Violations of quark-hadron duality in low-energy determinations of α_s

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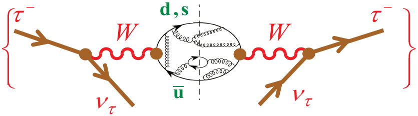
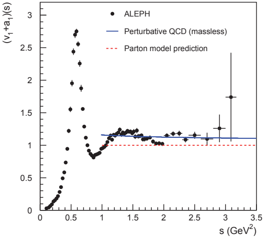
τ Hadronic Width: R_τ

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}$$



Davier et al, 1312.1501



$$\Pi^{(J)}(s) \;\equiv\; |V_{ud}|^2 \left(\Pi^{(J)}_{ud,V}(s) + \Pi^{(J)}_{ud,A}(s)\right) \;+\; |V_{us}|^2\; \Pi^{(J)}_{us,V+A}(s)$$

$$v_1 \;=\; 2\pi\;\text{Im}\Pi^{(1)}_{ud,V}(s) \quad , \quad a_1 \;=\; 2\pi\;\text{Im}\Pi^{(1)}_{ud,A}(s)$$

Braaten-Narison-Pich, 1992

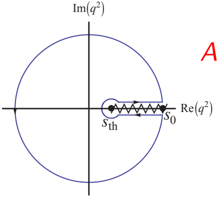
$$R_\tau \;=\; \frac{\Gamma[\tau^-\to\nu_\tau\text{hadrons}]}{\Gamma[\tau^-\to\nu_\tau e^-\overline{\nu}_e]} \;=\; R_{\tau,V}+R_{\tau,A}+R_{\tau,S}$$

$$=\; 12\pi\int_0^{m_\tau^2}\frac{ds}{m_\tau^2}\left(1-\frac{s}{m_\tau^2}\right)^2\left[\left(1+2\frac{s}{m_\tau^2}\right)\text{Im}\,\Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2}\,\text{Im}\,\Pi^{(0)}(s)\right]$$

Theoretical Framework

$$i \int d^4x\, e^{iqx}\, \langle 0|\, T\left[\mathcal{J}_{ij}^{\mu}(x)\, \mathcal{J}_{ij}^{\nu\,\dagger}(0)\right]\, |0\rangle \;=\; (-g^{\mu\nu} q^2 + q^\mu q^\nu)\, \Pi^{(1)}_{ij,\mathcal{J}}(q^2) + q^\mu q^\nu\; \Pi^{(0)}_{ij,\mathcal{J}}(q^2)$$

$$s\, \Pi^{(0)}_{ud,\mathcal{J}}(s) = \mathcal{O}(m_{u,d}) \approx 0 \qquad \Rightarrow \qquad \Pi_{\mathcal{J}}(s) \equiv \Pi^{(0+1)}_{ud,\mathcal{J}}(s)$$



$$A^{\omega}_{\mathcal{J}}(s_0) \;\equiv\; \int_{s_{th}}^{s_0} \frac{ds}{s_0}\, \omega(s)\; \mathrm{Im}\, \Pi_{\mathcal{J}}(s) \;=\; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s)\, \Pi_{\mathcal{J}}(s)$$

$$\Pi_{\mathcal{J}}(s) \;\approx\; \Pi^{\rm OPE}_{\mathcal{J}}(s) \;=\; \sum_{D=2n} \frac{\mathcal{O}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

Theoretical Framework

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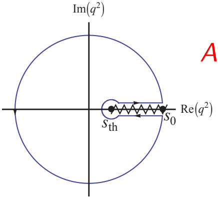

$$A^{\omega}_{\mathcal{J}}(s_0)\;\equiv\;\int_{s_{th}}^{s_0}\frac{ds}{s_0}\,\omega(s)\;\mathrm{Im}\,\Pi_{\mathcal{J}}(s)\;=\;\frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\Pi_{\mathcal{J}}(s)$$

$$\Pi_{\mathcal{J}}(s)\;\approx\;\Pi^{\rm OPE}_{\mathcal{J}}(s)\;=\;\sum_{D=2n}\frac{\mathcal{O}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

$$\rho_{\mathcal{J}}(s)\equiv\frac{1}{\pi}\,\mathrm{Im}\,\Pi_{\mathcal{J}}(s)\qquad\qquad,\qquad\qquad \Delta\rho^{\rm DV}_{\mathcal{J}}(s)\equiv\rho_{\mathcal{J}}(s)-\rho^{\rm OPE}_{\mathcal{J}}(s)$$

$$\longrightarrow\quad A^{\omega}_{\mathcal{J}}(s_0)\;\equiv\;\underbrace{\pi\int_{s_{th}}^{s_0}\frac{ds}{s_0}\,\omega(s)\,\rho_{\mathcal{J}}(s)}_{\text{Experiment}}\;=\;\underbrace{\frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\Pi^{\rm OPE}_{\mathcal{J}}(s)}_{\text{OPE}}+\underbrace{\Delta A^{\omega}_{\mathcal{J}}(s_0)}_{\text{DV}}$$

$$\Delta A^{\omega}_{\mathcal{J}}(s_0)\;=\;\frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\left\{\Pi_{\mathcal{J}}(s)-\Pi^{\rm OPE}_{\mathcal{J}}(s)\right\}\;=\;-\pi\int_{s_0}^{\infty}\frac{ds}{s_0}\,\omega(s)\,\Delta\rho^{\rm DV}_{\mathcal{J}}(s)$$

- **Standard Strategy:** Minimize $\Delta A^{\omega}_{\mathcal{J}}(s_0)$

Take s_0 large enough and/or pinched weights, $\omega(s_0) = 0$,
so that the tiny correction $\Delta A^{\omega}_{\mathcal{J}}(s_0)$ can be neglected

- **DV Approach:** Maximize and "measure" $\Delta A^{\omega}_{\mathcal{J}}(s_0)$ Boito et al

Non-protected weights. Usual default: $\omega(s) = 1$

Modelling of $\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s)$ needed ➔ Additional fit parameters

QCD precision can never be better than the reached control on $\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s)$
(but high accuracy on α_s claimed)

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LO OPE approximation: $\mathcal{P}_{D,\mathcal{J}} = 0$

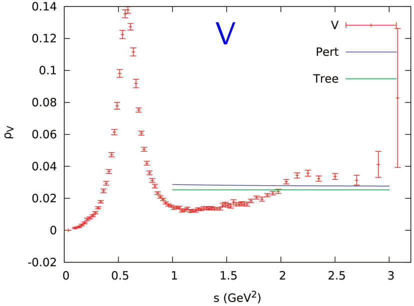
$$\Pi^{\mathrm{OPE}}_{\mathcal{J}}(s)\Big|^{\mathrm{NLO}}_{D>0} = \sum_{D>0} \frac{\mathcal{O}_{D,\mathcal{J}}(\mu) + \mathcal{P}_{D,\mathcal{J}}\;\log\left(-s/\mu^2\right)}{(-s)^{D/2}}$$

