Borel-Laplace sum rules for tau decay data

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(talk presented by G.C.)

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Introduction

- We use ALEPH data for the strangeless semihadronic tau decay in the sum rules, for the (V+A)-channel.
- For the (V+ee)-channel, ALEPH data complemented with OPAL and electroproduction data were used (Boito, Peris, et al. (2021)).
- The weight functions in the sum rules are Borel-Laplace, double-pinched.
- For the theoretical expression of the sum rules we use an improved version of the truncated OPE.
  1. For $D = 0$ part we use Adler function $d(Q^2)_{D=0}$ truncated series up to ${\alpha_s}^5$, with the fixed-order (FO) approach in the sum rule.
  2. The $D = 4$ term has the known structure $\sim 1/(Q^2)^2$, while $D = 6$ terms $\sim 1/(Q^2)^3$ have the form $\sim \alpha_s (Q^2)^{k_6}/(Q^2)^3$ with the known nonzero anomalous dimension $k_6 \approx 0.222$. We truncate OPE at $D = 10$ (V+A) and $D = 14$ (V).

The averaged extracted value of α_s will be presented.

Sum rules

The Adler function $\mathcal{D}(Q^2)$ is logarithmic derivative of the quark current polarisation function $\Pi(Q^2)$

$$\mathcal{D}(Q^2) \equiv -2\pi^2 \frac{d\Pi(Q^2)}{d \ln Q^2}, \qquad (1)$$

where $Q^2 \equiv -q^2$ $(= -(q^0)^2 + \vec{q}^2)$. We will consider either the total (V+A)-channel

$$\Pi(Q^2)_{\rm V+A} = \Pi_{\rm V}^{(1)}(Q^2) + \Pi_{\rm A}^{(1)}(Q^2) + \Pi_{\rm A}^{(0)}(Q^2), \qquad (2)$$

or only the V-channel.

According to the general principles of Quantum Field Theory, $\Pi(Q^2;\mu^2)$ and $\mathcal{D}(Q^2)$ are holomorphic (i.e., analytic) functions of Q^2 in the complex Q^2-plane with the exception of the real negative axis $(-\infty, -m_\pi^2)$. If $g(Q^2)$ is an (arbitrary) holomorphic function of Q^2, and we apply the Cauchy theorem to the integral $\oint dQ^2 g(Q^2)\Pi(Q^2;\mu^2)$ along a closed path in the complex Q^2-plane (cf. Figure), we obtain

Sum rules

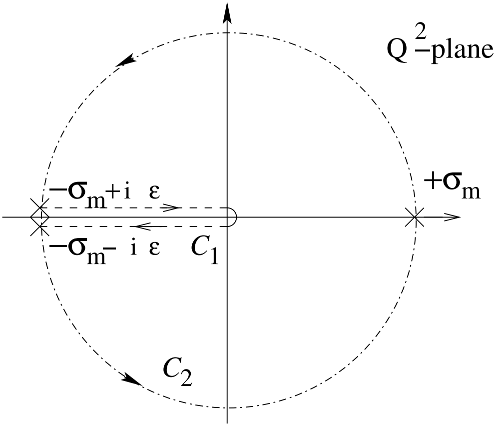

Figure: The closed integration path $C_1 + C_2$ for $\oint dQ^2 g(Q^2)\Pi(Q^2)$. The radius of the circle C_2 is $|Q^2| = \sigma_{\rm m}$ ($= 2.8\ {\rm GeV}^2$ or $3.057\ {\rm GeV}^2$). On the path C_1 we have $\varepsilon \to +0$.

Sum rules

$$\oint_{C_1+C_2} dQ^2 g(Q^2)\Pi(Q^2) = 0 \tag{3a}$$

$$\Rightarrow \int_0^{\sigma_{\rm m}} d\sigma g(-\sigma)\omega_{\rm exp}(\sigma) = -i\pi \oint_{|Q^2|=\sigma_{\rm m}} dQ^2 g(Q^2)\Pi_{\rm th}(Q^2), \tag{3b}$$

where $\omega(\sigma)$ is proportional to the discontinuity (spectral) function of the polarisation function

$$\omega(\sigma) \equiv 2\pi \, {\rm Im} \, \Pi(Q^2 = -\sigma - i\epsilon) \ . \tag{4}$$

