# Hadronic Tau Spectral Function Moments in the RF GC scheme

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arXiv:2008.00578 , arXiv:2105.11222

(with **Christoph Regner**)

arXiv:2202.10957, arXiv:2207.01116

(with **Miguel Benitez-Rathgeb, Diogo Boito** and **Matthias Jamin**)

arXiv:2305.10288

(with **Néstor Gracia** and **Vicent Mateu**)

$\int dk\,\Pi$ Doktoratskolleg Particles and Interactions

universität wien

FWF Der Wissenschaftsfonds.

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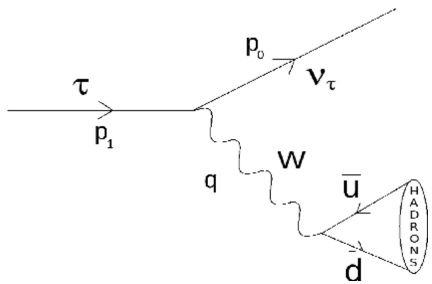
# Outline

- Review: FOPT-CIPT discrepancy problem
  → Why CIPT fails
- Mathematical view: [New!]
  → CIPT is a non-uniform asymptotic expansion
- Review: RF Gluon condensate scheme
  → Reconciliation of FOPT and CIPT
- Impact on determinations of $\alpha_s(m_\tau^2)$
- Outlook

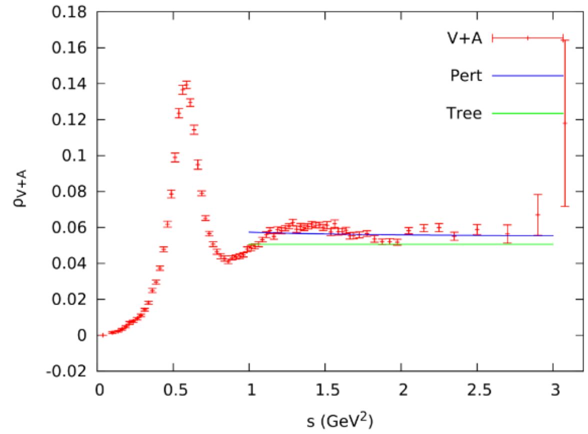
# Hadronic τ Spectral Function Moments

ALEPH: τ hadronic width (HFLAV 2019)

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\,\nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^-\overline{\nu}_e\nu_\tau(\gamma)]} = 3.6355 \pm 0.0081$$



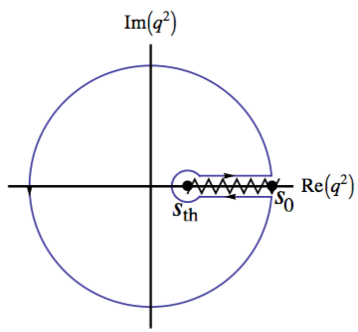

Inclusive hadronic mass spectrum




$$\left(p^\mu p^\nu - g^{\mu\nu}p^2\right)\Pi(p^2) \equiv i\int dx\, e^{ipx}\,\langle\Omega|\,T\{j^\mu_{v/av,jk}(x)\, j^\nu_{v/av,jk}(0)^\dagger\}\Omega\rangle$$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A^\omega_{V/A}(s_0) \equiv \int_{s_{\rm th}}^{s_0}\frac{ds}{s_0}\,\omega(s)\,\mathrm{Im}\,\Pi_{V/A}(s) = \frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\Pi_{V/A}(s)$$



# Hadronic τ Spectral Function Moments

Theory: Operator product expansion

Adler function: $$\frac{1}{4\pi^2}\Big(1+D(s)\Big) \equiv -s\frac{d\Pi(s)}{ds}$$

Braaten, Narison, Pich, Le Diberder, ... 90's

$$A_{W_i}(s_0) = \frac{N_c}{2}|V_{ud}|^2\left[\delta^{\rm tree}_{W_i} + \delta^{(0)}_{W_i}(s_0) + \sum_{d\geq 2}\delta^{(d)}_{W_i}(s_0) + \delta^{\rm DV}_{W_i}(s_0)\right]$$

$$W_i(x) = \sum_{n=0}^{m} a_n\, x^n$$

pQCD — OPE — Duality violation (→ Santi's und Toni's talks)

Shifman, Vainshtein, Sacharow 1978

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1}\left(\frac{\alpha_s(-s)}{\pi}\right)^n$$

← Perturbative

OPE non-pert. corrections

$$\hat{D}^{\rm OPE}(s) = \frac{C(\alpha_s(-s))}{(-s)^2}\langle\alpha_s G^2\rangle + \sum_{p=3}^{\infty}\frac{1}{(-s)^p}\Big[C_0(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_1}\rangle + C_1(\alpha_s(-s))\langle\mathcal{O}_{2p,\gamma_2}\rangle + \ldots\Big]$$

$$\delta^{(0)}_{W_i}(s_0) = \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\,W_i(\tfrac{s}{s_0})\,\hat{D}(s) \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0}\frac{ds}{s}\,W_i\Big(\frac{s}{s_0}\Big)\frac{\Lambda^d_{\rm QCD}}{s^{d/2}}$$

# FOPT-CIPT Discrepancy Problem

$$\hat{D}(s) = \sum_{n=1}^{\infty} c_{n,1} \left(\frac{\alpha_s(-s)}{\pi}\right)^n ,$$
$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(s_0)}{\pi}\right)^n \sum_{k=1}^{n+1} k\, c_{n,k} \ln^{k-1}\left(\frac{-s}{s_0}\right)$$