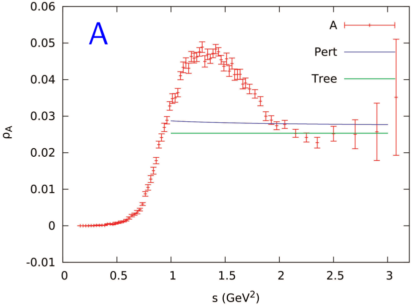
$$\omega_n(s) = \left(\frac{s}{s_0}\right)^n \quad ➔ \quad \Delta A^{(n)}_{\mathcal{J}}(s_0)\Big|^{\mathrm{OPE}}_{D>0} = -\pi \sum_{p=2} \frac{d^{(n)}_{p,\mathcal{J}}}{(-s_0)^p} \quad , \quad d^{(n)}_{p,\mathcal{J}} = \begin{cases} \mathcal{O}_{2p,\mathcal{J}}(s_0) & \text{if } p = n+1 \\ \dfrac{\mathcal{P}_{2p,\mathcal{J}}}{n-p+1} & \text{if } p \neq n+1 \end{cases}$$

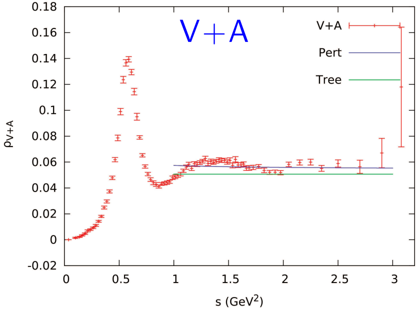
$\mathcal{P}_{D,\mathcal{J}} = 0$ justified if OPE is well behaved. Meaningless otherwise

ALEPH Spectral Functions

Davier et al. 2014







— $\alpha_s(m_\tau^2) = 0.329$

— Parton Model

➡ $\alpha_s(m_\tau^2) = 0.332 \pm 0.012$

(Standard approach)

Detailed analysis of ALEPH data

Standard Approach

Rodríguez-Sánchez, Pich, arXiv:1605.06830

Method (V + A)	$\alpha_s(m_\tau^2)$		
	CIPT	FOPT	Average
ALEPH moments [1]	$0.339\,^{+0.019}_{-0.017}$	$0.319\,^{+0.017}_{-0.015}$	$0.329\,^{+0.020}_{-0.018}$
Mod. ALEPH moments [2]	$0.338\,^{+0.014}_{-0.012}$	$0.319\,^{+0.013}_{-0.010}$	$0.329\,^{+0.016}_{-0.014}$
$A^{(2,m)}$ moments [3]	$0.336\,^{+0.018}_{-0.016}$	$0.317\,^{+0.015}_{-0.013}$	$0.326\,^{+0.018}_{-0.016}$
s_0 dependence [4]	0.335 ± 0.014	0.323 ± 0.012	0.329 ± 0.013
Borel transform [5]	$0.328\,^{+0.014}_{-0.013}$	$0.318\,^{+0.015}_{-0.012}$	$0.323\,^{+0.015}_{-0.013}$
Combined value	0.335 ± 0.013	0.320 ± 0.012	0.328 ± 0.013

→ $$\boxed{\alpha_s(M_Z^2) = 0.1197 \pm 0.0015}$$

1) $\omega_{kl}(x) = (1+2x)\,(1-x)^{2+k}x^l$ $\quad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$ $\quad x \equiv s/s_0$

2) $\tilde{\omega}_{kl}(x) = (1-x)^{2+k}x^l$ $\quad (k,l) = (0,0),\ (1,0),\ (1,1),\ (1,2),\ (1,3)$

3) $\omega^{(2,m)}(x) = (1-x)^2 \sum_{k=0}^{m}(k+1)\,x^k = 1-(m+2)\,x^{m+1}+(m+1)x^{m+2}$, $\quad 1 \le m \le 5$

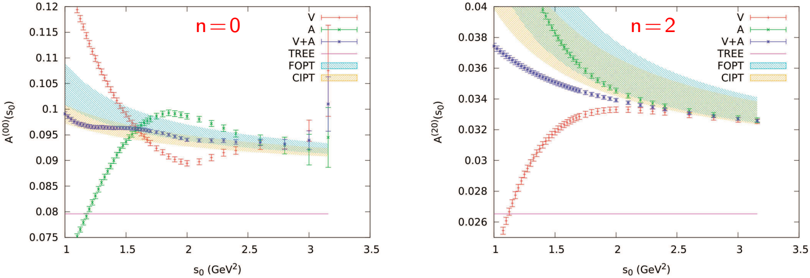
4) $\omega^{(2,m)}(x)$ $\quad 0 \le m \le 2$, 1 single moment in each fit

5) $\omega_a^{(1,m)}(x) = (1-x^{m+1})\,\mathrm{e}^{-ax}$ $\quad 0 \le m \le 6$

Experiment vs. (pinched) Perturbation Theory (only)

Rodríguez-Sánchez, A.P.

$\omega^{(n,0)}(s = s_0 x) \; = \; (1 - x)^n \; \longrightarrow \; \mathcal{O}_{D \leq 2\,(n+1)}$, $\alpha_s(m_\tau^2) \; = \; 0.329\,^{+\,0.020}_{-\,0.018}$



No OPE corrections at LO

Not protected against DV

Beautiful DV in V & A
(below 2.4 GeV^2)

DV largely cancels in $V{+}A$

Pinched weight (protected against DV)

No obvious signal of DV seen

Clear $D = 6$ OPE correction with opposite signs in V & A which cancels to a large extent in $V{+}A$

Non-Perturbative Contributions Neglected

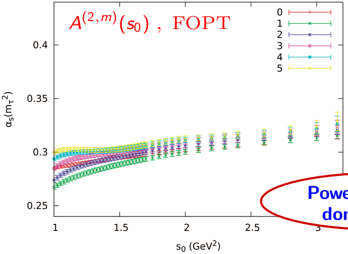
Rodríguez-Sánchez, A.P.

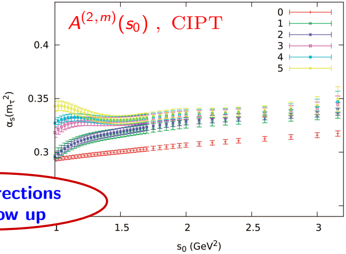
$$\omega^{(1,m)}(x) \;=\; 1-x^{m+1} \;\longrightarrow\; \mathcal{O}_{2m+4}$$

$$\omega^{(2,m)}(x) \;=\; (1-x)^2 \sum_{k=0}^{m} (k+1)\, x^k \;=\; 1-(m+2)\, x^{m+1} + (m+1)\, x^{m+2} \;\longrightarrow\; \mathcal{O}_{2m+4,2m+6}$$

