Extraction of α_s using energy of a static quark-antiquark pair within hyperasymptotic approximation

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work in collaboration with
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  - Add a NP power correction in a systematic way. Combined expansion of perturbation series and NP terms (Hyperasymptotic expansion [Proc.Roy.Soc.London A,430(1990)]).
  - The mixing between perturbative and NP effects may hinder estimating the real size of NP effects

Superasymptotics

- What is the optimal truncation order?

$$\sum_{n=0}^{N} r_n \alpha^{n+1}$$

- For factorially divergent series, convergence, plateau, divergence
- Truncate in the plateau: Minimize $|r_n \alpha^{n+1}|$
- Superasymptotics[1]

$$N_{\mathrm{optimal}} \sim \frac{\#}{\alpha}$$

- No fixed order. Exponentially suppressed ambiguity $\sim \alpha^{1/2} e^{\frac{-\#}{\alpha}}$

[1] M. V. Berry et al. Proc. R. Soc. A 430, 653 (1990)

Borel summation

- From a divergent series we can construct a Borel transform

$$R \sim \sum_{n=0}^{\infty} r_n \alpha^{n+1} \rightarrow \hat{R}(t) = \sum_{n=0}^{\infty} \frac{r_n}{n!} t^n$$

- The Borel transform $\hat{R}$ has a finite radius of convergence
- Analytic continuation and a Laplace transform of it

$$R_{\rm BS} \equiv \int_0^{\infty} dt\, e^{-t/\alpha} \hat{R}(t)$$

Renormalons

- There are singularities in the integration path
- Contour deformation needed to avoid them
- Principal value (PV) prescription

$$t = xe^{\pm i\eta}$$

$$R_{\rm PV} \equiv \frac{1}{2} \lim_{\eta \to 0^+} \left\{ \int_{C_+} dt\, e^{-t/\alpha} \hat{R}(t) + \int_{C_-} dt\, e^{-t/\alpha} \hat{R}(t) \right\}$$

The Borel plane

- Singularities$\rightarrow$ divergence
- Instantons, renormalons[2]
- $t_d = \frac{2\pi d}{\beta_0}$, $d = \pm 1, \pm 2, \pm 3, \ldots$
-

$$\Delta \hat{R}(t) = Z \frac{1}{(1 - t/t_d)^{1+l}} \sum_{j=0}^{\infty} w_j (1 - t/t_d)^j$$

$$r_n^{(\mathrm{as})}(\mu) = Z \left(\frac{\mu}{Q}\right)^d \frac{1}{t_d^n} \frac{\Gamma(n+1+l)}{\Gamma(1+l)} \sum_{j=0}^{\infty} w_j \prod_{k=1}^{j} \frac{(1+l-k)}{(n+1+l-k)}$$

[2] G. 't Hooft. Subnucl Ser. 15, 943 (1979)

PV Borel sums vs. truncation

- Assuming some properties PV Borel sum is renormalization scale and scheme independent
- One huge disadvantage, in principle we need to know *all* the coefficients r_n
- It is possible to relate truncated sums of perturbative series with their principal value Borel sums

Terminants

- Based on Dingle's theory of terminants[3]

$$R = \sum_{n=0}^{\infty} r_n \alpha^{n+1}$$

- We split the series

$$R = \sum_{n=0}^{N} r_n \alpha^{n+1} + \sum_{n=N+1}^{\infty} r_n \alpha^{n+1}$$

[3] R. B. Dingle. *Asymptotic Expansions: Their Derivation and Interpretation*. Academic Press, London (1973)

Terminants

- We will organize the contents in the remainder tail

$$R = \sum_{n=0}^{N} r_n \alpha^{n+1} + \sum_{n=N+1}^{\infty} r_n \alpha^{n+1}$$

- Displaying the divergent contributions to the coefficient

$$r_n = r_n^{(\mathrm{small}\, n)} + r_n^{(\mathrm{as}_{d_1})} + r_n^{(\mathrm{as}_{d_2})} + \ldots$$

$$r_n^{(\mathrm{as}_{d_i})} \equiv (\pm 1)^n K_i \left(\frac{\beta_0}{2\pi |d_i|}\right)^n \Gamma(n + 1 + l_i)$$

$$R = \sum_{n=0}^{N} r_n \alpha^{n+1} + \sum_{n=N+1}^{\infty} r_n^{(\mathrm{as}_{d_1})} \alpha^{n+1} + \sum_{n=N+1}^{\infty} \left(r_n - r_n^{(\mathrm{as}_{d_1})}\right) \alpha^{n+1}$$

$$\sum_{n=N+1}^{\infty} r_n^{(\mathrm{as}_{d_1})} \alpha^{n+1} \rightarrow T_{d_1}$$

