broadhurst,et al., Phys. Lett. B 329 (1994), [arXiv:hep-ph/9403274 [hep-ph]]

$$x \equiv \alpha_s(\mu)/\pi$$
$$m_q \equiv m_q(\mu)$$
$$L \equiv \log\left(\mu^2/m_q^2\right)$$

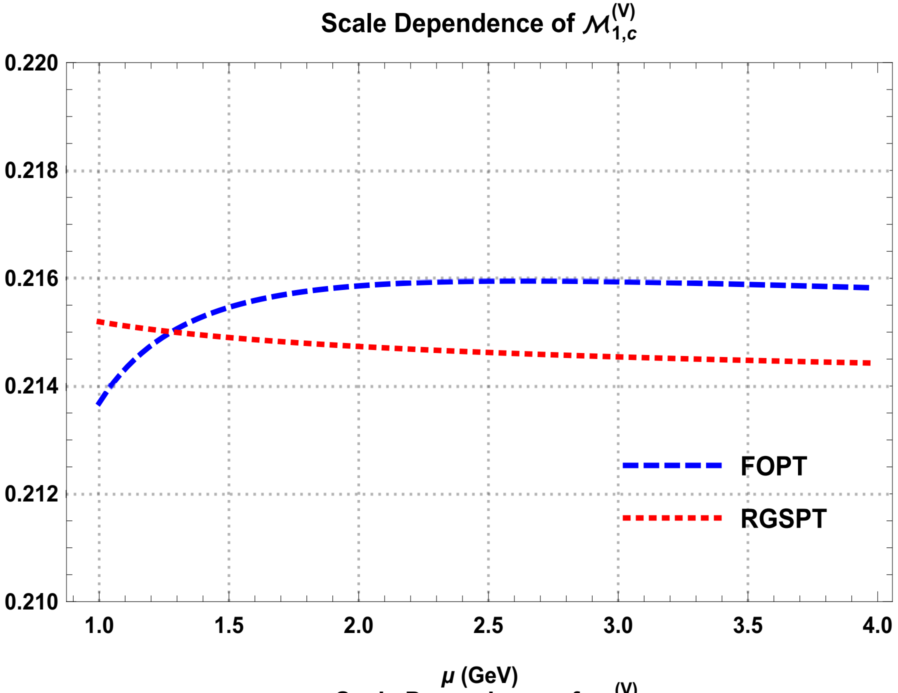
$$\left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle_{\mathrm{RGI}} = 0.006 \pm 0.012\,\mathrm{GeV}^4$$

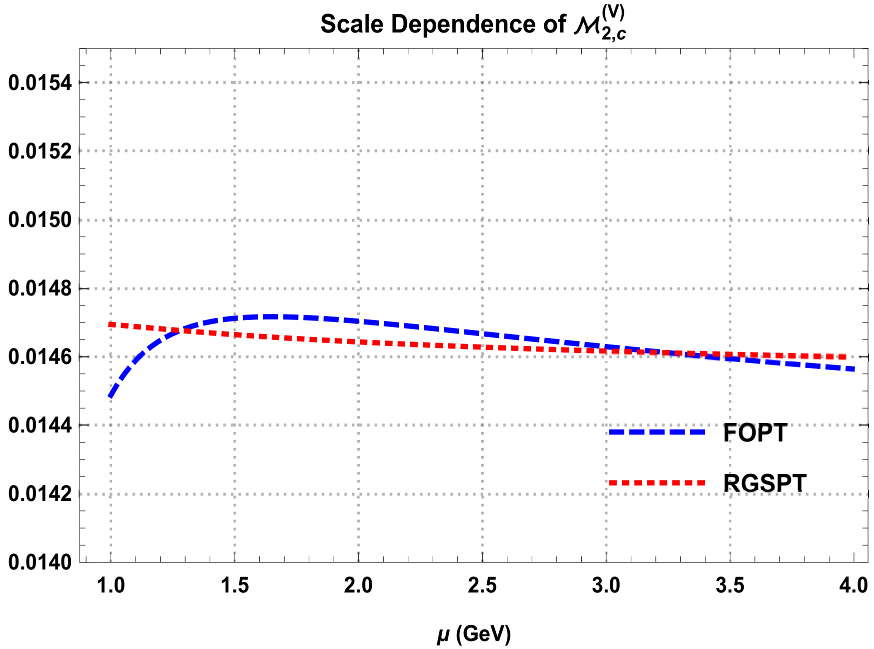
B. L. Ioffe, Prog. Part. Nucl. Phys. 56 (2006), [arXiv:hep-ph/0502148 [hep-ph]].

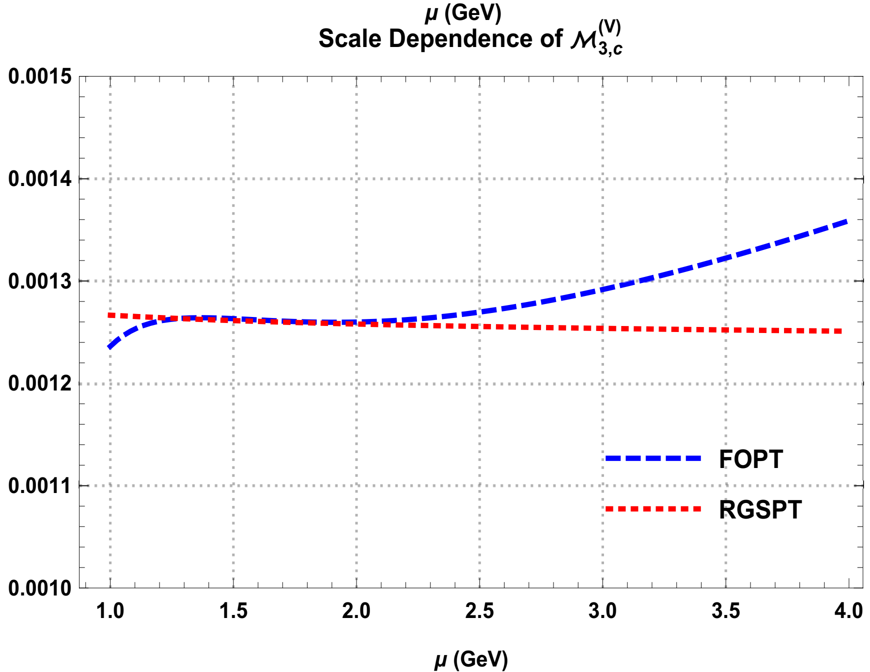
$$m_q(\mu) = M_q\left(1 - x(\mu)\left(\frac{4}{3} + \log\left(\frac{\mu^2}{M_q^2}\right)\right)\right) + \mathcal{O}\left(x^2\right)$$