Change of renormalization scale

$$c_{0,1} = c_{1,1} = 1 , \quad c_{2,1} = 1.640$$
$$c_{3.1} = 6.371$$
$$c_{4,1} = 49.076$$
$$c_{5,1} = 280 \pm 140$$

4-loop: Gorishni etal., Surguladze etal. 1991

5-loop: Baikov etal. 2008

6-loop estimate Beneke, Boito, Jamin; Caprini

Contour-improved perturbation theory (CIPT):

$$\delta_{W_i}^{(0),\mathrm{CIPT}}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} c_{n,1} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \boxed{\left(\frac{\alpha_s(-x s_0)}{\pi}\right)^n}$$

Fixed-order perturbation theory (FOPT):

$$x = \frac{s}{s_0}$$

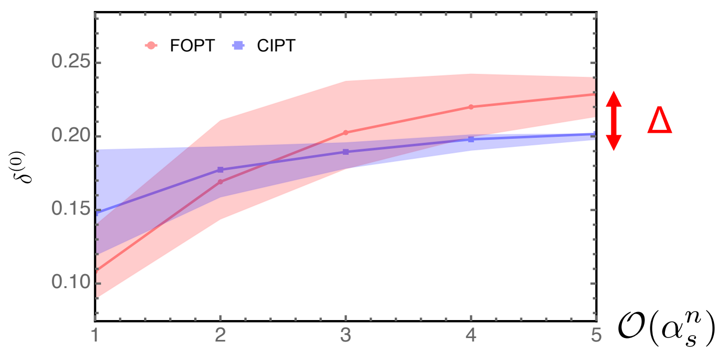
$$\delta_{W_i}^{(0),\mathrm{FOPT}}(s_0) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \boxed{\left(\frac{\alpha_s(s_0)}{\pi}\right)^n} \sum_{k=1}^{n+1} k\, c_{n,k} \oint_{|x|=1} \frac{\mathrm{d}x}{x} W_i(x) \ln^{k-1}(-x)$$

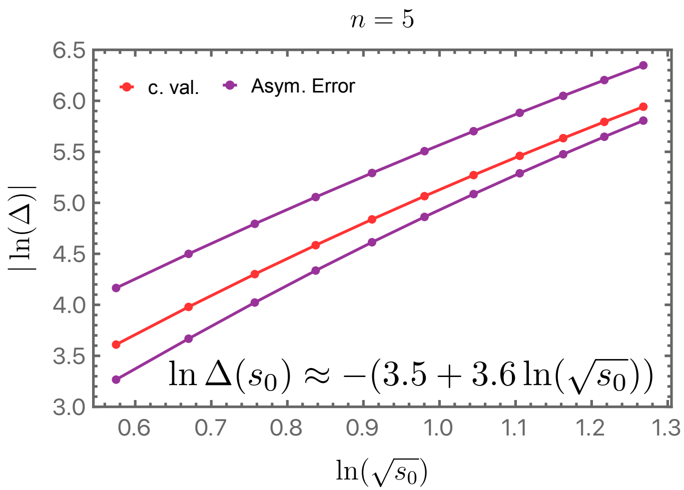
# Moments w/o $\Lambda^4$ Pcs : Total Decay Rate

$$W_\tau(x) = 1 - 2x + 2x^3 - x^4 \qquad \delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i}\oint_{|s|=s_0} \frac{ds}{s}\left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\,\delta_{m2}$$

→ Sensitivity to leading O($\Lambda^4_{\rm QCD}$) gluon condensate is strongly suppressed

Moment's perturbation series:




- Discrepancy consistent with $\Delta \sim (\Lambda/s_0)^{\approx 4}$ <u>which should not exist.</u>
- Either CIPT or FOPT inconsistent with the OPE

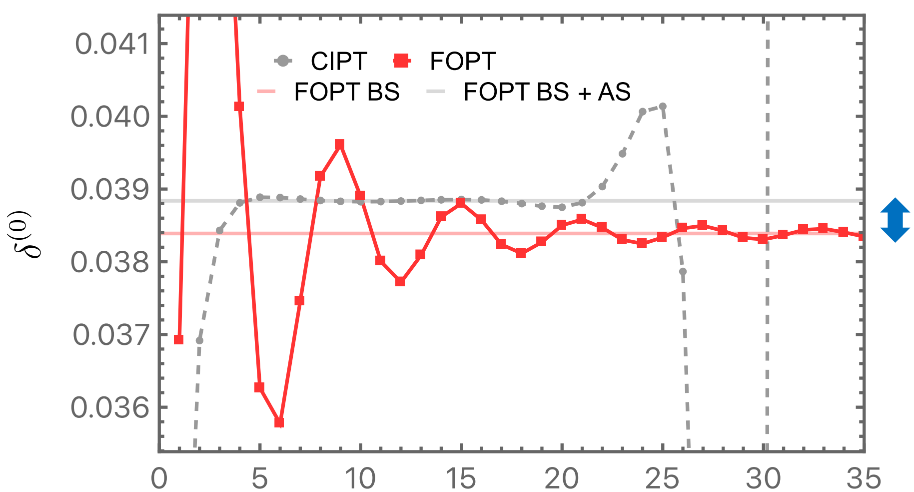
# CIPT is inconsistent

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

$$W(x) = 1$$

$\bar{c}^{(1)}_{4,0} = 0, \; R = 0.8 m_\tau, \; W(x) = 1$




$s_0 = m_\tau^2$

Regner, AHH 2008.00578 2105.11222

Golterman, Maltman, Peris 2305.10386

Pure O($\Lambda^4_{QCD}$) renormalon in Adler function

→ Gluon condensate corrections vanishes

→ Per. series should be convergent

$$\delta^{(d)}_{W_i}(s_0) \sim \frac{1}{2\pi i} \oint_{|s|=s_0} \frac{ds}{s} \left(\frac{s}{s_0}\right)^m \frac{\Lambda^4_{\rm QCD}}{s^2} = \frac{\Lambda^4_{\rm QCD}}{s_0^2}\, \delta_{m2}$$

- CIPT series is **divergent** !
  FOPT series convergent.
  → CIPT not compatible with standard OPE !
- Gluon condensate renormalon is numerically dominant
- Asymptotic separation can be determined analytically: $\Delta_W(s_0) \sim (\mu^2/s_0)^2$
  → CIPT problem only arises in the presence of renormalons in series for Π
- Why is CIPT ”better converging”.