Sum rules

Integration by parts replaces the theoretical polarisation function in the sum rule (3b) by the Adler function (1)

$$\int_0^{\sigma_{\rm m}} d\sigma g(-\sigma)\omega_{\rm exp}(\sigma) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, \mathcal{D}_{\rm th}(\sigma_{\rm m} e^{i\phi}) G(\sigma_{\rm m} e^{i\phi}), \qquad (5)$$

where $\mathcal{D}_{\rm th}(Q^2) = -2\pi^2 d\Pi_{\rm th}(Q^2)/d \ln Q^2$, it is given by the theoretical OPE expansion of the Adler function, and the (holomorphic) function G is an integral of g:

$$G(Q^2) = \int_{-\sigma_{\rm m}}^{Q^2} dQ'^2 g(Q'^2). \qquad (6)$$

Sum rules

The quantity $\omega(\sigma)$ was measured to a high precision by ALEPH Collaboration, in semihadronic strangeless τ-decays, cf. Figure.

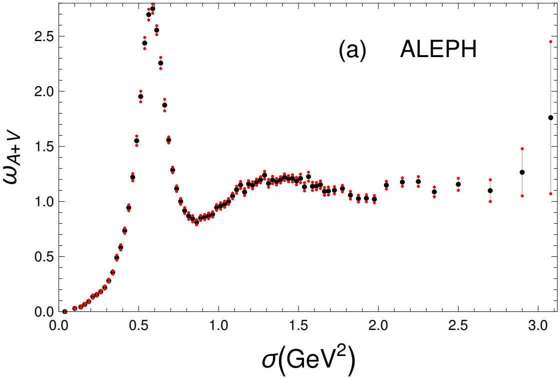

Figure: The spectral function $\omega(\sigma)$ for the (V+A)-channel, as measured by ALEPH Collaboration. The extremely narrow pion peak contribution $2\pi^2 f_\pi^2 \delta(\sigma - m_\pi^2)$ ($f_\pi = 0.1305$ GeV) has to be added to this. The last two bins have large uncertainties, so we exclude them, and this means that $\sigma_{\rm m} = 2.80 \ {\rm GeV}^2$ in the sum rules.

Sum rules

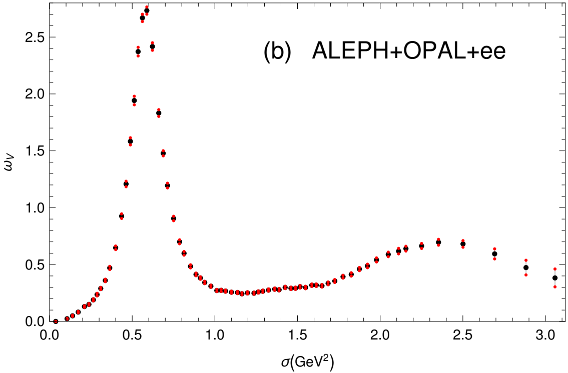

Figure: The spectral function $\omega(\sigma)$ for the (V+ee)-channel, based on a combination of ALEPH, OPAL and electroproduction data (Boito, Maltman, Peris, et al. (2021)); $\sigma_{\max} \equiv \sigma_{\mathrm{m}} = 3.0574\ \mathrm{GeV}^2$.

In the (V+A)-channel of ALEPH, we have data bins with reasonable experimental uncertainties up to $\sigma \leq \sigma_{\mathrm{m}} = 2.8\ \mathrm{GeV}^2$ (77 bins); in the V-channel $\sigma_{\mathrm{m}} = 3.057\ \mathrm{GeV}^2$.

Sum rules

The theoretical OPE expression of the polarisation function has the following form:

$$\Pi_{\rm th}(Q^2;\mu^2) = -\frac{1}{2\pi^2}\ln\left(\frac{Q^2}{\mu^2}\right) + \Pi(Q^2)_{D=0} + \frac{\langle O_4 \rangle}{(Q^2)^2}\left(1+\mathcal{O}(\alpha_s)\right)$$
$$+\frac{\langle O_6 \rangle a(Q^2)^{k_6}}{(Q^2)^3}\left(1+\mathcal{O}(\alpha_s)\right) + \sum_{2p\geq 8}^{D_{\rm max}} \frac{\langle O_{2p} \rangle}{(Q^2)^p}\left(1+\mathcal{O}(\alpha_s)\right),$$

where $k_6 \approx 0.222$ is a nonzero anomalous dimension for $D=6$ term (Boito, Hornung, Jamin (2015)).