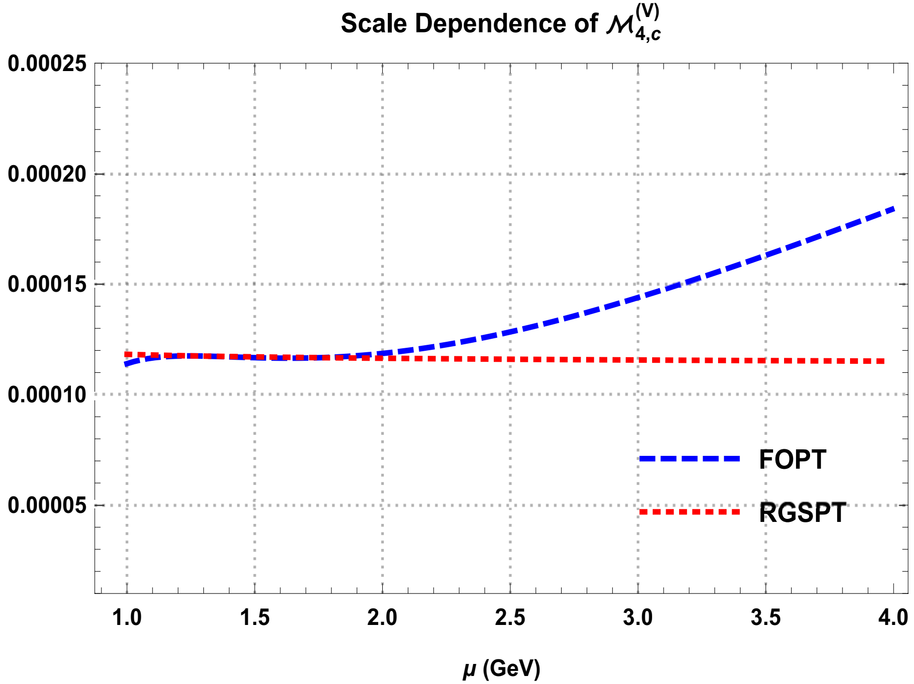
$$m_c(q) = \overline{m}_c \int_{x(\overline{m}_c)}^{x(q)} dx\, e^{\left(\frac{\gamma(x)}{\beta(x)}\right)}$$