# Mathematical Perspective

Gracia, AHH, Mateu, arXiv:2305.10288

Can we identify the mathematical reason, why CIPT is inconsistent with the OPE?

→ Inconsistency means: CIPT series is divergent for cases where OPE demands convergence

CIPT: $$\sum_{n=1}^{\infty} c_n H_{n,\ell}(a)$$

FOPT: $$\sum_{n=1}^{\infty} d_n a^n$$

$$H_{n,\ell}(a) \equiv \frac{1}{2i\pi} \oint_{|x|=1} \frac{\mathrm{d}x}{x} (-x)^\ell a^n(-x s_0)$$

$$a \equiv \frac{\beta_0 \alpha_(s_0)}{4\pi}$$

Series in non-trivial functions of a

Power series in a

First question: Is CIPT a consistent asymptotic expansion?

→ **Yes**, because the following relation holds:

$$\lim_{a\to 0} \frac{|H_{n+1,\ell}(a)|}{|H_{n,\ell}(a)|} = 0 \quad \text{for all } n$$

A. Erdelyi (1955)

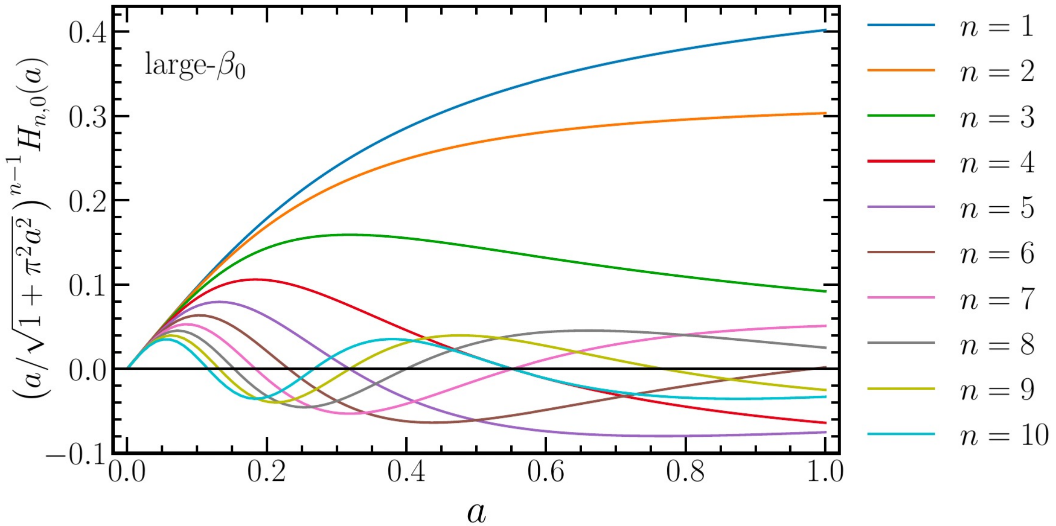
„The { $H_{n,l}$(a) } form an asymptotic sequence“

Ensures that coefficients $c_n$ can be determined unambiguously.

# Mathematical Perspective

But the asymptotic sequence { $H_{n,l}(a)$ } it non-uniform because the $H_{n,l}(a)$ has zeroes for real a

$$\frac{H_{n,0}(a)}{(a/\sqrt{1+a^2\pi^2})^{n-1}}$$




→ Zeros approach zero for large n

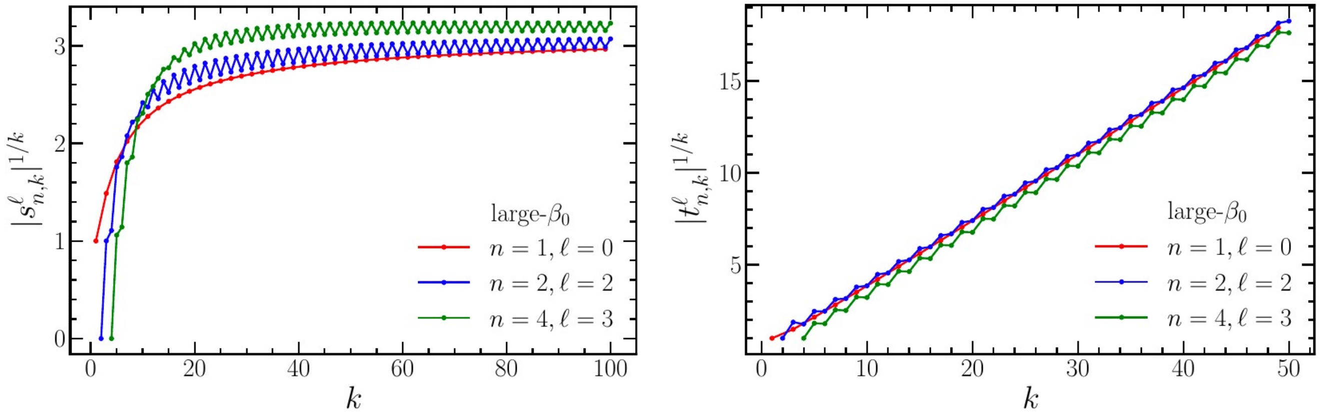
$$H_{n,\ell}(a) = \sum_{k=n}^{\infty} s^{\ell}_{n,k} a^k$$

← Has finite radius of convergence

$$a^n = \sum_{k=n}^{\infty} t^{\ell}_{n,k}\, H_{k,\ell}(a)$$

← Divergent for any a !