Sum rules

The corresponding Adler function (1) is then

$$\mathcal{D}_{\rm th}(Q^2) \equiv -2\pi^2 \frac{d\Pi_{\rm th}(Q^2)}{d \ln Q^2} = 1 + d(Q^2)_{D=0} + \delta d(Q^2)_{m_c} + \\ +4\pi^2 \frac{\langle O_4 \rangle}{(Q^2)^2} + 6\pi^2 \frac{\langle O_6 \rangle a(Q^2)^{k_6}}{(Q^2)^3} + 2\pi^2 \sum_{2p \geq 8}^{D_{\rm max}} \frac{p \langle O_{2p} \rangle}{(Q^2)^p}, \qquad (7)$$

where $a(Q^2) \equiv \alpha_s(Q^2)/\pi$, and the relative $\mathcal{O}(a)$-corrections were neglected.
In the (V+A)-channel, the above condensates are interpreted as $\langle O_{2p} \rangle \mapsto \langle O_{2p} \rangle_{\rm V+A}$, and in the V-channel as $\langle O_{2p} \rangle \mapsto 2\langle O_{2p} \rangle_{\rm V}$. .

$D = 0$ Adler function $d(Q^2)_{D=0}$

The massless $N_f = 3$ perturbation expansion of $d(Q^2)_{D=0}$ in powers of $a(\mu^2) \equiv \alpha_s(\mu^2)/\pi$ is truncated at a^5

$$d(Q^2)_{D=0} = a(\kappa Q^2) + d_1(\kappa)\, a(\kappa Q^2)^2 + \ldots + d_4(\kappa)\, a(\kappa Q^2)^5; \qquad (8)$$

Here $\kappa = \mu^2/Q^2$ (~ 1), the coefficients d_n ($n = 1, 2, 3$) are known exactly (Chetyrkin, Kataev, Baikov, and others (1979-2008)), and d_4 (with $\kappa = 1$) can be estimated

$$d_4 = 275. \pm 63., \qquad (9)$$

where the central value is from ECH approach (Kataev, Starshenko (1995)). Other estimates: $d_4 = 277 \pm 51$ (Boito et al. (2018)); $d_4 = 283$ (Beneke, Jamin (2008)); $d_4 = 338.19$ from a renormalon model (G.C. (2019)).

$$\delta d(Q^2)_{m_c} = \mathcal{C}_1(Q^2)_{m_c} a(Q^2)^2 \qquad (10)$$

are charm quark nondecoupling effects ($m_c \neq \infty$) that are known (Hoang et al. (1994); R.-Sánchez, Pich et al. (2023)) for the V-channel.

Specific Borel-Laplace sum rule

Weight functions $g(Q^2)$ used in the sum rules (5) are those corresponding to the double-pinched Borel-Laplace transforms $B(M^2)$ where M^2 is a complex squared energy parameter ($|M^2| \sim 1 \ \mathrm{GeV}^2$)

$$g_{M^2}(Q^2) = \left(1 + \frac{Q^2}{\sigma_{\rm m}}\right)^2 \frac{1}{M^2} \exp\left(\frac{Q^2}{M^2}\right) \quad \Rightarrow \tag{11}$$

$$G_{M^2}(Q^2) = \left\{ \left[\left(1 + \frac{Q^2}{\sigma_m}\right)^2 - 2\frac{M^2}{\sigma_m}\left(1 + \frac{Q^2}{\sigma_m}\right) + 2\left(\frac{M^2}{\sigma_m}\right)^2 \right] \exp\left(\frac{Q^2}{M^2}\right) - 2\left(\frac{M^2}{\sigma_m}\right)^2 \exp\left(-\frac{\sigma_m}{M^2}\right) \right\}. \tag{12}$$