RG behavior of perturbative moments in FOPT and RGSPT

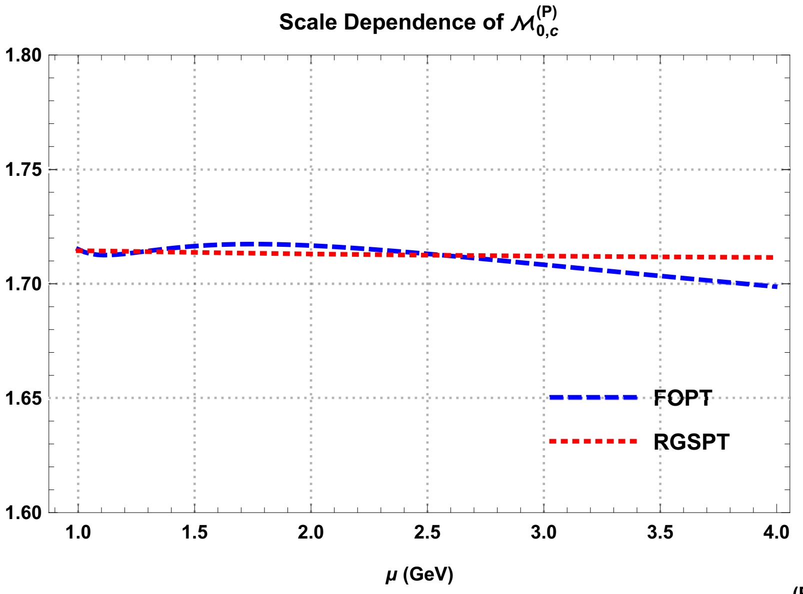


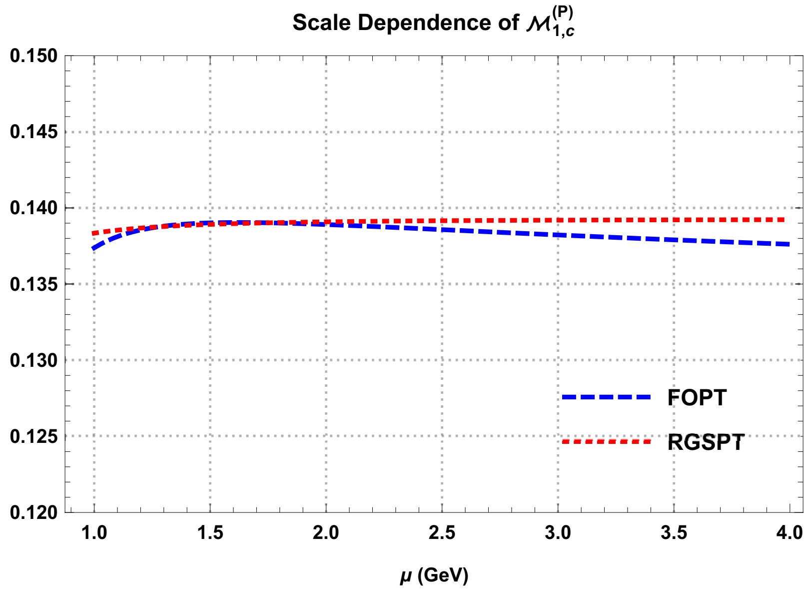


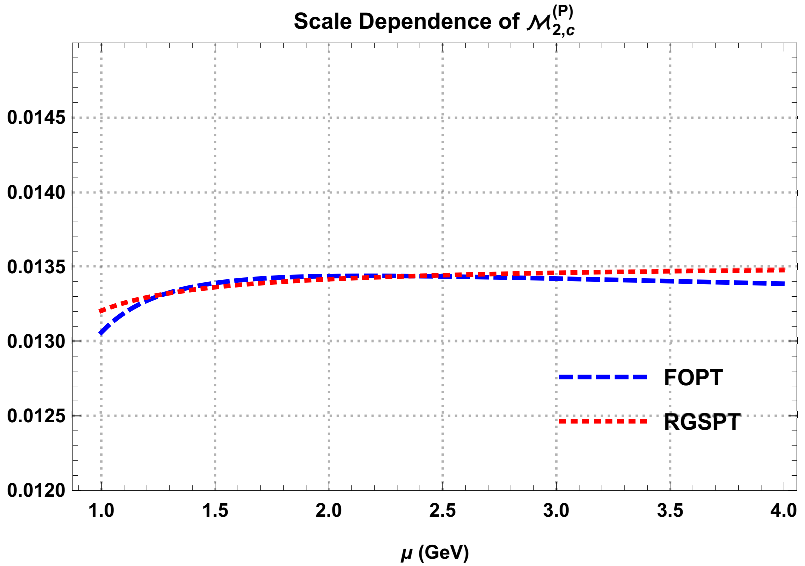


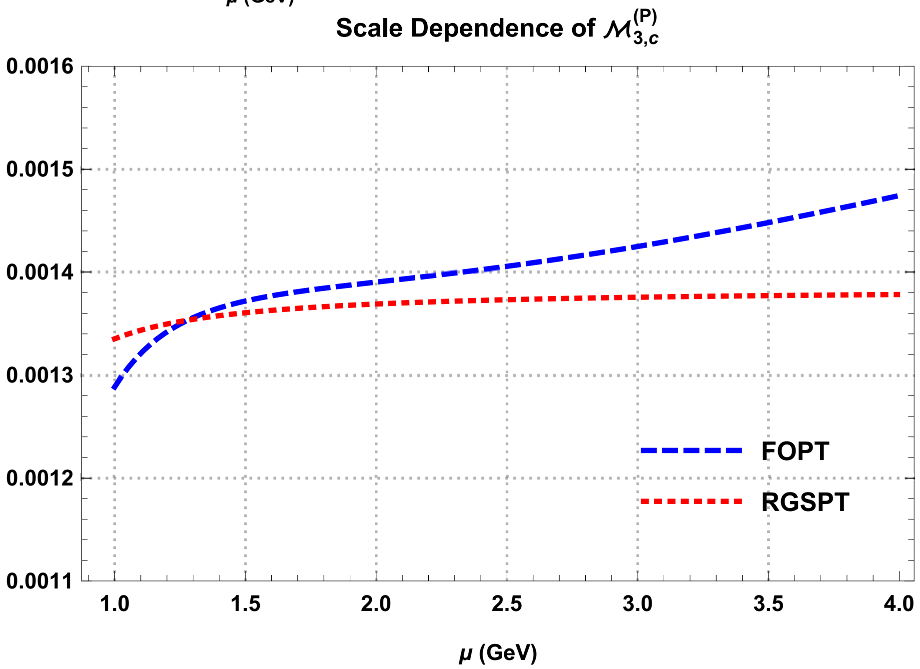


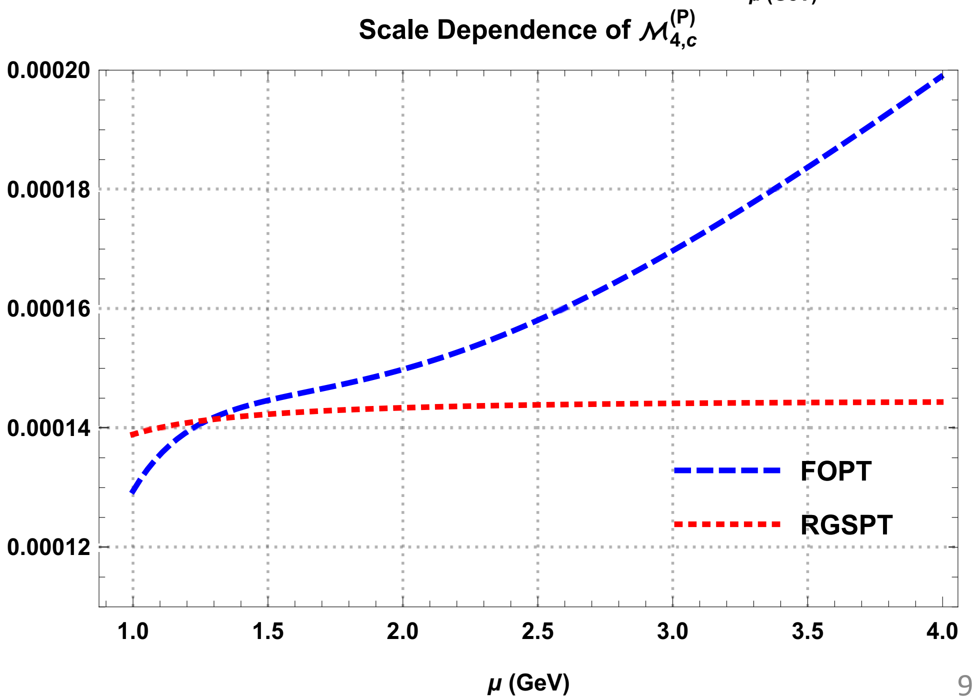
RG behavior of moments contd…











Experimental and lattice moments

Moments	Ref. [1]	Ref. [2]
$\mathcal{M}_1^V$	2.121 ± 0.036	2.154 ± 0.023
$\mathcal{M}_2^V$	1.478 ± 0.028	1.490 ± 0.017
$\mathcal{M}_3^V$	1.302 ± 0.027	1.308 ± 0.016
$\mathcal{M}_4^V$	1.243 ± 0.028	1.248 ± 0.016

Moments for the vector channel $10^{-n} GeV^{-2n}$.

Moments	Ref. [3]	Ref. [4]	Ref. [5]	Ref. [6]	Ref. [7]
$\mathcal{M}_1^P$	1.404 ± 0.019	1.395 ± 0.005	1.385 ± 0.007	1.386 ± 0.005	1.387 ± 0.004
$\mathcal{M}_2^P$	1.359 ± 0.041	1.365 ± 0.012	1.345 ± 0.032	1.349 ± 0.012	1.344 ± 0.010
$\mathcal{M}_3^P$	1.425 ± 0.059	1.415 ± 0.010	1.406 ± 0.048	1.461 ± 0.050	1.395 ± 0.022

Pseudoscalar moment from lattice QCD (units of $10^{-n} GeV^{-2n}$).}

1. B. Dehnadi, A. H. Hoang, V. Mateu and S. M. Zebarjad, JHEP **09** (2013), 103
2. K. G. Chetyrkin, J. H. Kuhn, A. Maier, P. Maierhofer, P. Marquard, M. Steinhauser and C. Sturm, PhysRevD.96.116007

Vector moments

3. I. Allison *et al.* [HPQCD], Phys. Rev. D **78** (2008), 054513
4. C. McNeile *et al.*, Phys. Rev. D **82** (2010), 034512