Terminants

- $r_n^{(\mathrm{as}_{d_i})} = (\pm 1)^n K_i A_i^n \Gamma(n+1+l_i) \qquad A_i \equiv \frac{\beta_0}{2\pi |d_i|}$

$$d_i < 0 \implies T_{d_i} \equiv (-1)^{N+1} \frac{1}{\alpha^{l_i}} K_i A_i^{N+1} \int_0^\infty dt\, e^{-t/\alpha} \frac{t^{N+1+l_i}}{1+A_i t}$$

$$d_i > 0 \implies T_{d_i} \equiv \frac{1}{\alpha^{l_i}} K_i A_i^{N+1} \mathrm{PV} \int_0^\infty dt\, e^{-t/\alpha} \frac{t^{N+1+l_i}}{1-A_i t}$$

Terminants

- $$R = \sum_{n=0}^{N} r_n \alpha^{n+1} + \sum_{n=N+1}^{\infty} r_n^{(\mathrm{as}_{d_1})} \alpha^{n+1} + \sum_{n=N+1}^{\infty} \left(r_n - r_n^{(\mathrm{as}_{d_1})}\right) \alpha^{n+1}$$

$$\sum_{n=N+1}^{\infty} r_n^{(\mathrm{as}_{d_1})} \alpha^{n+1} \to T_{d_1}$$

$$R_{\mathrm{PV}} \approx \sum_{n=0}^{N} r_n \alpha^{n+1} + T_{d_1}$$

- We can carry on

$$r_n = r_n^{(\mathrm{small}\ n)} + r_n^{(\mathrm{as}_{d_1})} + r_n^{(\mathrm{as}_{d_2})} + \ldots$$

Terminants

- Splitting again the series

$$R = \sum_{n=0}^{N_{d_1}} r_n \alpha^{n+1} + \sum_{n=N_{d_1}+1}^{\infty} r_n^{(\mathrm{as}_{d_1})} \alpha^{n+1} + \sum_{n=N_{d_1}+1}^{N_{d_2}} (r_n - r_n^{(\mathrm{as}_{d_1})}) \alpha^{n+1}$$

$$+ \sum_{n=N_{d_2}+1}^{\infty} r_n^{(\mathrm{as}_{d_2})} \alpha^{n+1} + \sum_{n=N_{d_2}+1}^{\infty} (r_n - r_n^{(\mathrm{as}_{d_1})} - r_n^{(\mathrm{as}_{d_2})}) \alpha^{n+1}$$

- $\sum_{n=N_{d_2}+1}^{\infty} r_n^{(\mathrm{as}_{d_2})} \alpha^{n+1} \sim T_{d_2}$

$$R_{\mathrm{PV}} \approx \sum_{n=0}^{N_{d_1}} r_n \alpha^{n+1} + T_{d_1} + \sum_{n=N_{d_1}+1}^{N_{d_2}} (r_n - r_n^{(\mathrm{as}_{d_1})}) \alpha^{n+1} + T_{d_2}$$

Hyperasymptotics

- In general ($N_{d_0} \equiv 0$)

$$R_{\mathrm{PV}} = \sum_{i=0}^{\cdots} \left\{ \sum_{n=N_{d_i}+1}^{N_{d_{i+1}}} \left(r_n - \sum_{j=1}^{i} r_n^{(\mathrm{as}:d_j)}\right)\alpha^{n+1} + T_{d_{i+1}} \right\}$$

- Truncating in the plateau $N \sim \frac{|t_{d_i}|}{\alpha}$ where $t_{d_i} = \frac{2\pi d_i}{\beta_0}$

$$N_{\mathrm{P}}(d_i) \equiv \frac{2\pi|d_i|}{\beta_0 \alpha}(1 - c\alpha)$$

The singlet static potential in the large β_0 approximation

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$$V_{\beta_0} = \frac{-2C_F\alpha(\mu)}{\pi}\sum_{n=0}^{\infty}\int_0^{\infty} dq\,\frac{\sin(qr)}{qr}\left\{\frac{\beta_0\alpha(\mu)}{4\pi}\log\left(\frac{\mu^2}{q^2}e^{-c_X}\right)\right\}^n \quad (1)$$