Specific Borel-Laplace sum rule

$$
\begin{aligned}
B_{\rm th}(M^2;\sigma_{\rm m}) &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi \; G_{M^2}\left(\sigma_{\rm m} e^{i\phi}\right) \mathcal{D}_{\rm th}\left(\sigma_{\rm m} e^{i\phi}\right) \\
&= \left[\left(1 - 2\frac{M^2}{\sigma_{\rm m}}\right) + 2\left(\frac{M^2}{\sigma_{\rm m}}\right)^2 \left(1 - \exp\left(-\frac{\sigma_{\rm m}}{M^2}\right)\right)\right] \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi G_{M^2}(\sigma_{\rm m} e^{i\phi}) d\left(\sigma_{\rm m} e^{i\phi}\right)_{D=0} \\
&\quad + \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\phi G_{M^2}(\sigma_{\rm m} e^{i\phi}) \delta d\left(\sigma_{\rm m} e^{i\phi}\right)_{m_c} \\
&\quad + \sum_{D=4,8,\ldots} B_{\rm th}(M^2;\sigma_{\rm m})_D + B_{\rm th}(M^2;\sigma_{\rm m})_{D=6},
\end{aligned}
\tag{13}
$$

Specific Borel-Laplace sum rule

where we have for $D > 0$ and $D \neq 6$ ($D(= 2p) = 4, 8, 10, \ldots$)

$$B_{\rm th}(M^2; \sigma_{\rm m})_{D=2p} = \frac{2\pi^2 \langle O_{2p} \rangle}{(p-1)!(M^2)^p} \left[1 + 2(p-1)\frac{M^2}{\sigma_{\rm m}} + (p-1)(p-2)\left(\frac{M^2}{\sigma_{\rm m}}\right)^2 \right], \quad (14)$$

$$B_{\rm th}(M^2; \sigma_{\rm m})_{D=6} = \frac{1}{2\pi} \frac{6\pi^2}{{\sigma_{\rm m}}^3} \sum_{j=1}^{2} \langle O_6^{(j)} \rangle \times \int_{-\pi}^{+\pi} d\phi \, e^{-i3\phi} G_{M^2}(\sigma_{\rm m} e^{i\phi}) \left(\frac{a(\sigma_{\rm m})}{1 + i\beta_0 \phi a(\sigma_{\rm m})} \right)^{k_6}.$$

The coupling $a(Q^2)^{k_6}$ here, and the coupling $a(Q^2)^2$ in the contributions of $\delta d(Q^2)_{\rm m_c}$ is evolved by 1-loop running along the contour, while the powers $a(Q^2)^n$ in the contributions of $d(Q^2)_{D=0}$ run by the 5-loop $\overline{\rm MS}$ running.

Method of evaluation of the $D = 0$ contribution

Fixed Order Perturbation Theory using powers (FO): The truncated power expansion $d(\sigma_{\rm m} e^{i\phi})_{D=0,{\rm pt}}$ [cf. Eq. (8)]

$$d(\sigma_{\rm m} e^{i\phi})_{D=0,{\rm pt}} = a(\sigma_{\rm m} e^{i\phi}) + \sum_{n=1}^{N_t - 1} d_n(\kappa) a(\sigma_{\rm m} e^{i\phi})^{n+1}, \qquad (15)$$

which appears in the contour integrals in the sum rules, is written as truncated Taylor expansion in powers of $a(\sigma_{\rm m})$ up to (and including) $a(\sigma_{\rm m})^{N_t}$ where $N_t = 5$.

Fitting

The Borel-Laplace sum rules are applied in practice to the Real parts

$$\mathrm{Re}B_{\mathrm{exp}}(M^2;\sigma_{\mathrm{m}}) = \mathrm{Re}B_{\mathrm{th}}(M^2;\sigma_{\mathrm{m}}), \quad (16)$$

where for the Borel-Laplace scale parameters M^2 we take $M^2 = |M^2|\exp(i\Psi)$, where $0 \le \Psi < \pi/2$. Specifically, we take $0.9\ \mathrm{GeV}^2 \le |M^2| \le 1.5\ \mathrm{GeV}^2$, and $\Psi = 0, \pi/6, \pi/4$.
We minimised the difference between the two quantities (16) by minimising, with respect to parameters ($a \equiv \alpha_s/\pi$, $\langle O_4 \rangle$, ..., $\langle O_{D_{\max}} \rangle$), the sum of squares