5. K. Nakayama, B. Fahy and S. Hashimoto, Phys. Rev. D **94** (2016) no.5, 054507
6. Y. Maezawa and P. Petreczky, Phys. Rev. D **94** (2016) no.3, 034507
7. P. Petreczky and J. H. Weber, Phys. Rev. D **100** (2019) no.3, 034519

Pseudoscalar Moments

Application to charm moments: Vector moments

Sources	Moments	FOPT $m_c(3\,\text{GeV})$	FOPT Theo. Unc. α_s	FOPT Theo. Unc. μ	FOPT Theo. Unc. n.p.	FOPT Theo. Unc. total	FOPT Exp. Unc.	RGSPT $m_c(3\,\text{GeV})$	RGSPT Theo. Unc. α_s	RGSPT Theo. Unc. μ	RGSPT Theo. Unc. n.p.	RGSPT Theo. Unc. total	RGSPT Exp. Unc.
Ref. [1]	$\mathcal{M}_1^V$	1005.4(13.9)	3.2	7.6	0.2	8.3	11.2	1000.2(12.3)	3.7	1.9	2.3	4.8	11.3
	$\mathcal{M}_2^V$	997.2(19.8)	4.7	11.3	14.4	18.9	6.1	988.5(9.2)	5.4	1.6	3.5	6.6	6.3
	$\mathcal{M}_3^V$	1022.1(127.8)	3.4	41.8	120.6	127.7	4.0	983.4(9.2)	6.7	1.4	3.9	7.8	4.9
	$\mathcal{M}_4^V$	1077.3(113.6)	1.0	100.5	52.9	113.6	2.8	980.5(8.9)	7.7	0.9	1.8	8.0	3.9
Ref. [2]	$\mathcal{M}_1^V$	995.4(10.8)	3.3	7.6	0.0	8.3	6.9	990.1(8.5)	3.6	2.0	2.3	4.8	7.0
	$\mathcal{M}_2^V$	994.6(19.5)	4.7	11.3	14.7	19.1	3.7	985.8(7.7)	5.4	1.6	3.5	6.6	3.8
	$\mathcal{M}_3^V$	1021.3(126.5)	3.4	41.9	126.5	133.3	2.3	982.3(8.3)	6.7	1.4	3.9	7.8	2.8
	$\mathcal{M}_4^V$	1076.8(113.8)	1.0	100.7	52.9	113.8	1.6	979.8(8.3)	7.7	0.9	1.8	8.0	2.2

The pole mass of the charm quark is used as input in the non-perturbative condensate terms. $\mu \in [1, 4]\ \text{GeV}$