$u = \frac{\beta_0}{4\pi}t$ and[4]

$$\hat{V}_{\beta_0}(t(u)) = \frac{-C_F}{\pi^{1/2}r}e^{-c_X u}\left(\frac{r^2\mu^2}{4}\right)^u\frac{\Gamma(1/2-u)}{\Gamma(1+u)}$$

IR renormalons at $u = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \ldots$

$$V_{\mathrm{PV}} \equiv \mathrm{PV}\int_0^{\infty} dt\, e^{-t/\alpha}\,\hat{V}_{\mathrm{large}\beta_0}(t)$$

[4] U. Aglietti et al. Phys.Lett.B 364 (1995) 75

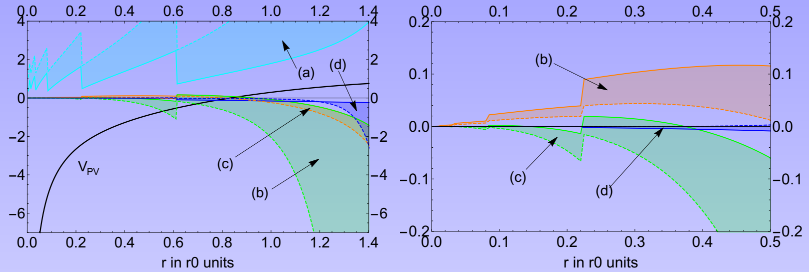
The singlet static potential in the large β_0 approximation

- $\overline{MS}$ and lattice scheme
  - $c_{\overline{\rm MS}} = -5/3$
  - $c_{\rm latt} \approx -8.38807$
- $n_f = 0$
- $\Lambda_{\rm QCD}^{\overline{\rm MS}} = 0.602 r_0^{-1} \approx 0.238\text{GeV}$[5]
- $\mu = 1/r$
- Two values of c

$$N_{\rm P}(1) = \frac{2\pi}{\beta_0 \alpha(1/r)} \left(1 - c\alpha(1/r)\right)$$

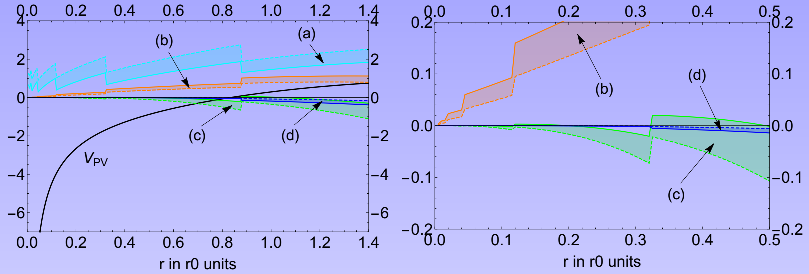
[5]S. Capitani et al. Nucl.Phys.B 544 (1999) 669-698

$\overline{\mathrm{MS}}$ $n_f = 0$



- V_{PV}
- (a) $V_{\mathrm{PV}} - V_{\mathrm{P}}$
- (b) $V_{\mathrm{PV}} - V_{\mathrm{P}} - T_1$
- (c) $V_{\mathrm{PV}} - V_{\mathrm{P}} - T_1 - \sum_{n=N_{\mathrm{P}}(1)+1}^{N_{\mathrm{P}}(3)} (V_n - V_n^{(\mathrm{as}_1)})\alpha^{n+1}$
- (d) $V_{\mathrm{PV}} - V_{\mathrm{P}} - T_1 - \sum_{n=N_{\mathrm{P}}(1)+1}^{N_{\mathrm{P}}(3)} (V_n - V_n^{(\mathrm{as}_1)})\alpha^{n+1} - T_3$