# Mathematical Perspective




**Theorem A.6** (Weierstrass' Double Series Theorem). *Consider an infinite set of functions* $f_i(x)$ *that are analytic for* $|x| < r$*, so that the power expansions* $f_i(x) = \sum_{k=0}^{\infty} a_k^{(i)} x^k$ *exist and converge at least for* $|x| < r$ *for all* $i$*. Furthermore, consider a convergent series of these functions* $F(x) = \sum_{i=0}^{\infty} f_i(x)$ *that is uniformly convergent for* $|x| \leq \rho$ *for every* $\rho < r$*, so that the series converges in particular everywhere within the interval* $|x| < r$ *and defines the function* $F(x)$ *there. Then, the infinite sums* $A_k = \sum_{i=0}^{\infty} a_k^{(i)}$ *are convergent and the infinite sum* $\sum_{k=0}^{\infty} A_k x^k$ *converges to* $F(x)$ *for* $|x| < r$*, so that* $F(x) = \sum_{i=0}^{\infty}(\sum_{k=0}^{\infty} a_k^{(i)} x^k) = \sum_{k=0}^{\infty}(\sum_{i=0}^{\infty} a_k^{(i)}) x^k$ *and is analytic for* $|x| < r$*.*

⇒ Any convergent CIPT series will be convergent in FOPT and sums to the same value

⇒ Any convergent FOPT series will be in general divergent in CIPT !

# Mathematical Perspective

## Models for expansion function

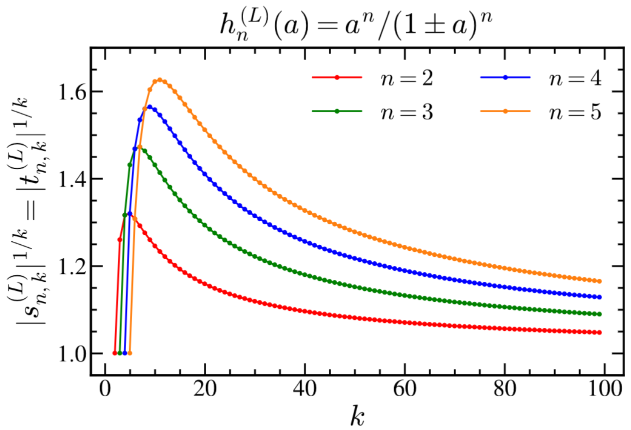
Change of renormalization scale:

zeros at a=-1/L

$$h_n^{(L)}(a) = [a(\mu^2)]^n = \frac{a^n}{(1+aL)^n} \qquad L = \log(\mu^2/s_0)$$

Expansion of $a^n$ in terms of $h_n^{(L)}(a)$ has same convergence radius as expansion of $h_n^{(L)}(a)$ in powers of n.

Change of renormalization scale does not change the radius of convergence.



# Mathematical Perspective

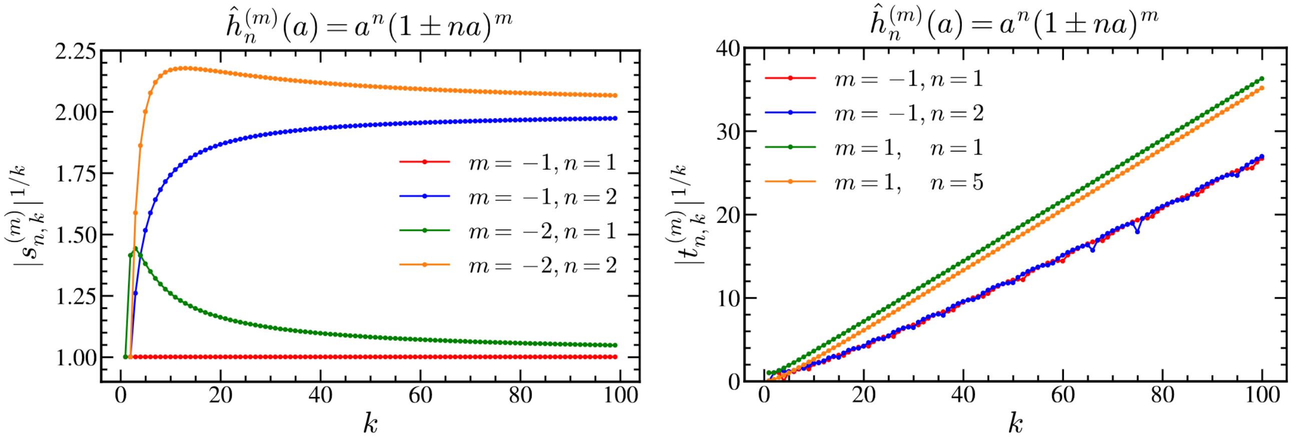
## Models for expansion function

Model with decreasing zeros/singuarity points:

m=-1: singularities at a=1/ξn

$$\hat{h}_n^{(m)}(a) = a^n(1-\xi n a)^m$$

m=1: zeros at a=1/ξn




Expansion of $a^n$ in terms of $h_n^{(m)}(a)$ is divergent for any a !