Moment	$\alpha_s(m_\tau^2)$		Moment	$\alpha_s(m_\tau^2)$	
(n,m)	FOPT	CIPT	(n,m)	FOPT	CIPT
(1,0)	$0.315\,^{+0.012}_{-0.007}$	$0.327\,^{+0.012}_{-0.009}$	(2,0)	$0.311\,^{+0.015}_{-0.011}$	$0.314\,^{+0.013}_{-0.009}$
(1,1)	$0.319\,^{+0.010}_{-0.006}$	$0.340\,^{+0.011}_{-0.009}$	(2,1)	$0.311\,^{+0.011}_{-0.006}$	$0.333\,^{+0.009}_{-0.007}$
(1,2)	$0.322\,^{+0.010}_{-0.008}$	$0.343\,^{+0.012}_{-0.010}$	(2,2)	$0.316\,^{+0.010}_{-0.005}$	$0.336\,^{+0.011}_{-0.009}$
(1,3)	$0.324\,^{+0.011}_{-0.010}$	$0.345\,^{+0.013}_{-0.011}$	(2,3)	$0.318\,^{+0.010}_{-0.006}$	$0.339\,^{+0.011}_{-0.008}$
(1,4)	$0.326\,^{+0.011}_{-0.011}$	$0.347\,^{+0.013}_{-0.012}$	(2,4)	$0.319\,^{+0.009}_{-0.007}$	$0.340\,^{+0.011}_{-0.009}$
(1,5)	$0.327\,^{+0.015}_{-0.013}$	$0.348\,^{+0.014}_{-0.012}$	(2,5)	$0.320\,^{+0.010}_{-0.008}$	$0.341\,^{+0.011}_{-0.009}$

Amazing stability





Power corrections don't show up

V + A

Exp. errors only

Modelling Duality Violations

$$\Delta A^{\omega}_{\mathcal{J}}(s_0) \;=\; \frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\left\{\Pi_{\mathcal{J}}(s)-\Pi_{\mathcal{J}}(s)^{\mathrm{OPE}}\right\} \;=\; -\pi\int_{s_0}^{\infty}\frac{ds}{s_0}\,\omega(s)\,\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s)$$

Ansatz: $\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \;=\; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\;\sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\, s\right)$, $s>\hat{s}_0$

Algorithmic procedure:

- $\omega_0(x)=1$ → no OPE corrections at LO (assumes well-behaved OPE)
- Fit s_0 dependence in $\hat{s}_0<s_0<m_\tau^2$ (bin by bin) → Direct fit of $\rho(s)_{\mathrm{exp}}$

$$\omega_n(x)=x^n \;\longrightarrow\; \mathrm{Im}\,\Pi_{\mathcal{J}}(s_0)=\frac{1}{s_0^n}\,\frac{d\left[s_0^{n+1}A^{\omega_n}_{\mathcal{J}}(s_0)\right]}{ds_0}$$

Direct fit of ansatz parameters (largely insensitive to α_s, OPE not valid in $\mathbb{R}$ axis)

Modelling Duality Violations

$$\Delta A^{\omega}_{\mathcal{J}}(s_0) \;=\; \frac{i}{2}\oint_{|s|=s_0}\frac{ds}{s_0}\,\omega(s)\,\left\{\Pi_{\mathcal{J}}(s)-\Pi_{\mathcal{J}}(s)^{\rm OPE}\right\} \;=\; -\pi\int_{s_0}^{\infty}\frac{ds}{s_0}\,\omega(s)\,\Delta\rho^{\rm DV}_{\mathcal{J}}(s)$$

Ansatz: $\Delta\rho^{\rm DV}_{\mathcal{J}}(s) \;=\; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\,s)}\;\sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\,s\right) \quad , \quad s>\hat{s}_0$

Algorithmic procedure:

- $\omega_0(x)=1$ ➡ no OPE corrections at LO (assumes well-behaved OPE)
- Fit s_0 dependence in $\hat{s}_0 < s_0 < m_\tau^2$ (bin by bin) ➡ Direct fit of $\rho(s)_{\rm exp}$

  $\omega_n(x)=x^n$ ➡ $\mathrm{Im}\,\Pi_{\mathcal{J}}(s_0) = \dfrac{1}{s_0^n}\,\dfrac{d\left[s_0^{n+1}A^{\omega_n}_{\mathcal{J}}(s_0)\right]}{ds_0}$

  Direct fit of ansatz parameters (largely insensitive to α_s, OPE not valid in $\mathbb{R}$ axis)
- $\forall\omega$ & $\forall s_0\geq\hat{s}_0$, the fitted ansatz determines $\Delta A^{\omega}_{\mathcal{J}}(s_0)$ **Modelling error?**
- α_s mainly extracted from $A^{\omega_0}_{\mathcal{J}}(\hat{s}_0)_{\rm pert}\approx A^{\omega_0}_{\mathcal{J}}(\hat{s}_0)_{\rm exp}-\Delta A^{\omega_0}_{\mathcal{J}}(\hat{s}_0)$

  Low scale $\hat{s}_0^{1/2}\sim 1.2\ \mathrm{GeV} < m_\tau$ ➡ **Large theory uncertainties**

Modelling Duality Violations

$$\Delta A^{\omega}_{\mathcal{J}}(s_0) \; = \; \frac{i}{2} \oint_{|s|=s_0} \frac{ds}{s_0}\, \omega(s) \left\{ \Pi_{\mathcal{J}}(s) - \Pi_{\mathcal{J}}(s)^{\mathrm{OPE}} \right\} \; = \; -\pi \int_{s_0}^{\infty} \frac{ds}{s_0}\, \omega(s)\, \Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s)$$

Ansatz: $\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \; = \; \mathcal{G}_{\mathcal{J}}(s) \; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)} \, \sin\left(\alpha_{\mathcal{J}} + \beta_{\mathcal{J}}\, s\right) \quad , \quad s > \hat{s}_0$

Algorithmic procedure:

- $\omega_0(x) = 1$ ➔ no OPE corrections at LO (assumes well-behaved OPE)
- Fit s_0 dependence in $\hat{s}_0 < s_0 < m_\tau^2$ (bin by bin) ➔ Direct fit of $\rho(s)_{\mathrm{exp}}$

  $$\omega_n(x) = x^n \;\; \text{➔} \;\; \mathrm{Im}\,\Pi_{\mathcal{J}}(s_0) = \frac{1}{s_0^n}\, \frac{d\left[s_0^{n+1} A^{\omega_n}_{\mathcal{J}}(s_0)\right]}{ds_0}$$

  Direct fit of ansatz parameters (largely insensitive to α_s, OPE not valid in $\mathbb{R}$ axis)
- $\forall\omega$ & $\forall s_0 \geq \hat{s}_0$, the fitted ansatz determines $\Delta A^{\omega}_{\mathcal{J}}(s_0)$ **Modelling error?**
- α_s mainly extracted from $A^{\omega_0}_{\mathcal{J}}(\hat{s}_0)_{\mathrm{pert}} \approx A^{\omega_0}_{\mathcal{J}}(\hat{s}_0)_{\mathrm{exp}} - \Delta A^{\omega_0}_{\mathcal{J}}(\hat{s}_0)$