$$\chi^2 = \sum_{\alpha=1}^{n} \left(\frac{\mathrm{Re}B_{\mathrm{th}}(M_\alpha^2;\sigma_{\mathrm{m}}) - \mathrm{Re}B_{\mathrm{exp}}(M_\alpha^2;\sigma_{\mathrm{m}})}{\delta_B(M_\alpha^2)} \right)^2, \quad (17)$$

where M_α^2 is a set of $n = 9$ points along the chosen rays with $\Psi = 0, \pi/6, \pi/4$ and $0.9\ \mathrm{GeV}^2 \le |M|^2 \le 1.5\ \mathrm{GeV}^2$. Further, $\delta_B(M_\alpha^2)$ are the experimental standard deviations of $\mathrm{Re}B_{\mathrm{exp}}(M_\alpha^2;\sigma_{\mathrm{m}})$. We usually get very small $\chi^2 \lesssim 10^{-5}$.

Fitting

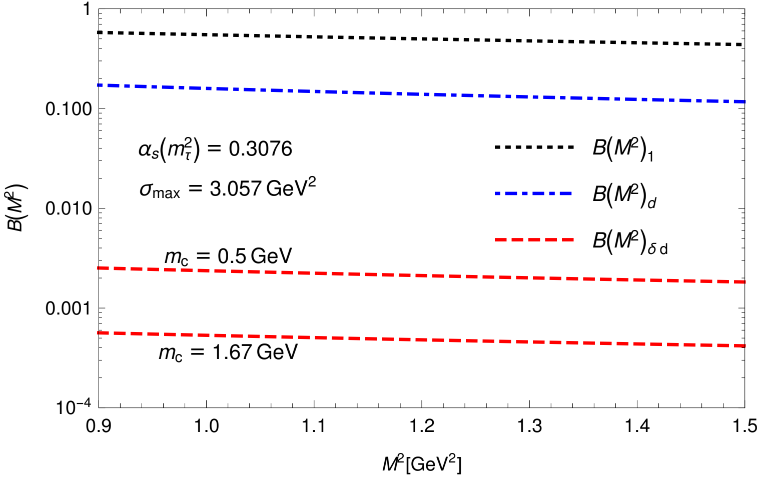

Figure: $D = 0$ part of the Borel-Laplace $B(M^2)$ for real M^2: the contribution of unity of the Adler function $(B(M^2)_1)$; of the massless truncated perturbation series $(B(M^2)_d)$; and of $\delta d(Q^2)_{m_c}$ $((B(M^2)_{\delta d})$ for $m_c = 1.67$ GeV and 0.5 GeV.

Fitting

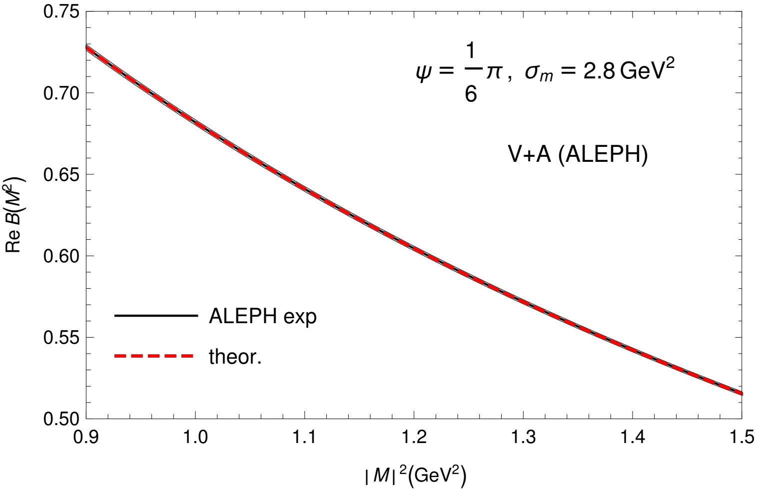

Figure: The values of $\mathrm{Re}B(M^2;\sigma_\mathrm{m})$ along the ray $M^2 = |M^2|\exp(i\Psi)$ with $\Psi = \pi/6$. The narrow grey band are the experimental predictions. The red dashed line is the result of the FO global fit with truncation index $N_t = 5$.