The zeros are one reason why CIPT has good apparent convergence at low orders, but they are also the reason why CIPT has the bad divergence property.

The fact that the Adler function contains inifinitely many IR renormalons ensures that the inconsistency of CIPT is mathematically uncurable.

**But:** For tau only the GC renormalon is relevant.

# RF (Renormalon-free) GC Scheme

Aim:
- Method to reconcile (to excellent approximation) CIPT with the OPE
- Take advantage of the faster CIPT convergence property.

Idea: “short-distance“ scheme for the gluon condensate

Benitez-Rathgeb, Boito, Jamin, AHH
2207.01116, 2202.10957

$$\langle \bar{G}^2\rangle^{\overline{\rm MS},(n)}$$

Original $\overline{\rm MS}$ GC contains
pure O($\Lambda^4_{QCD}$) renormalon (scale invariant)

$$\langle \bar{G}^2\rangle^{\overline{\rm MS},(n)} \equiv \underbrace{\langle G^2\rangle(R^2)}_{\text{renormalon-free, R-dependent}} - R^4 \overbrace{\sum_{\ell=1}^{n} \underbrace{N_g}_{\text{renormalon norm (approximately known)}} r_\ell^{(4,0)}\, \bar{a}^\ell(R^2)}\,,$$

Expand perturbatively with Adler function

IR factorization scale R

$$r_\ell^{(4,0)} = \left(\frac{1}{2}\right)^{\ell+4\hat{b}_1} \frac{\Gamma(\ell+4\hat{b}_1)}{\Gamma(1+4\hat{b}_1)} \qquad \bar{a}(R^2) = \frac{\beta_0\,\bar{\alpha}_s(R^2)}{4\pi}$$

→ Alternative GC scheme: subtraction series from gradient flow

Beneke, Takaura 2309.10853

# Renormalon-Free GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH 2207.01116, 2202.10957

$$\langle G^2\rangle(R^2) - \langle G^2\rangle(R'^2)$$

Renormalon-free (convergent series)

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\langle G^2\rangle(R^2) = \frac{N_g}{2^{4\hat{b}_1}}\frac{R^4\,\bar{a}(R^2)}{1-2\hat{b}_1\bar{a}(R^2)}$$

(R-evolution equation)

Convergent series!

We can define an R-independent „short-distance“ GC:

treated like a tree-level term (Do not expand !)

$$\langle G^2\rangle(R^2) \equiv \langle G^2\rangle^{\mathrm{RF}} + N_g\,\bar{c}_0(R^2)\,.$$

→ R-invariance of scheme at infinite truncation order

$$\bar{c}_0(R^2) \equiv R^4\ \mathrm{PV}\int_0^\infty \frac{\mathrm{d}u\; e^{-\frac{u}{\bar{a}_R}}}{(2-u)^{1+4\hat{b}_1}} = -\frac{R^4\,e^{-\frac{2}{\bar{a}(R^2)}}}{(\bar{a}(R^2))^{4\hat{b}_1}}\,\mathrm{Re}\left[e^{4\pi\hat{b}_1 i}\,\Gamma\Big(-4\hat{b}_1,-\frac{2}{\bar{a}(R^2)}\Big)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln R^2}\langle G^2\rangle^{\mathrm{RF}} = 0$$

Scale-invariant “short-distance“ scheme for the gluon condensate

→ Borel sum of original series unchanged ($N_g$-independence) ! („minimal scheme“)

# CIPT and FOPT: RF GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH 2207.01116, 2202.10957

Single renormalon model:

Pure O($\Lambda^4_{QCD}$) renormalon in Adler function

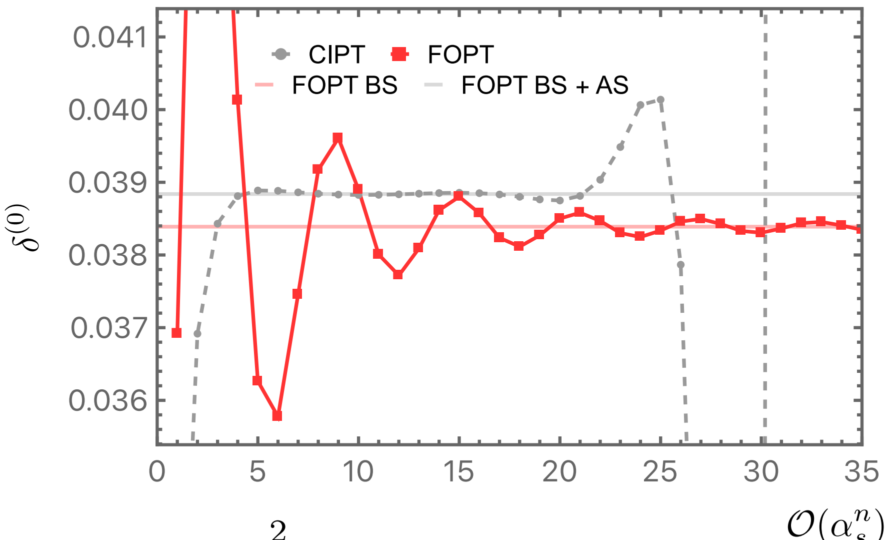
$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

$$W(x) = 1 \qquad N_g = \frac{3}{2\pi^2}$$

→ Gluon condensate corrections vanishes !