  Low scale $\hat{s}_0^{1/2} \sim 1.2\ \mathrm{GeV} < m_\tau$ ➔ **Large theory uncertainties**
- Taking additional weights $\omega_n(x)$: $A^{\omega_n}_{\mathcal{J}}(\hat{s}_0)$ ➔ $\mathcal{O}_{2n}$

Boito et al ansatz: $\mathcal{G}_{\mathcal{J}}(s) = 1$, $\hat{s}_0 = 1.55\ \mathrm{GeV}^2$, $\mathcal{J} = V$

ALEPH data, FOPT

$\alpha_s^{(n_f=3)}(m_\tau^2)$	δ_V	γ_V	α_V	β_V	p-value (%)
0.298 ± 0.010	3.6 ± 0.5	0.6 ± 0.3	-2.3 ± 0.9	4.3 ± 0.5	5.3

Numerical size of duality violations

	$A_V^{\omega_n}(\hat{s}_0)$			
	$\hat{s}_0 = 1.55\ \mathrm{GeV}^2$		$s_0 = 2.8\ \mathrm{GeV}^2$	
n	DV	Exp	DV	Exp
0	0.00288	0.09735 (51)	0.00142	0.09340 (114)
1	0.00228	0.04447 (34)	0.00143	0.04529 (100)
4	−0.00032	0.01049 (20)	0.00133	0.01896 (77)

$\omega_n(x) = x^n$

	$A_V^{\hat{\omega}_n}(\hat{s}_0)$			
	$\hat{s}_0 = 1.55\ \mathrm{GeV}^2$		$s_0 = 2.8\ \mathrm{GeV}^2$	
n	DV	Exp	DV	Exp
1	0.00061	0.05288 (24)	−0.00001	0.04810 (24)
2	0.00139	0.07356 (36)	0.00000	0.06342 (36)
4	0.00320	0.08686 (52)	0.00008	0.07444 (51)

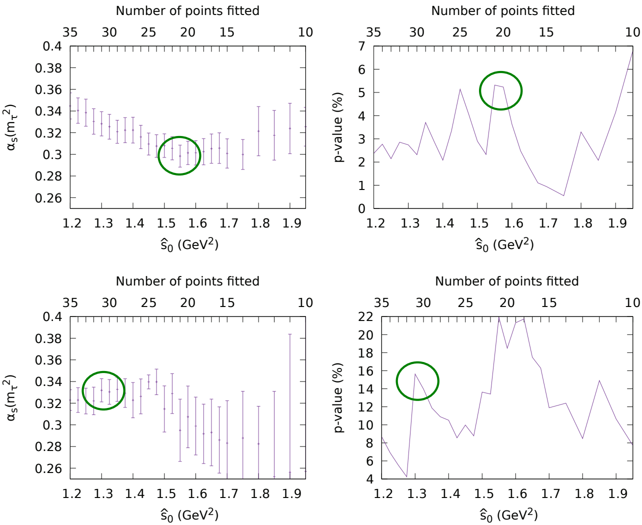
$\hat{\omega}_n(x) = 1 - x^n$

Pinching suppresses duality violations very efficiently at $\hat{s}_0 \sim m_\tau^2$

Boito et al ansatz: $\mathcal{G}_{\mathcal{J}}(s) = 1 \quad , \quad \omega_0(x) = 1$

ALEPH data, FOPT

$$\Delta\rho_{\mathcal{J}}^{\mathrm{DV}}(s) \; = \; e^{-(\delta_{\mathcal{J}} + \gamma_{\mathcal{J}}\, s)} \; \sin\left(\alpha_{\mathcal{J}} + \beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0$$



Rodríguez-Sánchez, A.P.
2205.07587

$\mathcal{J} = V$

$\hat{s}_0^V = 1.55\,\mathrm{GeV}^2$

$\alpha_s^V(m_\tau^2) = 0.298 \pm 0.010$

(Boito et al. value)

$\mathcal{J} = A$

$\hat{s}_0^A = 1.30\,\mathrm{GeV}^2$

$\alpha_s^A(m_\tau^2) = 0.332 \pm 0.011$

Bad quality fit **(Model dependence. Instabilities. Very low p-value)**

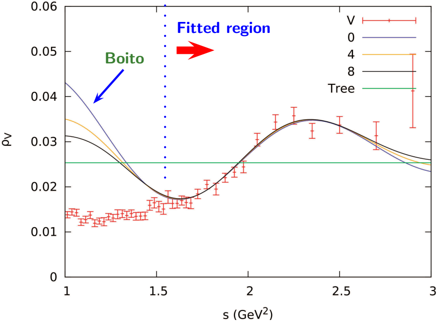
Sensitivity to the Assumed Ansatz

$$\boxed{\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \;=\; s^{\lambda_{\mathcal{J}}}\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\; \sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\, s\right) \quad , \quad s>\hat{s}_0}$$

$\lambda_V \geq 0 \quad , \quad \hat{s}_0 \sim 1.55\ \mathrm{GeV}^2 \quad , \quad \omega_0(x) = 1$

Rodríguez-Sánchez, A.P.

	λ_V	$\alpha_s(m_\tau^2)^{\mathrm{FOPT}}$	δ_V	γ_V	α_V	β_V	p-value
Boito	0	0.298 ± 0.010	3.6 ± 0.5	0.6 ± 0.3	-2.3 ± 0.9	4.3 ± 0.5	5.3 %
	2	0.302 ± 0.011	2.9 ± 0.5	1.6 ± 0.3	-2.2 ± 0.9	4.2 ± 0.5	6.0 %
	4	0.306 ± 0.013	2.3 ± 0.5	2.6 ± 0.3	-1.9 ± 0.9	4.1 ± 0.5	6.6 %
	8	0.314 ± 0.015	1.0 ± 0.5	4.6 ± 0.3	-1.5 ± 1.1	3.9 ± 0.6	7.7 %



- **Fitted α_s is model dependent**
- **$\lambda_V = 0$ (Boito) gives the worse fit**
- **Fit quality & α_s increase with λ_V**

  ➡ closer to data at $s < \hat{s}_0$
- **$\Delta\hat{s}_0$ ➡ 3 times larger errors**

Not competitive & unreliable

Sensitivity to the Assumed Ansatz

$$\boxed{\Delta\rho_{\mathcal{J}}^{\mathrm{DV}}(s) \;=\; s^{\lambda_{\mathcal{J}}}\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\; \sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0}$$

$\lambda_V \;\geq\; 0 \quad , \quad \hat{s}_0 \sim 1.55\;\mathrm{GeV}^2 \quad ,$ additional weights

Rodríguez-Sánchez, A.P.