Results of fitting and Conclusions

The extracted values for α_s, for (V+A) and (V+ee)-channel fits, are

$$
\begin{aligned}
\alpha_s(m_\tau^2)^{(\mathrm{V+A})} &= 0.3155 \pm 0.0018(\mathrm{exp})^{+0.0079}_{-0.0018}(\kappa) \\
&\mp 0.0007(d_4) \pm 0.0021(O_{14}) \pm 0.0036(N_{\mathrm{bin}}) \\
&= 0.3155^{+0.0091}_{-0.0049} \\
\alpha_s(m_\tau^2)^{(\mathrm{V+ee})} &= 0.3076 \pm 0.0049(\mathrm{exp})^{+0.0052}_{-0.0013}(\kappa) \\
&\mp 0.0007(d_4) \pm 0.0008(O_{16}) \\
&= 0.3076^{+0.0072}_{-0.0050}. \qquad (18)
\end{aligned}
$$

Experimental uncertainties were obtained by the method of Boito, Maltman, Peris et al. (2011). Renormalisation scale parameter κ varies as $\kappa = 1^{+1}_{-0.5}$.

We see that the theoretical uncertainties due to the truncation and/or renormalisation scale variation (κ) are dominant in the ALEPH (V+A)-channel; while in the combined (V+ee)-channel, the experimental (exp) and the theoretical (κ) uncertainties are competing.

Results of fitting and Conclusions

At the scale M_Z^2 this gives

$$\begin{aligned} \alpha_s(M_Z^2)^{(\mathrm{V+A})} &= 0.1181^{+0.0012}_{-0.0006} \\ \alpha_s(M_Z^2)^{(\mathrm{V+ee})} &= 0.1171^{+0.0009}_{-0.0006}. \end{aligned} \tag{19}$$

If we did not include the m_c-nondecoupling contribution $\delta d(Q^2)_{\mathrm{m_c}}$ in the Adler function, then the results for $\alpha_s(m_\tau^2)$ would increase by about $+0.0010$, and for $\alpha_s(M_Z^2)$ by $+0.0001$.

The above result for (V+ee)-data is very close to a different sum rule analysis of (Boito, Golterman, Maltman, Peris, et al.(2021)), with polynomial weight functions $g(Q^2)$ and a DV-violation model, and using the same set of data (V+ee): $\alpha_s(M_Z^2)^{(\mathrm{V+ee})} = 0.1171 \pm 0.0010$.

Results of fitting and Conclusions

Table: Comparison of values of $\alpha_s(m_\tau^2)$, extracted by various groups applying sum rules and various methods to the ALEPH τ-decay data:

group	sum rule	FO	CI	PV
Baikov et al.,2008	$a^{(2,1)} = r_\tau$	0.322 ± 0.020	0.342 ± 0.011	—
Beneke&Jamin, 2008	$a^{(2,1)} = r_\tau$	$0.320^{+0.012}_{-0.007}$	—	0.316 ± 0.006
Caprini, 2020	$a^{(2,1)} = r_\tau$	—	—	0.314 ± 0.006
Davier et al., 2013	$a^{(i,j)}$	0.324	0.341 ± 0.008	—
Pich&R.Sánchez, 2016	$a^{(i,j)}$	0.320 ± 0.012	0.335 ± 0.013	—
Boito et al., 2014	DV in $a^{(i,j)}$	0.296 ± 0.010	0.310 ± 0.014	—
our work (ren.mod.)*, 2023	BL	0.319 ± 0.012	—	$0.321^{+0.005}_{-0.010}$
present work	BL	$0.316^{+0.009}_{-0.005}$	—	—

Thank you for your attention.

Table: The results for $\alpha_s(m_\tau^2)$ and the condensates of the (V+A)-channel $\langle O_D \rangle_{\rm V+A}$ in units of $10^{-3}\ {\rm GeV}^D$, and ξ^2. Added are also central values obtained when $\delta d_{m_c} = 0$.