→ Nevertheless dramatic impact of changing to the RF GC scheme

$\bar{c}^{(1)}_{4,0} = 0, \; R = 0.8 m_\tau, \; W(x) = 1$




$s_0 = m_\tau^2$

- FOPT same as in the original GC scheme
- CIPT$^{\text{RS-GC}}$ series is convergent
- CIPT$^{\text{RS-GC}}$ consistent with FOPT !
- CIPT$^{\text{RS-GC}}$ compatible with standard OPE !
- CIPT$^{\text{RS-GC}}$ Borel sum = FOPT Borel sum
- CIPT$^{\text{RS-GC}}$ still converges much faster than FOPT (oscillating behavior absent)
  → CIPT still interesting to consider

# CIPT and FOPT: RF GC Scheme

Benitez-Rathgeb, Boito, Jamin, AHH 2207.01116, 2202.10957

Single renormalon model:

$$B(u) \sim \frac{1}{(2-u)^{1+4\hat{b}_1}} + \ldots \quad \Longleftrightarrow \quad \frac{\langle \bar{G}^2 \rangle}{s^2}$$

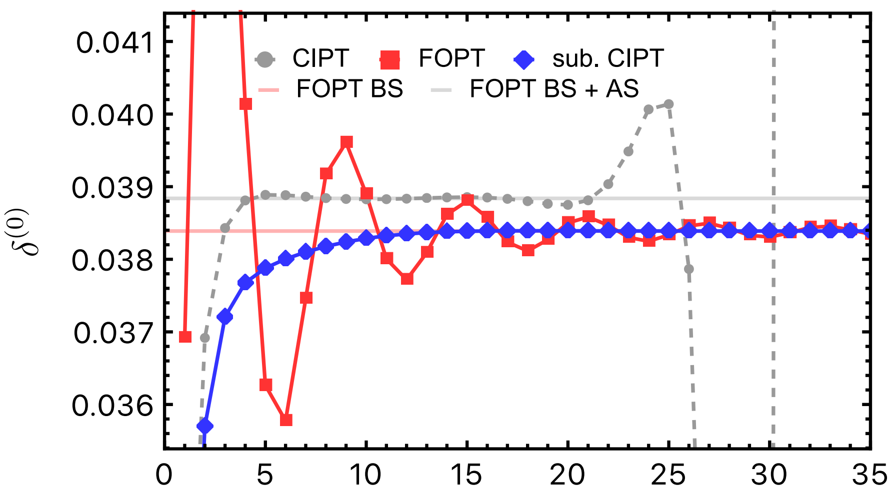
$$W(x) = 1 \qquad N_g = \frac{3}{2\pi^2}$$

Pure O($\Lambda^4_{QCD}$) renormalon in Adler function

→ Gluon condensate corrections vanishes !

→ Nevertheless dramatic impact of changing to the RF GC scheme

$\bar{c}^{(1)}_{4,0} = 0, \; R = 0.8 m_\tau, \; W(x) = 1$

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# CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, $O(\Lambda^4_{QCD}, \Lambda^6_{QCD})$ + UV renormalons in Adler function

Beneke, Jamin 2008

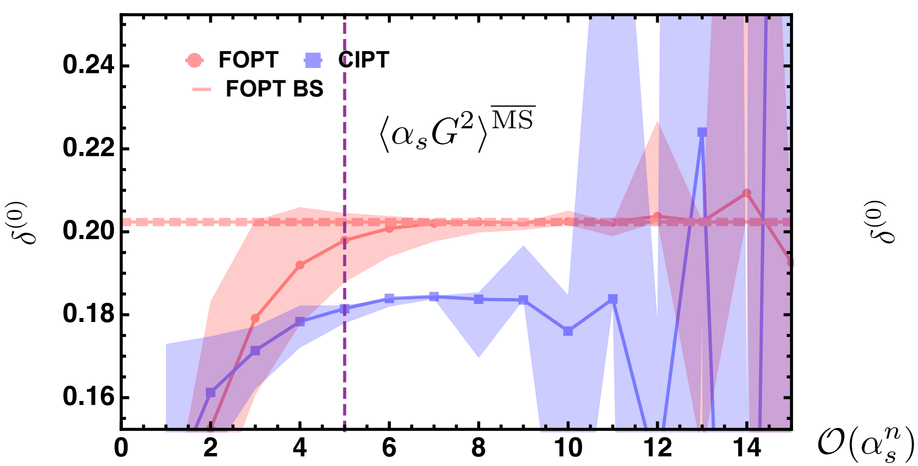
$$W(x) = 1 - 2x + 2x^3 - x^4$$

→ GC suppressed

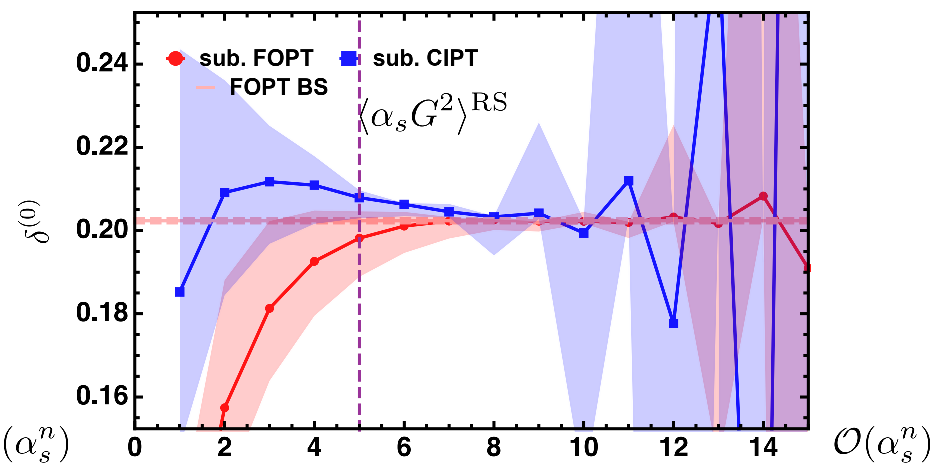
$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