Sources	Moments	FOPT $m_c(3\,\text{GeV})$	FOPT Theo. Unc. α_s	FOPT Theo. Unc. μ	FOPT Theo. Unc. n.p.	FOPT Theo. Unc. total	FOPT Exp. Unc.	RGSPT $m_c(3\,\text{GeV})$	RGSPT Theo. Unc. α_s	RGSPT Theo. Unc. μ	RGSPT Theo. Unc. n.p.	RGSPT Theo. Unc. total	RGSPT Exp. Unc.
Ref. [1]	$\mathcal{M}_1^V$	1004.8(13.7)	3.3	7.1	1.4	7.9	11.2	1000.9(12.1)	3.7	1.9	0.9	4.3	11.3
	$\mathcal{M}_2^V$	989.4(9.2)	5.4	3.5	2.0	6.7	6.3	989.6(8.6)	5.5	1.8	1.0	5.8	6.3
	$\mathcal{M}_3^V$	990.9(13.5)	5.4	10.9	2.5	12.6	4.7	984.8(8.7)	6.8	2.4	1.0	7.3	4.8
	$\mathcal{M}_4^V$	1014.5(37.7)	3.8	37.2	2.9	37.5	3.4	980.9(9.8)	8.0	3.9	0.9	9.0	3.9
Ref. [2]	$\mathcal{M}_1^V$	995.1(10.3)	3.3	6.8	0.7	7.6	7.0	990.8(8.2)	3.8	2.0	0.9	4.3	7.0
	$\mathcal{M}_2^V$	987.3(7.4)	5.4	3.2	0.8	6.3	3.8	986.9(7.0)	5.5	1.8	1.0	5.9	3.8
	$\mathcal{M}_3^V$	990.7(12.7)	5.8	10.9	2.7	12.4	2.7	983.7(7.8)	6.8	2.4	1.1	7.3	2.8
	$\mathcal{M}_4^V$	1014.9(37.0)	3.8	36.7	0.8	36.9	2.0	980.2(9.3)	8.1	3.9	1.0	9.0	2.2

Application to charm moments: pseudoscalar moments

Sources	Moments	FOPT $m_c(3\,\mathrm{GeV})$	FOPT Theo. Unc. α_s	FOPT Theo. Unc. μ	FOPT Theo. Unc. n.p.	FOPT Theo. Unc. total	FOPT Exp. Unc.	RGSPT $m_c(3\,\mathrm{GeV})$	RGSPT Theo. Unc. α_s	RGSPT Theo. Unc. μ	RGSPT Theo. Unc. n.p.	RGSPT Theo. Unc. total	RGSPT Exp. Unc.
Ref. [3]	$\mathcal{M}_1^P$	983.6(10.0)	1.1	5.0	2.4	5.7	8.2	989.3(9.0)	1.4	3.5	0.7	3.8	8.1
	$\mathcal{M}_2^P$	988.3(12.5)	1.7	6.8	3.6	12.5	9.8	990.5(11.4)	1.5	5.8	0.9	6.0	9.7
	$\mathcal{M}_3^P$	998.9(29.9)	2.2	26.6	10.5	28.6	8.5	985.4(11.6)	3.2	6.4	2.0	7.4	9.0
Ref. [4]	$\mathcal{M}_1^P$	987.1(6.1)	1.1	5.0	2.3	5.5	2.4	992.8(4.5)	1.4	3.5	3.5	0.7	2.3
	$\mathcal{M}_2^P$	986.9(8.3)	1.7	6.7	3.6	7.8	2.7	989.1(6.6)	1.5	5.8	0.9	6.0	2.7
	$\mathcal{M}_3^P$	1000.2(28.6)	2.2	26.5	10.4	28.6	1.4	986.9(7.6)	3.2	6.4	2.0	7.4	1.5
Ref. [5]	$\mathcal{M}_1^P$	991.7(6.4)	1.1	4.9	2.2	5.5	3.2	997.3(5.0)	1.4	3.5	0.7	3.8	3.2
	$\mathcal{M}_2^P$	991.5(10.9)	1.6	6.8	3.5	7.9	7.5	993.6(9.6)	1.5	5.8	0.9	6.1	7.5
	$\mathcal{M}_3^P$	1001.5(29.4)	2.2	26.5	10.3	28.5	7.1	988.2(10.5)	3.2	6.4	2.0	7.4	7.5
Ref. [6]	$\mathcal{M}_1^P$	991.2(6.0)	1.1	4.9	2.3	5.5	2.4	996.8(4.5)	1.4	3.5	0.7	3.8	2.3
	$\mathcal{M}_2^P$	990.5(8.3)	1.6	6.8	3.5	7.9	2.8	992.7(6.6)	1.5	5.8	0.9	6.0	2.7
	$\mathcal{M}_3^P$	993.8(29.7)	2.2	26.7	10.9	28.9	7.0	980.0(10.5)	3.3	6.3	2.0	7.4	7.4
Ref. [7]	$\mathcal{M}_1^P$	990.6(5.9)	1.1	4.9	2.3	5.5	1.9	996.2(4.2)	1.4	3.5	0.7	3.8	1.9
	$\mathcal{M}_2^P$	991.6(8.2)	1.6	3.5	7.9	2.3	9.8	993.7(6.5)	1.5	5.8	0.9	6.1	2.3
	$\mathcal{M}_3^P$	1003.2(28.6)	2.2	26.5	10.2	28.6	3.2	989.9(8.2)	3.2	6.4	2.0	7.4	3.4

Final Value: $m_c(3\,\mathrm{GeV}) = 0.9962(42)\ \mathrm{GeV}$

$$\Longrightarrow\ m_c(m_c) = 1.2811(38)\ \mathrm{GeV}$$

PDG Value: $m_c(m_c) = 1.27 \pm 0.02\ \mathrm{GeV}$

REvolver:

A. H. Hoang, C. Lepenik and V.Mateu,
Comput. Phys. Commun. 270 (2022), 108145
[arXiv:2102.01085 [hep-ph]].

Strong coupling determination

Vector moments

n	$\mathcal{R}_n^V$ Refs. [1,2]
1	1.770 ± 0.017
2	1.1173 ± 0.0023
3	1.03536 ± 0.00084