Lattice $n_f = 0$



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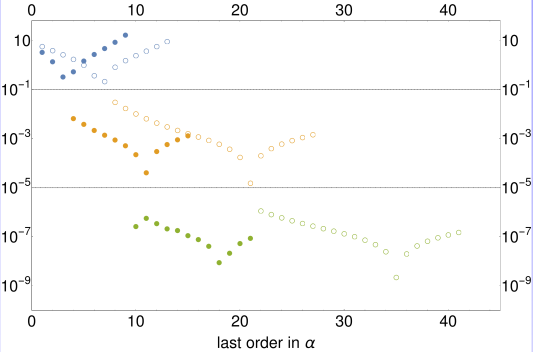
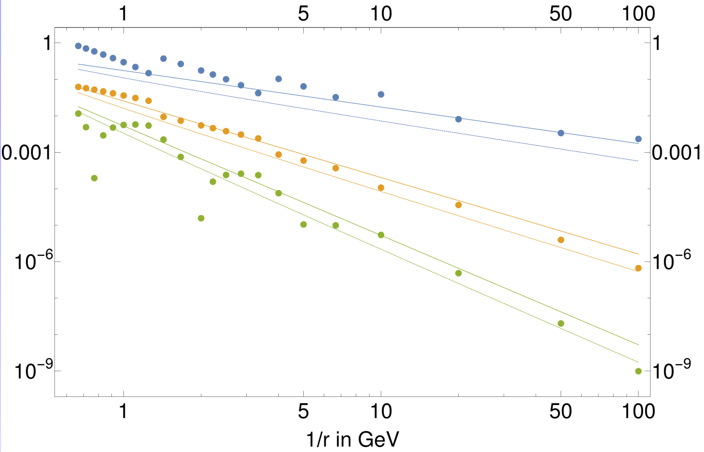
Behavior of the expansion for $r = 0.04r_0$



- $|V_{\rm PV} - \sum_{n=0} V_n \alpha^{n+1}|$
- $|V_{\rm PV} - \sum_{n=0}^{N} V_n \alpha^{n+1} - T_1 - \sum_{n=N_{\rm P}(1)+1} (V_n - V_n^{(\rm as_1)}) \alpha^{n+1}|$
- $|V_{\rm PV} - \sum_{n=0}^{N_{\rm P}(1)} V_n \alpha^{n+1} - T_1 - \sum_{n=N_{\rm P}(1)+1}^{N_{\rm P}(3)} (V_n - V_n^{(\rm as_1)}) \alpha^{n+1} - T_3 - \sum_{n=N_{\rm P}(3)+1} (V_n - V_n^{(\rm as_1)} - V_n^{(\rm as_3)}) \alpha^{n+1}|$

Exponential scaling in α. $\overline{\mathrm{MS}}$ with $n_f = 3$



- $r\left(V_{\mathrm{PV}} - \sum_{n=0}^{N_{\mathrm{P}}(1)} V_n \alpha^{n+1}\right)$ vs $\alpha^{1/2} e^{-\frac{2\pi}{\beta_0 \alpha}}$
- $r\left(V_{\mathrm{PV}} - \sum_{n=0}^{N_{\mathrm{P}}(1)} V_n \alpha^{n+1} - T_1\right)$ vs $\alpha^{1/2} e^{-\frac{2\pi}{\beta_0 \alpha}(1+\log 3)}$
- $r\left(V_{\mathrm{PV}} - \sum_{n=0}^{N_{\mathrm{P}}(1)} V_n \alpha^{n+1} - T_1 - \sum_{n=N_{\mathrm{P}}(1)+1}^{N_{\mathrm{P}}(3)} (V_n - V_n^{(\mathrm{as}_1)}) \alpha^{n+1}\right)$ vs $\alpha^{1/2} e^{-\frac{6\pi}{\beta_0 \alpha}}$

The static energy of a quark-antiquark pair

- $E^{\rm latt}(r) - E^{\rm latt}(r_{\rm ref}) = E^{\rm th}(r) - E^{\rm th}(r_{\rm ref})$
- In pNRQCD the singlet static energy admits the expression

$$E^{\rm th}(r) = V(r, \nu_{\rm us} = \nu_{\rm s}) + \delta V_{\sf RG}(r, \nu_{\rm s}, \nu_{\rm us}) + \delta E_{\rm us}(r, \nu_{\rm us})$$

- $E^{\rm th}$ has a $u = 1/2$ renormalon
- This renormalon is r independent
- We use the derivative of the static energy ($\frac{d}{dr} V \equiv F$)

$$\begin{aligned}\mathcal{F} \equiv & \frac{d}{dr} E^{\rm th}(r) \\ = & F(r, \nu_{\rm us} = \nu_{\rm s}) + \frac{d}{dr} \delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us}) + \frac{d}{dr} \delta E_{\rm us}(r, \nu_{\rm us})\end{aligned}$$