Boito et al ⟶

λ_V	$\mathcal{O}_{6,V}$	$\mathcal{O}_{8,V}$	$\mathcal{O}_{10,V}$	$\mathcal{O}_{12,V}$
0	-0.0082	0.014	-0.019	0.023
4	-0.0064	0.010	-0.012	0.014
8	-0.0037	0.004	0.001	-0.0099

GeV units

Fitted condensates are model dependent

The size of $\mathcal{O}_{2\mathrm{n,V}}$ decreases when α_s (& fit quality) increases

Huge condensates claimed by Boito et al (OPE breakdown), but:

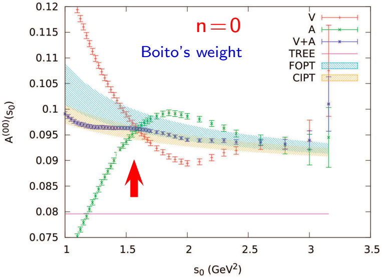
- $|\mathcal{O}_{6,V/A}| \;<\; |\mathcal{O}_{6,V-A}| \;=\; |(-3.5 \pm 0.9)| \times 10^{-3}\;\mathrm{GeV}^6$
- Opposite signs found with e^+e^- data: $\mathcal{O}_6^{\mathrm{EM}} \sim +0.008 \quad , \quad \mathcal{O}_8^{\mathrm{EM}} \sim -0.03$

(Boito et al, 1805.08176)

Experiment vs. (pinched) Perturbation Theory (only)

Rodríguez-Sánchez, A.P.

$\omega^{(n,0)}(s = s_0 x) \; = \; (1-x)^n \; \rightarrow \; \mathcal{O}_{D\leq 2\,(n+1)}$, $\alpha_s(m_\tau^2) \; = \; 0.329\,^{+\,0.020}_{-\,0.018}$

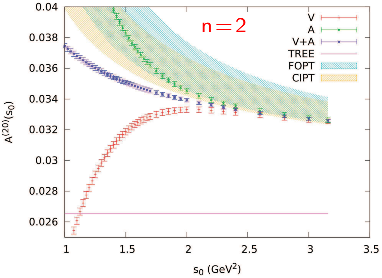


No OPE corrections at LO

Not protected against DV

Beautiful DV in V & A
(below 2.4 GeV^2)

DV largely cancels in $V+A$



Pinched weight (protected against DV)

No obvious signal of DV seen

Clear $D = 6$ OPE correction with opposite signs in V & A which cancels to a large extent in $V+A$

Sensitivity to the Assumed Ansatz

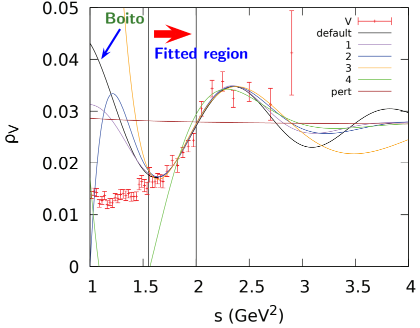
$\omega_0(x) = 1$

$$\boxed{\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \; = \; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}} + \gamma_{\mathcal{J}}\, s)}\; \sin\left(\alpha_{\mathcal{J}} + \beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0}$$

GeV^2 units

Rodríguez-Sánchez, A.P.

	Variation	$\mathcal{G}_V(s)$	$\hat{s}_0$	$\alpha_s(m_\tau^2)^{\mathrm{FOPT}}$	δ_V	γ_V	α_V	β_V	p-value
Boito	Default	1	1.55	0.298	3.6	0.6	-2.3	4.3	5.3 %
	1	s^8	1.55	0.314	1.0	4.6	-1.5	3.9	7.7 %
	2	$1 - \frac{1.35}{s}$	1.55	0.319	-0.19	1.8	-0.8	3.5	7.8 %
	3	$1 - \frac{2}{s}$	1.55	0.260	0.23	1.2	3.2	2.1	6.4 %
	4	1	2	0.320	0.56	1.9	0.15	3.1	6.9 %



- **Fitted α_s is model dependent**
- **Higher p-values than default fit**
- **Bad behaviour outside fitted region**

  Below $\hat{s}_0$ data deviate from DV models much more than from OPE at $\sim m_\tau^2$
- **Models with low values of α_s (def, 3) have large bumps above m_τ^2**

Sensitivity to the Assumed Ansatz

$\omega_n(x) = x^n$

$$\boxed{\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \; = \; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\; \sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0}$$

GeV^2 units

Rodríguez-Sánchez, A.P.

Variation	$\mathcal{O}_{4,V}$	$\mathcal{O}_{6,V}$	$\mathcal{O}_{8,V}$	$\mathcal{O}_{10,V}$	$\mathcal{O}_{12,V}$	$\mathcal{O}_{14,V}$	$\mathcal{O}_{16,V}$	α_s
Default	0.0016	−0.0082	0.014	−0.019	0.023	−0.028	0.037	0.298
1	−0.0003	−0.0037	0.004	0.001	−0.010	0.006	0.079	0.314
2	−0.0009	−0.0023	0.001	0.008	−0.026	0.046	−0.032	0.319
3	0.0079	−0.0318	0.091	−0.25	0.59	−1.04	0.24	0.260
4	−0.0009	−0.0012	−0.003	0.021	−0.06	0.13	−0.20	0.320

Variation	$\mathcal{O}_{4,A}$	$\mathcal{O}_{6,A}$	$\mathcal{O}_{8,A}$	$\mathcal{O}_{10,A}$	$\mathcal{O}_{12,A}$	$\mathcal{O}_{14,A}$	$\mathcal{O}_{16,A}$	α_s
Default	0.0006	−0.0016	0.016	−0.052	0.11	−0.11	−0.28	0.298
1	−0.0015	0.0039	0.0024	−0.023	0.060	−0.084	−0.037	0.314
2	−0.0021	0.0055	−0.0012	−0.015	0.047	−0.074	−0.000	0.319
3	0.0054	−0.017	0.058	−0.16	0.31	−0.16	−2.15	0.260
4	−0.0014	0.0048	−0.0005	−0.016	0.048	−0.08	0.028	0.320

Minor ansatz modifications can generate huge changes of $\mathcal{O}_{2n,\mathcal{J}}$

Once the ansatz and α_s get fixed, $\mathcal{O}_{2n,\mathcal{J}}$ must reabsorb all perturbative (through a slightly off α_s) **and DV deformations introduced by the models**

Numerical Size of Duality Violations: $\hat{\omega}_N(x) = 1 - x^N$

$$\Delta\rho_{\mathcal{J}}^{\mathrm{DV}}(s) \;=\; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\;\sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0$$

Rodríguez-Sánchez, A.P.