2202.10957

$\bar{c}^{(1)}_{4,0} = -22/81,\ W(x) = (1-x)^3(1+x)$




$\bar{c}^{(1)}_{4,0} = -22/81,\ R = 0.8m_\tau,\ W(x) = (1-x)^3(1+x)$




- Discrepancy between CIPT and FOPT removed
- CIPT becomes consistent with FOPT (which is only slightly modified)
- Higher precision for $\alpha_s$ determinations from hadronic tau decays

# CIPT and FOPT: RF GC Scheme

Realistic Multi renormalon model: GC, $O(\Lambda^4_{QCD}, \Lambda^6_{QCD})$ + UV renormalons in Adler function

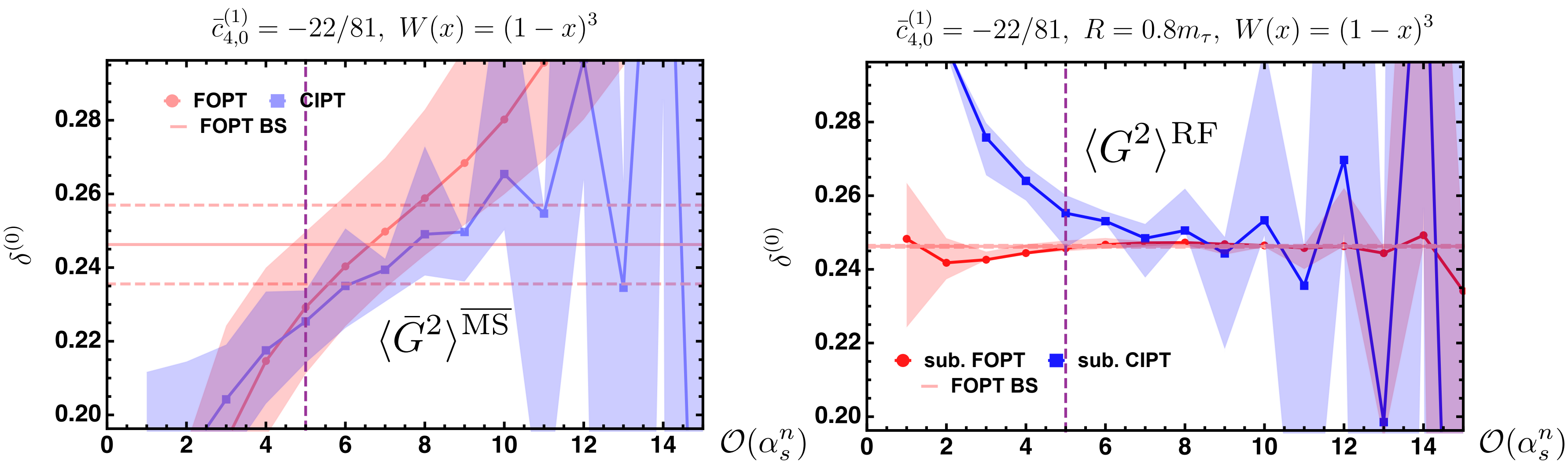
Beneke, Jamin 2008

$$W(x) = (1-x)^3$$

→ GC enhanced

$$s_0 = m_\tau^2\,, \quad \frac{1}{2} \le \xi \le 2\,, \quad N_g = 0.64$$

2202.10957




- FOPT and CIPT expansions both get improved substantially
- Spectral function moments with high sensitivity to the GC can now be used for high-precision determinations of the strong coupling and the GC

# GC Renormalon Normalization

Benitez-Rathgeb, Boito, Jamin, AHH: 2207.01116

GC Norm in the Adler function's Borel function:

$$B[\hat{D}(s)]_{\rm GC}(u) = \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} \qquad \bar{a}(\mu^2) \equiv \frac{\beta_0\,\bar{\alpha}_s(\mu^2)}{4\pi} \qquad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

## Multi-renormalon model approach

Beneke, Jamin 2008

$$B[\hat{D}(s)]_{\rm mr}(u) = b^{(0)} + b^{(1)}u + \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} + \frac{N_6}{(3-u)^{1+6\hat{b}_1}} + \frac{N_{-2}}{(1+u)^{2-2\hat{b}_1}}$$

$$c_{0,1} = c_{1,1} = 1\,, \quad c_{2,1} = 1.640$$
$$c_{3.1} = 6.371$$
$$c_{4,1} = 49.076$$
$$c_{5,1} = 280 \pm 140$$

→ $N_g = 0.64 \pm 0.27$

## Conformal mapping approach

Lee 2012

$$\tilde{B}(u) \equiv \frac{3\,(2-u)^{1+4\hat{b}_1}}{2\pi^2}\,B[\hat{D}(s)](u)$$

GC renormalon-free

$$w(u,p) = \frac{\sqrt{1+u}-\sqrt{1-\frac{u}{p}}}{\sqrt{1+u}+\sqrt{1-\frac{u}{p}}}$$

u=2 closest to the origin in the w plane $\quad N_g = \tilde{B}(w(2,p))$

Use $c_{1,1}$ to $c_{5,1}$ and $w$-expansion → $N_g = 0.71 \pm 0.26$

# GC Renormalon Normalization

GC Norm in the Adler function‘s Borel function:

$$B[\hat{D}(s)]_{\rm GC}(u) = \frac{2\pi^2}{3}\,\frac{N_g\left[1-\frac{22}{81}\bar{a}(-s)\right]}{(2-u)^{1+4\hat{b}_1}} \qquad \bar{a}(\mu^2) \equiv \frac{\beta_0\,\bar{\alpha}_s(\mu^2)}{4\pi} \qquad \hat{b}_1 \equiv \frac{\beta_1}{2\beta_0^2}$$