Pseudoscalar moments

Moments	Ref. [3]	Ref. [4]	Ref. [5]	Ref. [6]	Ref. [7]	Ref. [8]
$\mathcal{M}_0^P$	1.708 ± 0.007	1.708 ± 0.005	–	1.699 ± 0.008	1.705 ± 0.005	1.7037 ± 0.0027
$\mathcal{R}_1^P$	1.197 ± 0.004	–	1.188 ± 0.004	1.199 ± 0.004	1.1886 ± 0.013	1.1881 ± 0.0007
$\mathcal{R}_2^P$	1.033 ± 0.004	–	1.0341 ± 0.0018	1.0344 ± 0.0013	1.0324 ± 0.0016	–

These $\mathcal{M}_n^P$ is in the units of 10^{-n} GeV^{-2n}

Refs.:

1. D. Boito and V. Mateu, Phys. Lett. B **806** (2020), 135482
2. D. Boito and V. Mateu, JHEP **03** (2020), 094

Vector moments

3. I. Allison *et al.* [HPQCD], Phys. Rev. D **78** (2008), 054513
4. C. McNeile *et al.* , Phys. Rev. D **82** (2010), 034512
5. Y. Maezawa and P. Petreczky, Phys. Rev. D **94** (2016) no.3, 034507
6. K. Nakayama, B. Fahy and S. Hashimoto, Phys. Rev. D **94** (2016) no.5, 054507
7. P. Petreczky and J. H. Weber, Phys. Rev. D **100** (2019) no.3, 034519
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Pseudoscalar Moments

Strong coupling determination: Vector moments

$\overline{\text{MS}}$ value of the quark mass in the condensate terms:

$$m_c(q) = \overline{m}_c \int_{x(\overline{m}_c)}^{x(q)} dx\, e^{\left(\frac{\gamma(x)}{\beta(x)}\right)}$$

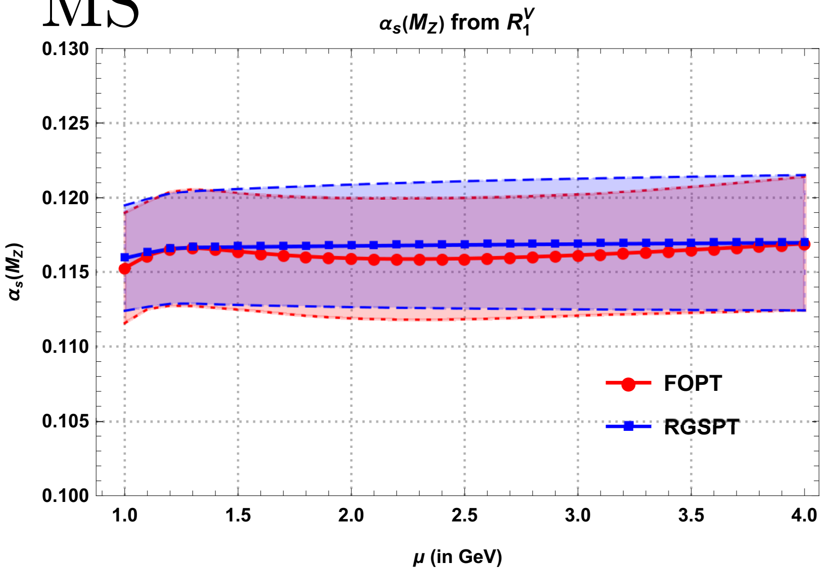
Moment	FOPT						RGSPT					
	$\alpha_s(M_Z)$	Theo. Unc.				Exp. Unc.	$\alpha_s(M_Z)$	Theo. Unc.				Exp. Unc.
		m_c	μ	n.p.	total			m_c	μ	n.p.	total	
$\mathcal{R}_1^V$	0.1167(39)	3	13	8	16	36	0.1167(38)	3	7	8	11	36
$\mathcal{R}_2^V$	0.1163(31)	4	27	12	29	11	0.1163(18)	3	8	12	15	11
$\mathcal{R}_3^V$	0.1159(60)	4	58	14	60	5	0.1159(17)	3	6	14	16	5

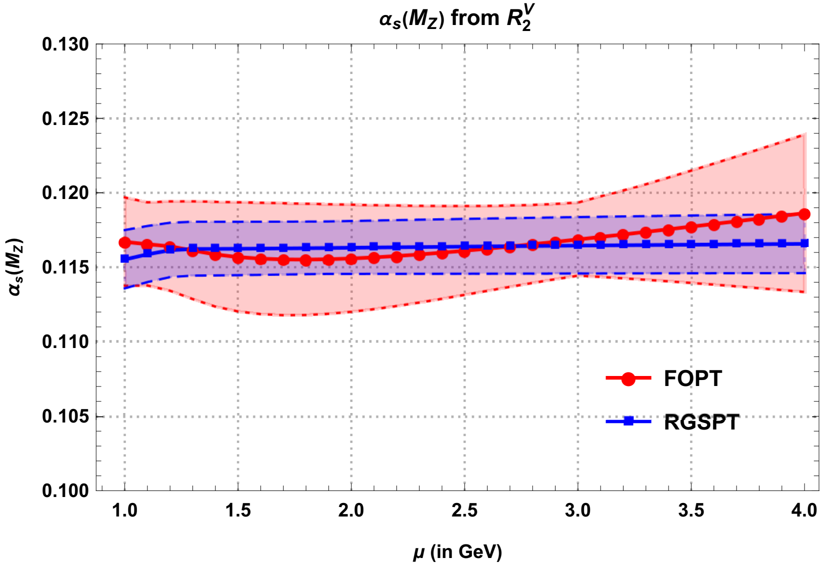
Pole value of the quark mass in the condensate terms:

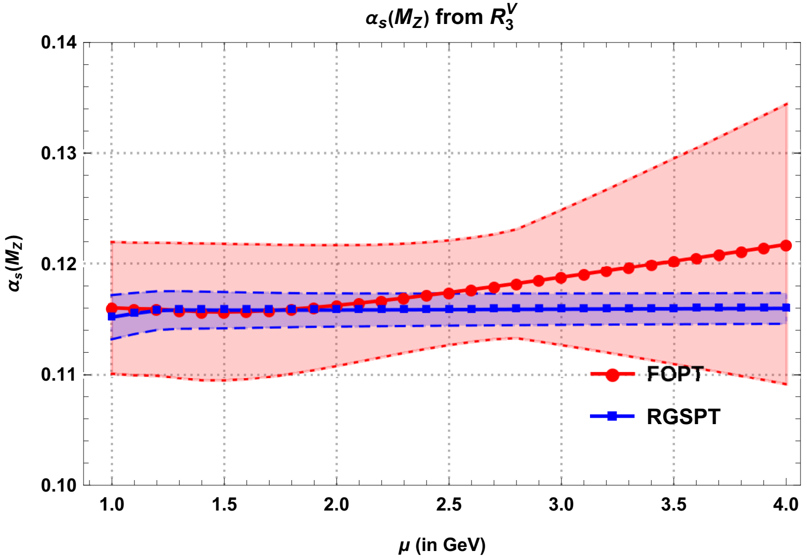
Moment	FOPT						RGSPT					
	$\alpha_s(M_Z)$	Theo. Unc.				Exp. Unc.	$\alpha_s(M_Z)$	Theo. Unc.				Exp. Unc.
		m_c	μ	n.p.	total			m_c	μ	n.p.	total	
$\mathcal{R}_1^V$	0.1169(38)	3	13	5	15	35	0.1169(36)	2	6	5	8	35
$\mathcal{R}_2^V$	0.1164(28)	4	24	9	26	10	0.1164(15)	2	5	6	8	10
$\mathcal{R}_3^V$	0.1159(30)	3	27	13	30	5	0.1159(14)	2	2	6	6	5