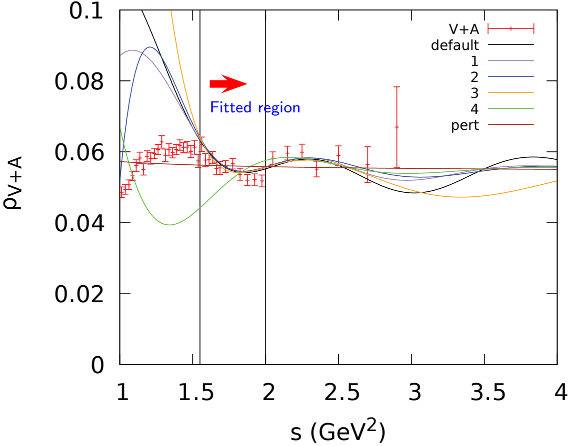
		$A_V^{\hat{\omega}_N}(s_0)$			
		$\hat{s}_0$		$s_0 = 2.8\ \mathrm{GeV}^2$	
Ansatz	N	DV	Exp	DV	Exp
Default $\mathcal{G}_V(s) = 1$ $\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	1	0.00061	0.05288 (24)	−0.00001	0.04810 (24)
	2	0.00139	0.07356 (36)	0.00000	0.06342 (36)
	4	0.00320	0.08686 (52)	0.00008	0.07444 (51)
1 $\mathcal{G}_V(s) = s^8$ $\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	1	0.00151	0.05288 (24)	−0.00008	0.04810 (24)
	2	0.00358	0.07356 (36)	−0.00016	0.06342 (36)
	4	0.00859	0.08686 (52)	−0.00031	0.07444 (51)
2 $\mathcal{G}_V(s) = 1 - \frac{1.35}{s}$ $\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	1	0.00177	0.05288 (24)	−0.00011	0.04810 (24)
	2	0.00421	0.07356 (36)	−0.00021	0.06342 (36)
	4	0.01053	0.08686 (52)	−0.00040	0.07444 (51)
3 $\mathcal{G}_V(s) = 1 - \frac{2}{s}$ $\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	1	−0.00470	0.05288 (24)	−0.00071	0.04810 (24)
	2	0.00358	0.07356 (36)	−0.00137	0.06342 (36)
	4	−0.073411	0.08686 (52)	−0.00191	0.07444 (51)
4 $\mathcal{G}_V(s) = 1$ $\hat{s}_0 = 2\ \mathrm{GeV}^2$	1	0.00064	0.05019 (24)	−0.00009	0.04810 (24)
	2	0.00126	0.06707 (32)	−0.00018	0.06342 (36)
	4	0.00214	0.07720 (38)	−0.00039	0.07444 (51)

Pinching suppresses duality violations very efficiently at $s_0 \sim m_\tau^2$

DV modelling of the $V + A$ Spectral Function

Rodríguez-Sánchez, A.P.

$$\Delta\rho_{\mathcal{J}}^{\mathrm{DV}}(s) \;=\; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\;\sin\left(\alpha_{\mathcal{J}}+\beta_{\mathcal{J}}\, s\right)$$

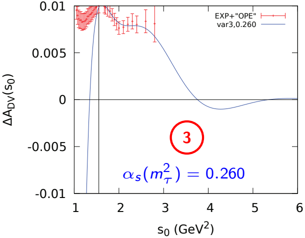


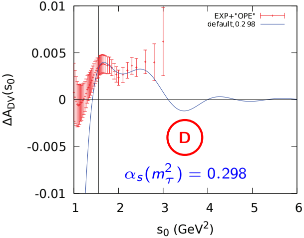
- **Big disagreement with data below fitted region**
- **Fake violation of duality at $s \gtrsim m_\tau^2$** (larger than a_1 resonance peak) **in models 'default' & '3'** (the models giving low values of α_s)

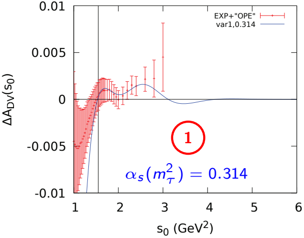
Predicted DV Shapes

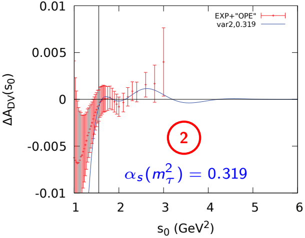
$$\Delta A^{\omega_0}_{V+A}(s_0) = -\pi \int_{s_0}^{\infty} \frac{ds}{s_0}\, \Delta\rho^{\rm DV}_{V+A}$$

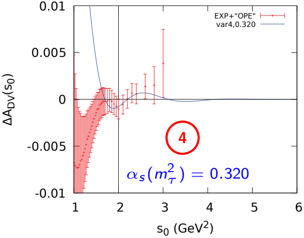
$$= A^{\omega_0}_{V+A}(s_0)_{\rm exp} - A^{\omega_0}_{V+A}(s_0)_{\rm OPE}$$











- **Unphysical behaviour outside fitted region for models D & 3**

  Huge DV at m_τ^2 needs to fall down abruptly to zero to comply with Asymptotic Freedom
- **The non-pathological scenarios 1, 2 & 4 give values of α_s within our estimated 1σ interval** (standard approach)

Numerical Size of Power Corrections

$$\boxed{\Delta\rho_{\mathcal{J}}^{\mathrm{DV}}(s) \; = \; \mathcal{G}_{\mathcal{J}}(s)\; e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)} \;\sin\left(\alpha_{\mathcal{J}} + \beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0}$$

$\omega^{(2,n)}(x) = 1 - (n+2)\, x^{n+1} + (n+1)\, x^{n+2}$

Rodríguez-Sánchez, A.P.

Weight	Model	Pert	$\mathcal{O}_{2(n+2),V+A}$	$\mathcal{O}_{2(n+3),V+A}$	DV	Exp
$A_{V+A}^{\omega^{(2,1)}}(m_\tau^2)$	D	0.0938 (5)	0.0029	−0.0019	−0.0001	0.0954 (3)
	1	0.0952 (7)	−0.0001	−0.0004	−0.0000	0.0954 (3)
	2	0.0957 (8)	−0.0010	0.0000	−0.0000	0.0954 (3)
	3	0.0908 (2)	0.0145	−0.0095	−0.0007	0.0954 (3)
	4	0.0958 (8)	−0.0011	−0.0005	−0.0000	0.0954 (3)
$A_{V+A}^{\omega^{(2,4)}}(m_\tau^2)$	D	0.1316 (4)	0.0025	−0.0007	0.0001	0.1344 (8)
	1	0.1331 (5)	0.0009	−0.0004	0.0001	0.1344 (8)
	2	0.1336 (5)	0.0004	−0.0001	0.0000	0.1344 (8)
	3	0.1282 (2)	0.0171	−0.0061	−0.0056	0.1344 (8)
	4	0.1337 (5)	−0.0002	0.0002	0.0001	0.1344 (8)

Weight	Model	Pert	$\mathcal{O}_{2(n+2),V+A}$	$\mathcal{O}_{2(n+3),V+A}$	DV	Exp
$A_{V+A}^{\omega^{(2,1)}}(\hat{s}_0)$	D	0.1010 (18)	0.0248	−0.0326	0.0062	0.0994 (4)
	1	0.1043 (28)	−0.0006	−0.0071	0.0028	0.0994 (4)
	2	0.1054 (32)	−0.0081	0.0003	0.0018	0.0994 (4)
	3	0.0948 (06)	0.1221	−0.1629	0.0452	0.0994 (4)
	4	0.1010 (18)	−0.0042	0.0015	−0.0001	0.0980 (3)
$A_{V+A}^{\omega^{(2,4)}}(\hat{s}_0)$	D	0.1391 (10)	0.1808	−0.1012	−0.0787	0.1401 (5)
	1	0.1424 (14)	0.0676	−0.0572	−0.0128	0.1401 (5)
	2	0.1434 (16)	0.0281	−0.0203	−0.0112	0.1401 (5)
	3	0.1327 (05)	1.2216	−0.8833	−0.3309	0.1401 (5)
	4	0.1392 (11)	−0.0036	0.0058	−0.0034	0.1378 (4)