- We fit

$$E^{\rm latt}(r) - E^{\rm latt}(r_{\rm ref}) = \int_{r_{\rm ref}}^{r} dr' \, \mathcal{F}(r')$$

Terminants

- The leading singularity $d = 3$ and the terminant takes the form

$$T = \sqrt{\alpha(\mu)} K^{(\mathrm{P})} r\mu^3 e^{-\frac{6\pi}{\beta_0\alpha(\mu)}} \left(\frac{\beta_0\alpha(\mu)}{4\pi}\right)^{-3b} \left(1 + \bar{K}_1^{(\mathrm{P})}\alpha(\mu) + \mathcal{O}\left(\alpha^2(\mu)\right)\right)$$

where $\eta_c = -3b + \frac{6\pi c}{\beta_0} - 1$; $b = \frac{\beta_1}{2\beta_0^2}$ and

$$\begin{aligned} K^{(\mathrm{P})} &= -\frac{Z_3^F 2^{1-3b}\pi 3^{3b+1/2}}{\Gamma(1+3b)}\beta_0^{-1/2}\left[-\eta_c + \frac{1}{3}\right] \\ \bar{K}_1^{(\mathrm{P})} &= \frac{\beta_0/(3\pi)}{-\eta_c + \frac{1}{3}}\left[-3b_1 b\left(\frac{1}{2}\eta_c + \frac{1}{3}\right) - \frac{1}{12}\eta_c^3 + \frac{1}{24}\eta_c - \frac{1}{1080}\right] \end{aligned}$$

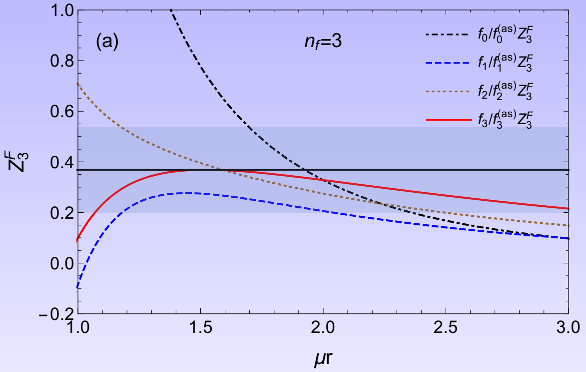
Normalization of the $u = 3/2$ renormalon

- The PV Borel sum of the force requires knowledge of Z_3^F (recall $Z_3^F = 2Z_3^V$)

$$r^2 f_n^{\rm as} = Z_3^F (r\mu)^3 \left(\frac{\beta_0}{6\pi}\right)^n \frac{\Gamma(n+1+3b)}{\Gamma(1+3b)} \left\{1 + \frac{3b}{n+3b} b_1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}$$

- We estimate it dividing by the exact value ($x = 1.52$)

$$Z_3^F\big|_{n_f=3} = 0.37^{-0.06}_{-0.16}(\Delta x) + 0.02(\mathrm{N^2LO}) - 0.05(\mathcal{O}(1/n)) + 0.005(\mathrm{us}) = 0.37(17)$$



$$F = \frac{d}{dr} V$$

$$V = \sum_{n=0}^{\infty} V_n(r, \nu_s, \nu_{us}) \alpha^{n+1}(\nu_s)$$

$$V_n(r, \nu_s, \nu_{us}) = -\frac{C_F}{r} \frac{1}{(4\pi)^n} a_n(r, \nu_s, \nu_{us})$$

$$a_0 = 1$$

$$a_1(r, \nu_s) = a_1 + 2\beta_0 \log\left(\nu_s e^{\gamma_E} r\right)$$

$$a_2(r, \nu_s) = a_2 + \frac{\pi^2}{3}\beta_0^2 + (4a_1\beta_0 + 2\beta_1) \log\left(\nu_s e^{\gamma_E} r\right) + 4\beta_0^2 \log^2\left(\nu_s e^{\gamma_E} r\right)$$

$$\begin{aligned} a_3(r, \nu_s, \nu_{us}) = {} & a_3 + a_1\beta_0^2\pi^2 + \frac{5\pi^2}{6}\beta_0\beta_1 + 16\zeta_3\beta_0^3 \\ & + \left(2\pi^2\beta_0^3 + 6a_2\beta_0 + 4a_1\beta_1 + 2\beta_2\right) \log\left(\nu_s e^{\gamma_E} r\right) + \frac{16}{3} C_A^3 \pi^2 \log\left(\nu_{us} e^{\gamma_E} r\right) \\ & + \left(12a_1\beta_0^2 + 10\beta_0\beta_1\right) \log^2\left(\nu_s e^{\gamma_E} r\right) + 8\beta_0^3 \log^3\left(\nu_s e^{\gamma_E} r\right) \end{aligned}$$