Behavior of strong coupling determination from moments

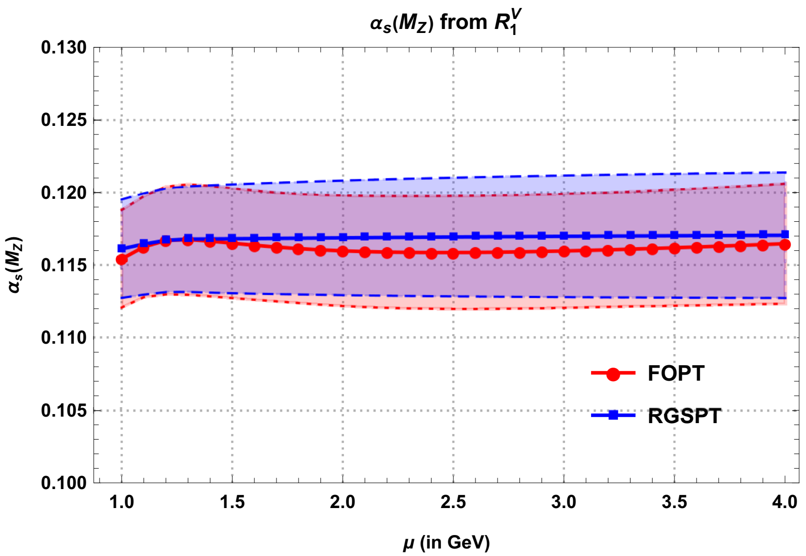
$\overline{\mathrm{MS}}$

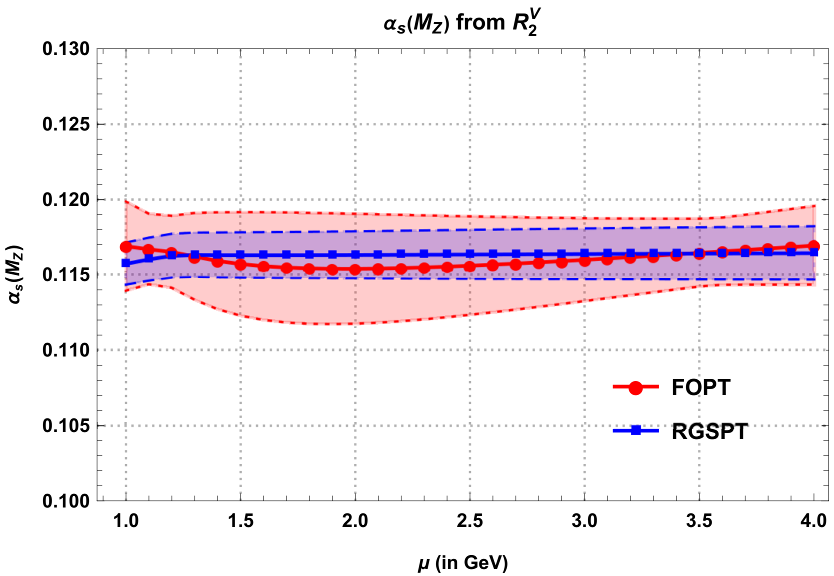


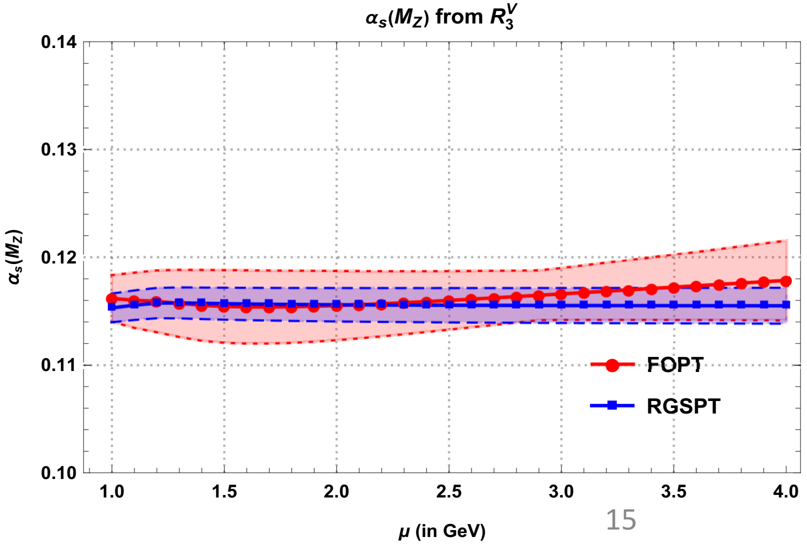




Pole mass in the condensate terms







Strong coupling determination from lattice moments

Sources	Moments	FOPT $\alpha_s(M_Z)$	FOPT Theo. Unc. m_c	FOPT Theo. Unc. μ	FOPT Theo. Unc. n.p.	FOPT Theo. Unc. total	FOPT Exp. Unc.	RGSPT $\alpha_s(M_Z)$	RGSPT Theo. Unc. m_c	RGSPT Theo. Unc. μ	RGSPT Theo. Unc. n.p.	RGSPT Theo. Unc. total	RGSPT Exp. Unc.
Ref. [3]	$\mathcal{M}_0^P$	0.1172(20)	3	19	3	19	6	0.1172(8)	3	3	3	5	6
	$\mathcal{R}_1^P$	0.1182(43)	4	42	5	43	6	0.1181(15)	3	12	5	13	6
	$\mathcal{R}_2^P$	0.1150(53)	4	50	9	51	15	0.1149(18)	3	7	9	11	15
Ref. [4]	$\mathcal{M}_0^P$	0.1172(20)	3	19	3	19	5	0.1172(7)	3	3	3	5	5
Ref. [5]	$\mathcal{M}_0^P$	0.1168(48)	3	8	6	47	7	0.1168(13)	3	9	6	11	7
	$\mathcal{R}_1^P$	0.1152(51)	4	50	8	50	6	0.1152(13)	3	7	8	11	6
Ref. [6]	$\mathcal{M}_0^P$	0.1164(20)	3	18	4	19	7	0.1164(9)	3	3	4	5	7
	$\mathcal{R}_1^P$	0.1182(43)	4	42	5	43	6	0.1184(15)	3	13	5	14	6
	$\mathcal{R}_2^P$	0.1153(50)	4	49	8	50	5	0.1153(12)	3	7	8	7	5
Ref. [7]	$\mathcal{M}_0^P$	0.1169(20)	3	19	3	19	5	0.1169(7)	3	13	3	5	5
	$\mathcal{R}_1^P$	0.1169(47)	4	47	6	47	2	0.1169(12)	3	10	6	12	2
	$\mathcal{R}_2^P$	0.1146(53)	4	52	9	53	6	0.1146(13)	3	6	9	11	6
Ref. [8]	$\mathcal{M}_0^P$	0.1168(19)	3	19	3	19	2	0.1168(13)	3	9	6	11	7
	$\mathcal{R}_1^P$	0.1168(47)	4	47	6	47	1	0.1168(12)	3	1	6	11	1

Final value: $\alpha_s(M_Z) = \{0.1172(7), 0.1169(7)\}$

$$\implies \alpha_s(M_Z) = 0.1171(7)$$

PDG value: $\alpha_s(M_Z) = 0.1179(9)$

Refs.:

3 . I. Allison *et al.* [HPQCD], Phys. Rev. D **78** (2008), 054513

4 . C. McNeile, C. T. H. Davies, E. Follana, K. Hornbostel and G. P. Lepage, Phys. Rev. D **82** (2010), 034512

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Future directions

1. Proper correlation of different moments.
2. Truncation uncertainty.
3. Imroved continuum contributions.

$$\mathcal{M}_q^{V,n} = \int_0^\infty \frac{ds}{s^{n+1}} R_{q\bar{q}}(s) \quad = \int_0^{s_{\mathrm{th.}}} \frac{ds}{s^{n+1}} R_{q\bar{q}}^{\mathrm{exp.}}(s) + \int_{s_{\mathrm{th.}}}^\infty \frac{ds}{s^{n+1}} R_{q\bar{q}}^{\mathrm{cont.}}(s)$$

$$R_{\bar{q}q}^{\mathrm{cont.}} \equiv 12\pi\,\mathrm{Im}\,\Pi^{\mathrm{V}}(s+i\epsilon) = 12\pi \times \frac{1}{2i}\left(\Pi^{\mathrm{V}}(s+i\epsilon) - \Pi^{\mathrm{V}}(s-i\epsilon)\right)$$

Analytic continuation: M. S. A. Alam Khan, “Renormalization group summation and analytic continuation from spacelike to timeline regions,” Phys. Rev. D 108 (2023) no.1, 014028 [arXiv:2306.10262 [hep-ph]].

Other applications in the determination of α_S using finite energy sum rules as well as Borel-Laplace and Laplace sum rules for $e^+e^- \rightarrow \mathrm{hadrons}$

Thank you