Optimal subtraction approach

Benitez-Rathgeb, Boito, Jamin, AHH (new)

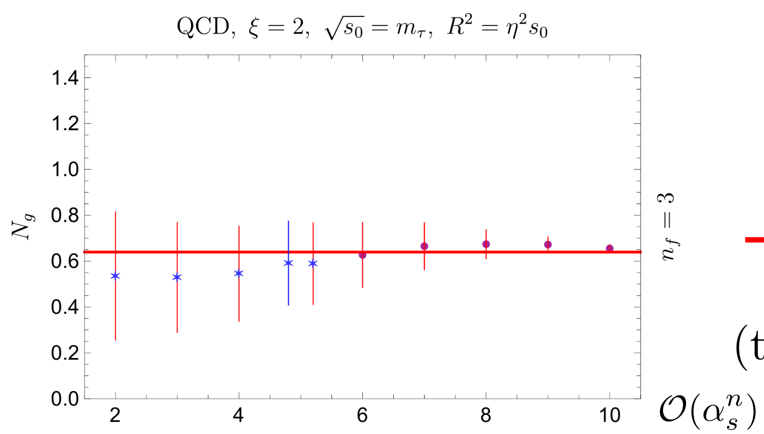
Use quantiative measure for improvements for GC suppresed and GC enhanced moments in the RF GC scheme

$$\chi^2_m(N_g) = \chi^2_{m,\rm GCS}(N_g) + \chi^2_{m,\rm GCE}(N_g)$$

Good convergence of 5 GC enhanced moments

Small discrepancy for 5 GC supressed moments

Test: Can precisely determine $N_g$ for the Beneke-Jamin model




$$\longrightarrow \quad N_g = 0.57 \pm 0.23$$

(take $\mathcal{O}(\alpha_s^4)$ result)

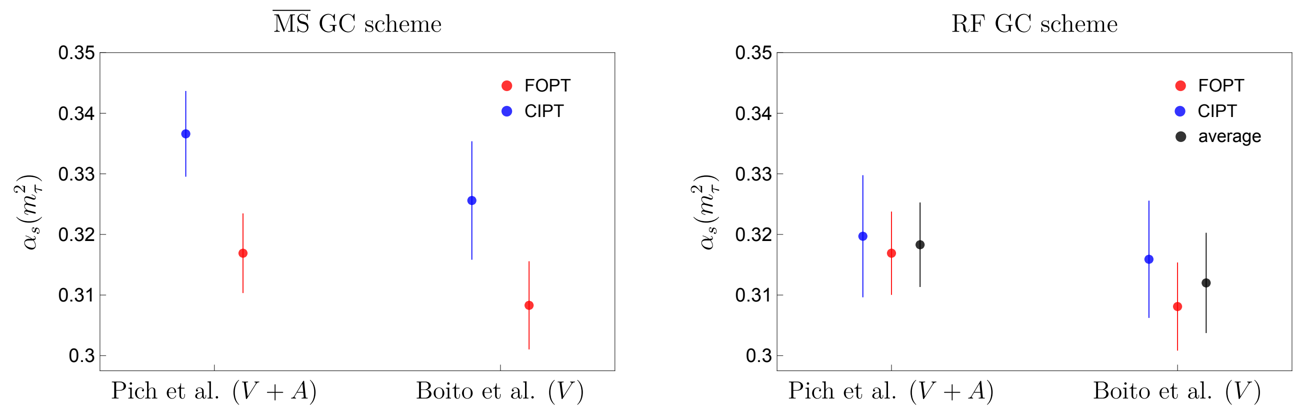
# Impact on Strong Coupling Determinations

We repeat (in detail!) two state-of-the-art determination methods in the RF GC scheme:

Truncated OPE approach: Pich, Rodriguez-Sanchez 2016

Duality Violation model approach: Boito, Golterman, Maltman, Peris, Rodriguesz, Scharf 2021

→ Include uncertainties: $N_g = 0.57 \pm 0.23 \qquad 0.7 m_\tau \leq R \leq m_\tau$




Benitez-Rathgeb, Boito, Jamin, AHH: 2207.01116

- FOPT-CIPT for GC suppressed moments remedied
- Taking average of FOPT and CIPT$^{\text{RF-GC}}$ results now meaningful
- Uncertainties due GC renormalon norm $N_g$ and R variations quite small !
- We may take advantage of CIPT$_{\text{RF scheme}}$ convergence properties

# Summary and Outlook

- FOPT-CIPT discrepancy resolved and understood:
  → CIPT inconsistent with standard OPE (canonical renormalon formalism fails)

- CIPT can be (practically) reconciled with the OPE using a renormalon-free GC scheme

- RF-GC scheme (practically) remedies CIPT inconsistency

- New updated $\alpha_S$ full fledged updated analyses may be carried out by groups experienced on tau spectral functions using FIPT and CIPT in the RF-GC scheme.

- Future improvements:
  → 6-loop corrections for Π  → improved $N_g$ GC renormalon norm
  → OPE corrections
  → Duality violation effects
  → improved data

Upcoming work: → 3-loop corrections to dimension-4 OPE Wilson coefficients (GC renormalon norm update)
→ Gluon condensate determination