Summary

- **Violations of quark-hadron duality are interesting per se, but they cannot be used to determine α_s precisely**
  - Strong sensitivity to the assumed DV model
  - Unstable algorithmic procedure
  - Accuracy necessarily limited by our lack of control of DV effects

- **A precise determination of α_s requires minimizing DV**
  - Use inclusive observables (moments)
  - Take s_0 large enough (exponential suppression)
  - Use pinched weights (with low values of n)

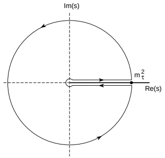
- ***R_τ* ideally suited** (doubly pinched, m_τ^2 large enough, $V+A$)

- **Accurate, robust and reliable determination:** (standard approach)

$$\boxed{\alpha_s(m_\tau^2) = \begin{cases} 0.320 \pm 0.012 \\ 0.335 \pm 0.013 \end{cases} \Rightarrow \alpha_s(M_Z^2) = \begin{cases} 0.1187 \pm 0.0015 & \text{(FOPT)} \\ 0.1205 \pm 0.0015 & \text{(CIPT)} \end{cases}}$$

Backup

R_τ suitable for a precise α_s determination



$$R_\tau = 6\pi i \oint_{|x|=1} (1-x)^2 \left[(1+2x)\, \Pi^{(0+1)}(m_\tau^2 x) - 2x\, \Pi^{(0)}(m_\tau^2 x) \right]$$

$$\Pi^{(J)}_{\mathcal{J}}(s) \approx \Pi^{(J)}_{\mathcal{J}}(s)^{\rm OPE} = \sum_D \frac{\mathcal{O}^{(J)}_{D,\mathcal{J}}}{(-s)^{D/2}}$$

- **Known to $\mathcal{O}(\alpha_s^4)$. Sizeable $\delta_P \sim 20\%$. Strong sensitivity to α_s**
- **m_τ large enough to safely use the OPE. Flat $V+A$ distribution**
- **OPE only valid away from real axis: $(1-x)^2$ pinched at $s = m_\tau^2$**
- $m_{u,d} = 0$ ➡ $s\,\Pi^{(0)} = 0$ ➡ $R_{\tau,V+A} = 6\pi i \oint_{|x|=1} (1-3x^2+2x^3)\, \Pi^{(0+1)}_{ud,V+A}(m_\tau^2 x)$

➡ $\delta_{NP} \sim 1/m_\tau^6$ **Strong suppression of non-perturbative effects**

- **$D=6$ OPE contributions have opposite sign for V & A. Cancellation**
- **δ_{NP} can be determined from data: $\delta_{NP} = -0.0064 \pm 0.0013$** Davier et al

R_τ ➡ $\boxed{\alpha_s(m_\tau^2) = 0.331 \pm 0.013}$ Pich 2014

Unjustified complaints (M. Golterman, TAU 2023)

- **Criticism 1: DV model is not stable against variations**

$$\Delta\rho^{\mathrm{DV}}(s) \;=\; \left(1+\frac{c}{s}\right)\, e^{-(\delta+\gamma\, s)}\,\sin\left(\alpha+\beta\, s\right) \qquad , \qquad s>\hat{s}_0$$

$c\sim -2\ \mathrm{GeV}^2$ is not a small correction for $s\sim 2\ \mathrm{GeV}^2$

None of the analysed ansatzs is theoretically justified at $\hat{s}_0\sim 1.55\ \mathrm{GeV}^2$

The large variations exhibit the very strong dependence on the assumed model

Ansatz can be adapted to reproduce small changes of data and input value of α_s

Regge-inspired model: (Boito et al, 1711.10316)

$$\Delta\rho^{\mathrm{DV}}(s) \;=\; e^{-(\delta+\gamma\, s)}\,\sin\left(\alpha+\beta\, s\right)\,\left\{1+\frac{a}{\log\left(\frac{\beta s}{2\pi}\right)}+\frac{b}{\log^2\left(\frac{\beta s}{2\pi}\right)}+\cdots\right\}$$

$$\frac{1}{\log\left(\frac{\beta s}{2\pi}\right)}<\infty \;\longrightarrow\; s>\frac{2\pi}{\beta} \;\longrightarrow\; \begin{cases} s>1.7\ \mathrm{GeV}^2 & (\beta_V=3.8\ \mathrm{GeV}^2;\ \text{Boito et al, 2012.10440}) \\ s>1.5\ \mathrm{GeV}^2 & (\beta_V=4.3\ \mathrm{GeV}^2;\ \text{Boito et al, 1410.3528}) \end{cases}$$

$$\frac{1}{\log\left(\frac{\beta s}{2\pi}\right)}<\frac{1}{2} \;\longrightarrow\; s>\frac{2\pi}{\beta}\,\mathrm{e}^2 \;\longrightarrow\; \begin{cases} s>12.2\ \mathrm{GeV}^2 & (\beta_V=3.8\ \mathrm{GeV}^2;\ \text{Boito et al, 2012.10440}) \\ s>10.8\ \mathrm{GeV}^2 & (\beta_V=4.3\ \mathrm{GeV}^2;\ \text{Boito et al, 1410.3528}) \end{cases}$$

Unjustified complaints (M. Golterman, TAU 2023)

- **Criticism 2: Logarithmic corrections to OPE**

Higher-order logarithms ($\mathcal{P}_{D,\mathcal{J}}$) affect sum rules with low-degree (n) polynomials

$$\Pi^{\rm OPE}_{\mathcal{J}}(s)\Big|^{\rm NLO}_{D>0} = \sum_{D>0} \frac{\mathcal{O}_{D,\mathcal{J}}(\mu) + \mathcal{P}_{D,\mathcal{J}}\;\log\left(-s/\mu^2\right)}{(-s)^{D/2}}$$

$\omega_n(s) = \left(\frac{s}{s_0}\right)^n$ ➡ $$\Delta A^{(n)}_{\mathcal{J}}(s_0)\Big|^{\rm OPE}_{D>0} = -\pi \sum_{p=2} \frac{d^{(n)}_{p,\mathcal{J}}}{(-s_0)^p} \quad , \quad d^{(n)}_{p,\mathcal{J}} = \begin{cases} \mathcal{O}_{2p,\mathcal{J}}(s_0) & \text{if } p = n+1 \\ \dfrac{\mathcal{P}_{2p,\mathcal{J}}}{n-p+1} & \text{if } p \neq n+1 \end{cases}$$

$\mathcal{P}_{\rm D,\mathcal{J}} = 0$ **justified if OPE is well behaved. Meaningless otherwise:**