Details on the fits I

- Data from Phys. Rev. D100, 114511
- $\beta = 8.4$ lattice spacing $a = 0.025\,\mathrm{fm} = 0.125\,\mathrm{GeV}^{-1}$
- We consider the ranges
  - Set I: $0.353\ \text{GeV}^{-1} \leq r \leq 0.499\ \text{GeV}^{-1}$; 8 points
  - Set II: $0.353\ \text{GeV}^{-1} \leq r \leq 0.612\ \text{GeV}^{-1}$; 17 points
  - Set III: $0.353\ \text{GeV}^{-1} \leq r \leq 0.8002\ \text{GeV}^{-1}$; 31 points
  - Set IV: $0.353\ \text{GeV}^{-1} \leq r \leq 1\ \text{GeV}^{-1}$; 50 points
- $r_{\mathrm{ref}} = 0.353\mathrm{GeV}^{-1}$
- The central values for the soft and ultrasoft scales are
  - $\nu_{\mathrm{s}} = 1/r$
  - $\nu_{\mathrm{us}} = \frac{C_A \alpha(\nu_{\mathrm{s}})}{2r}$

Details on the fits II

We will explore the following orders

- LL/LO

$$\mathcal{F}_{\mathrm{LO}}(r) = F(r, \nu_{\mathrm{us}} = \nu_{\mathrm{s}})\Big|_{\mathrm{LO\,in}\,\alpha(\nu_{\mathrm{s}})}$$

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- $\mathrm{N^2LL}$

$$\mathcal{F}_{\rm N^2LL}(r) = F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm N^2LO\,in\,\alpha(\nu_s)} + \frac{d}{dr}\delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us})\Big|_{\rm N^2LL}$$

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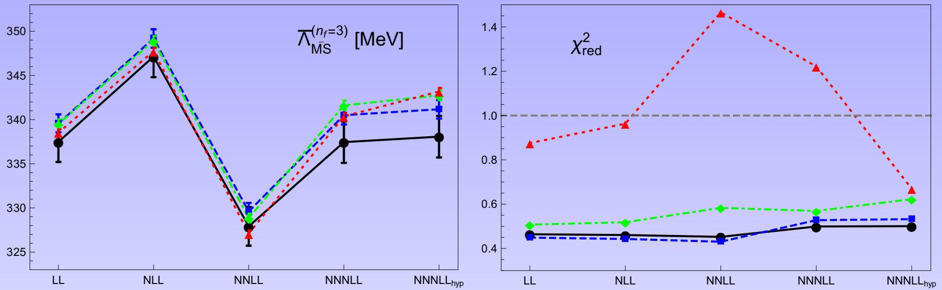
- $\mathrm{N^3LL_{hyp}}$

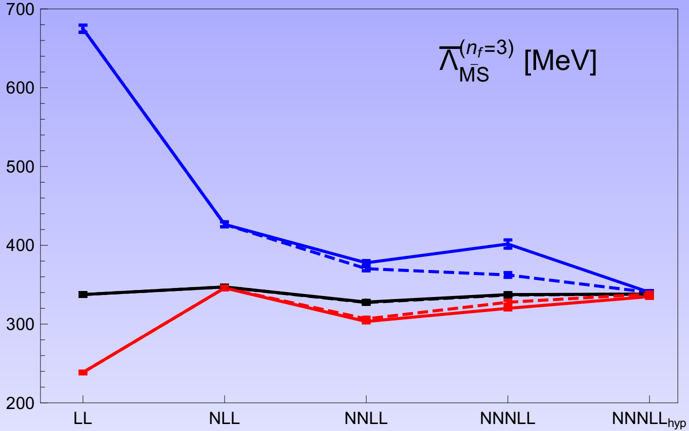
$$\begin{aligned}\mathcal{F}_{\rm N^3LL}(r) =& F(r, \nu_{\rm us} = \nu_{\rm s})\Big|_{\rm N^3LO\,in\,\alpha(\nu_s)} + T(d=3, N_{\rm P}=3, \nu_{\rm s}) + \frac{d}{dr}\delta V_{\rm RG}(r, \nu_{\rm s}, \nu_{\rm us})\Big|_{\rm N^3LL} \\ &+ \frac{d}{dr}\delta E_{\rm us}(r, \nu_{\rm us})\Big|_{\rm LO\,in\,\alpha(\nu_{us})} - T(d=3, N_{\rm P}=0, \nu_{\rm us})\end{aligned}$$