	$\alpha_s(m_\tau^2)$	$\langle O_4 \rangle$	$\langle O_6 \rangle$	$\langle O_8 \rangle$	$\langle O_{10} \rangle$	χ^2
δd_{m_c} incl.	$0.3155^{+0.0091}_{-0.0049}$	$-0.8^{+0.9}_{-2.0}$	$+2.6^{+3.3}_{-2.2}$	$-1.2^{+1.8}_{-1.9}$	$+0.5^{+2.3}_{-2.2}$	6.7×10^{-6}
no δd_{m_c}	0.3164	-0.8	$+2.7$	-1.2	$+0.5$	7.5×10^{-6}

Table: The results for $\alpha_s(m_\tau^2)$ and the condensates of the (V+ee)-channel $2\langle O_D \rangle_{\rm V}$ in units of $10^{-3}\ {\rm GeV}^D$, and ξ^2. Added are also central values obtained when $\delta d_{m_c} = 0$.

	$\alpha_s(m_\tau^2)$	$2\ \langle O_4 \rangle$	$2\ \langle O_6 \rangle$	$2\ \langle O_8 \rangle$	$2\ \langle O_{10} \rangle$	$2\ \langle O_{12} \rangle$	$2\ \langle O_{14} \rangle$	χ^2
δd_{m_c} incl.	$0.3076^{+0.0072}_{-0.0050}$	$+2.8^{+1.1}_{-2.1}$	$-24.0^{+4.5}_{-2.8}$	$+23.0^{+2.5}_{-3.6}$	$-23.8^{+3.2}_{-4.2}$	$+18.3^{+4.2}_{-4.1}$	$-8.0^{+10.1}_{-10.0}$	2.6×10^{-11}
no δd_{m_c}	0.3086	-0.8	$+2.7$	-1.2	$+0.5$	7.5×10^{-6}		

Appendix A: Further details of the results

Table: (V+A)-channel: The extracted values of $\alpha_s(m_\tau^2)$, for different values of truncation of the OPE: $D_{\rm max}$ means that the term with maximal dimension $D = D_{\rm max}$ is included in the OPE.

$D_{\rm max} = 6$	$D_{\rm max} = 8$	$D_{\rm max} = 10$	$D_{\rm max} = 12$	$D_{\rm max} = 14$	$D_{\rm max} = 16$
0.3096	0.3131	0.3155	0.3166	0.3176	0.3186

Table: (V+ee)-channel: The extracted values of $\alpha_s(m_\tau^2)$, for different values of truncation of the OPE.

$D_{\rm max} = 8$	$D_{\rm max} = 10$	$D_{\rm max} = 12$	$D_{\rm max} = 14$	$D_{\rm max} = 16$	$D_{\rm max} = 18$
0.3438	0.3226	0.3131	0.3076	0.3068	0.3075

Appendix A: Further details of the results

Table: The values of $\alpha_s(m_\tau^2)$, extracted by various groups applying sum rules and various methods to the ALEPH τ-decay data.

group	sum rule	FO	CI	PV	average
Baikov et al.,2008	$a^{(2,1)} = r_\tau$	0.322 ± 0.020	0.342 ± 0.011	—	0.332 ± 0.016
Beneke&Jamin, 2008	$a^{(2,1)} = r_\tau$	$0.320^{+0.012}_{-0.007}$	—	0.316 ± 0.006	0.318 ± 0.006
Caprini, 2020	$a^{(2,1)} = r_\tau$	—	—	0.314 ± 0.006	0.314 ± 0.006
Davier et al., 2013	$a^{(i,j)}$	0.324	0.341 ± 0.008	—	0.332 ± 0.012
Pich&R.Sánchez, 2016	$a^{(i,j)}$	0.320 ± 0.012	0.335 ± 0.013	—	0.328 ± 0.013
Boito et al., 2014	DV in $a^{(i,j)}$	0.296 ± 0.010	0.310 ± 0.014	—	0.303 ± 0.012
our work (ren.mod.)*, 2023	BL	0.319 ± 0.012 (V+A)		$0.321^{+0.005}_{-0.010}$	$0.320^{+0.005}_{-0.010}$ (V+A)
our work (ren.mod.)*, 2023	BL	$0.312^{+0.008}_{-0.010}$ (V+ee)		$0.312^{+0.006}_{-0.009}$	$0.312^{+0.006}_{-0.009}$ (V+ee)
present work	BL	$0.316^{+0.009}_{-0.005}$ (V+A)			$0.316^{+0.009}_{-0.005}$ (V+A)
present work	BL	$0.308^{+0.007}_{-0.005}$ (V+ee)			$0.308^{+0.007}_{-0.005}$ (V+ee)