Results



- $r \in [0.353, 0.499] \times \mathrm{GeV}^{-1}$
- $r \in [0.353, 0.612] \times \mathrm{GeV}^{-1}$
- $r \in [0.353, 0.8002] \times \mathrm{GeV}^{-1}$
- $r \in [0.353, 1] \times \mathrm{GeV}^{-1}$

Variations on $\nu_s \equiv x_s/r$ and $\nu_{us} \equiv x_{us}\frac{C_A\alpha(\nu_s)}{2r}$



$x_s = 1$

$x_s = 2$

$x_s = 1/2^{1/2}$

Continuous $x_{us} = 1$; Dashed $\nu_{us} = 1$ GeV

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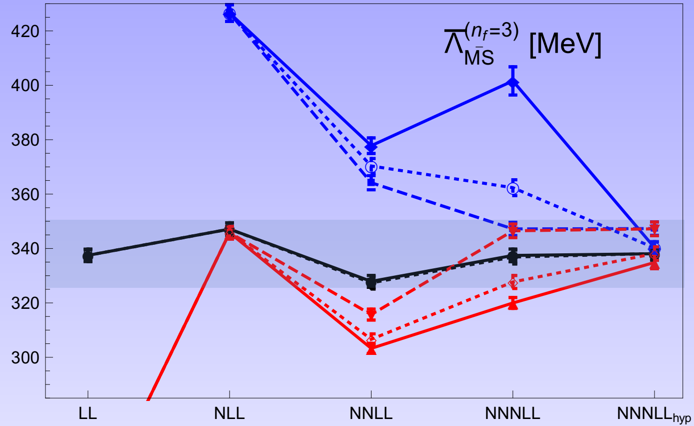

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$x_s = 2$

$x_s = 1/2^{1/2}$

Continuous $x_{us} = 1$; Dashed $x_{us} = x_s$; Dotted $\nu_{us} = 1$ GeV

Final numbers

$$\text{Set I} \quad \Lambda_{\overline{\rm MS}}^{(n_f=3)} = 338(2)_{\rm stat}(10)_{\rm h.o.}(8)_{r_{\rm ref}}\ \text{MeV} = 338(12)\ \text{MeV}$$

$$\text{Set II} \quad \Lambda_{\overline{\rm MS}}^{(n_f=3)} = 341(1)_{\rm stat}(11)_{\rm h.o.}(6)_{r_{\rm ref}}\ \text{MeV} = 341(14)\ \text{MeV}$$

$$\text{Set III} \quad \Lambda_{\overline{\rm MS}}^{(n_f=3)} = 343(1)_{\rm stat}(13)_{\rm h.o.}(7)_{r_{\rm ref}}\ \text{MeV} = 343(14)\ \text{MeV}$$

$$\text{Set IV} \quad \Lambda_{\overline{\rm MS}}^{(n_f=3)} = 343(0)_{\rm stat}(13)_{\rm h.o.}(9)_{r_{\rm ref}}\ \text{MeV} = 343(16)\ \text{MeV}$$

Therefore our central value result for the strong coupling

- $\alpha^{(n_f=3)}(M_\tau) = 0.3151(65)$
- $\alpha^{(n_f=5)}(M_z) = 0.1181(8)_{\Lambda_{\overline{\rm MS}}}(4)_{M_\tau \to M_z} = 0.1181(9)$

Conclusions

We have constructed an hyperasymptotic expansion for the static energy and for the force regulated with PV prescription. Here we included the second $d = 3$ IR renormalon.

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$$Z_3^F \Big|_{n_f=3} = 0.37(17)$$

- Making use of the hyperasymptotic expansion of principal value Borel sums and $\mathrm{N^3LL}$ resummation we have obtained an estimate of the QCD strong coupling

$$\alpha^{(n_f=5)}(M_z) = 0.1181(9)$$

THANKS!

General Expression

And

$$\Omega_{\substack{IR\\UV}} = \int_0^\infty dt\, e^{-t/\alpha_s(\mu)} \sum_{n=N_P+1}^{\infty} \frac{r_n^{(\mathrm{as})}}{n!} t^n$$
$$\approx \sqrt{\alpha(\mu)} K_X^{(P,\substack{IR\\UV})} \left(\frac{\mu}{Q}\right)^{\substack{+\\-}|d|} e^{\frac{-2\pi|d|}{\beta_0\alpha(\mu)}} \left(\frac{\beta_0\alpha(\mu)}{4\pi}\right)^{-b'} \Big\{1 + \bar{K}_{X,1}^{(P,\substack{IR\\UV})}\alpha(\mu) + \bar{K}_{X,2}^{(P,\substack{IR\\UV})}\alpha^2(\mu) + \mathcal{O}\left(\alpha^3(\mu)\right)\Big\}$$
$$\equiv \Delta\Omega_{\substack{IR\\UV}}(db) + c_1\Delta\Omega_{\substack{IR\\UV}}(db) + \omega_2\Delta\Omega_{\substack{IR\\UV}}(db) + \ldots$$

where

$$\Delta\Omega_{\substack{IR\\UV}}(db) = Z_{\mathcal{O}_d}^X \left(\frac{\mu}{Q}\right)^{\substack{+\\-}|d|} \frac{1}{\Gamma(1+b')} \left(\frac{\beta_0}{2\pi d}\right)^{(N_P+1)} \alpha_X^{(N_P+2)}(\mu) \times I_{\substack{IR\\UV}} \tag{2}$$

General Expression

$$I_{\substack{IR \\ UV}} = \int_0^\infty dx\ x^{N_P+1+b'} \frac{e^{-x}}{1 \mp x^{\frac{\beta_0 \alpha_X(\mu)}{2\pi|d|}}} \tag{3}$$

In case 2), it is possible to show that:

$$S_{\rm PV}(Q) = S_A + \int_0^{\frac{4\pi}{\beta_0 \chi}} dt e^{-t/\alpha_X(\overline{m})} B[S_{\rm PV} - S_A](t)\,, \tag{4}$$

where

$$S_A = \sum_{n=0}^{N_A(|d_{min}|)} p_n \alpha^{n+1}(\mu)\,. \tag{5}$$

General Expression

$$m_{\rm PV} = m_A + K_X^{(A)}\Lambda_X + \mathcal{O}(\alpha\Lambda_X)\,, \tag{6}$$

where

$$m_A = \overline{m} + \lim_{\mu\to\infty;2)} \sum_{n=0}^{N_A} r_n \alpha^{n+1}(\mu) \tag{7}$$

and

$$K_X^{(A)} = \frac{2\pi}{\beta_0} Z_m^X \left(\frac{\beta_0}{4\pi}\right)^b \int_{-c',{\rm PV}}^{\infty} dx\, e^{\frac{-2\pi dx}{\beta_0}} \frac{1}{(-x)^{1+b}}\,. \tag{8}$$

It is also possible to show that

$$m_A = \overline{m} + \int_0^{\frac{4\pi}{\beta_0 \chi}} dt e^{-t/\alpha_X(\overline{m})} B[m_{\rm PV} - \overline{m}](t)\,. \tag{9}$$

Large β_0: comparing method (1) and (2)

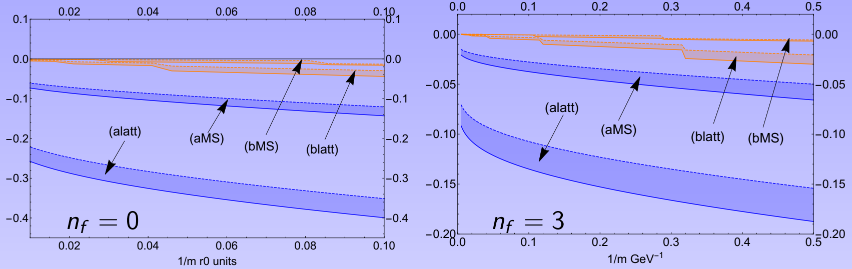

Figure: We plot (a) $m_{\rm PV} - m_A - K_X^{(A)}\Lambda_X$ for $n_f = 0$ (left panel) and $n_f = 3$ (right panel) in the lattice and $\overline{MS}$ scheme. For each case, we generate bands by computing m_A with $c' = 1$ and $c' = c'_{\rm min}$. We also compare with (b) $m_{PV} - m_P - \overline{m}\Omega_m$ obtained with method 1).