OPE validity ➡ $\frac{|\mathcal{O}_{D+2k,\mathcal{J}}|}{s_0^k} \lesssim |\mathcal{O}_{D,\mathcal{J}}| \quad , \quad |\mathcal{P}_{D,\mathcal{J}}| \equiv \xi_{D,\mathcal{J}}\, |\mathcal{O}_{D,\mathcal{J}}| \quad , \quad \xi_{D,\mathcal{J}} \sim O(\frac{\alpha_s}{\pi})$

$|\mathcal{P}_{D,\mathcal{J}}| \ll \frac{|\mathcal{O}_{D+2k,\mathcal{J}}|}{s_0^k} \quad \forall k$ ➡ $$\left|\frac{\mathcal{O}_{D+2k,\mathcal{J}}}{\mathcal{O}_{D,\mathcal{J}}}\right|^{1/(2k)} < \sqrt{s_0} \ll \underbrace{\xi_{D,\mathcal{J}}^{-1/(2k)}}_{\approx 1} \left|\frac{\mathcal{O}_{D+2k,\mathcal{J}}}{\mathcal{O}_{D,\mathcal{J}}}\right|^{1/(2k)}$$

Educated guess estimate: $\xi_{D,\mathcal{J}} \sim O(0.1 - 0.2)$ [$|\xi_{6,V+A}| \sim 0.15$, (Boito-Hornung-Jamin, 1510.03812)]

Golterman's justification based on a large-N_C estimate $\xi_6 \sim 0.03$, while assuming $|\mathcal{O}_{6,V+A}| \sim 200\, |\mathcal{O}_6|_{N_C\to\infty}$

Unjustified complaints

(M. Golterman, TAU 2023)

- **Criticism 3: DV strategy is "tautological" - additional weights do not add information**

- $\omega_0(x) = 1$ → no OPE corrections at LO (assumes well-behaved OPE)
- Fit s_0 dependence in $\hat{s}_0 < s_0 < m_\tau^2$ (bin by bin) → Direct fit of $\rho(s)_{\mathrm{exp}}$

$$\omega_n(x) = x^n \quad \rightarrow \quad \mathrm{Im}\,\Pi_{\mathcal{J}}(s_0) = \frac{1}{s_0^n}\,\frac{d\left[s_0^{n+1}A_{\mathcal{J}}^{\omega_n}(s_0)\right]}{ds_0}$$

Direct fit of ansatz parameters (largely insensitive to α_s, OPE not valid in $\mathbb{R}$ axis)

- $\forall\omega$ & $\forall s_0 \geq \hat{s}_0$, the fitted ansatz determines $\Delta A_{\mathcal{J}}^{\omega}(s_0)$ **Modelling error?**
- α_s mainly extracted from $A_{\mathcal{J}}^{\omega_0}(\hat{s}_0)_{\mathrm{pert}} \approx A_{\mathcal{J}}^{\omega_0}(\hat{s}_0)_{\mathrm{exp}} - \Delta A_{\mathcal{J}}^{\omega_0}(\hat{s}_0)$

Low scale $\hat{s}_0^{1/2} \sim 1.2\ \mathrm{GeV} < m_\tau$ → **Large theory uncertainties**

- Taking additional weights $\omega_n(x)$: $A_{\mathcal{J}}^{\omega_n}(\hat{s}_0)$ → $\mathcal{O}_{2n}$

Golterman takes $n = 0, 2$ and claims that the wrong correction $\hat{C}_6/s_0^5$ (why doesn't call it C_{10}?) modifies α_s. He is actually enforcing $C_6 = 0$ ($\hat{C}_6 \equiv C_{10}$ does not contribute to those moments).
Therefore, **he constraints α_s with two sum rules without any other free parameters**

Unjustified complaints (M. Golterman, TAU 2023)

- **Criticism 4: DV strategy determines α_s at** $1.55\ \text{GeV}^2$

  - $\omega_0(x) = 1$ ➔ no OPE corrections at LO (assumes well-behaved OPE)
  - Fit s_0 dependence in $\hat{s}_0 < s_0 < m_\tau^2$ (bin by bin) ➔ Direct fit of $\rho(s)_{\mathrm{exp}}$

    $$\omega_n(x) = x^n \quad \Longrightarrow \quad \mathrm{Im}\,\Pi_{\mathcal{J}}(s_0) = \frac{1}{s_0^n}\,\frac{d\left[s_0^{n+1} A_{\mathcal{J}}^{\omega_n}(s_0)\right]}{ds_0}$$

    Direct fit of ansatz parameters (largely insensitive to α_s, OPE not valid in $\mathbb{R}$ axis)

  - $\forall\omega$ & $\forall s_0 \geq \hat{s}_0$, the fitted ansatz determines $\Delta A_{\mathcal{J}}^{\omega}(s_0)$ **Modelling error?**
  - α_s mainly extracted from $A_{\mathcal{J}}^{\omega_0}(\hat{s}_0)_{\mathrm{pert}} \approx A_{\mathcal{J}}^{\omega_0}(\hat{s}_0)_{\mathrm{exp}} - \Delta A_{\mathcal{J}}^{\omega_0}(\hat{s}_0)$

    Low scale $\hat{s}_0^{1/2} \sim 1.2\ \mathrm{GeV} < m_\tau$ ➔ **Large theory uncertainties**

  - Taking additional weights $\omega_n(x)$: $A_{\mathcal{J}}^{\omega_n}(\hat{s}_0)$ ➔ $\mathcal{O}_{2n}$

All parameters obtained from fits using all data. True, but…

Data above $\hat{s}_0$ fix the ansatz. $A_{\mathcal{J}}^{\omega_n}(\hat{s}_0)$ are the ONLY additional information

Numerical Size of Duality Violations: $\omega_N(x) = x^N$

$$\Delta\rho^{\mathrm{DV}}_{\mathcal{J}}(s) \; = \; \mathcal{G}_{\mathcal{J}}(s)\, e^{-(\delta_{\mathcal{J}}+\gamma_{\mathcal{J}}\, s)}\; \sin\left(\alpha_{\mathcal{J}} + \beta_{\mathcal{J}}\, s\right) \qquad , \qquad s > \hat{s}_0$$

Rodríguez-Sánchez, A.P.

		$A_V^{\omega_N}(s_0)$			
		$\hat{s}_0$		$s_0 = 2.8\ \mathrm{GeV}^2$	
Ansatz	N	DV	Exp	DV	Exp
Default	0	0.00288	0.09735 (51)	0.00142	0.09340 (114)
$\mathcal{G}_V(s) = 1$	1	0.00228	0.04447 (34)	0.00143	0.04529 (100)
$\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	4	-0.00032	0.01049 (20)	0.00133	0.01896 (77)
1	0	0.00132	0.09735 (51)	0.00088	0.09340 (114)
$\mathcal{G}_V(s) = s^8$	1	-0.00019	0.04447 (34)	0.00096	0.04529 (100)
$\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	4	-0.00726	0.01049 (20)	0.00119	0.01896 (77)
2	0	0.00082	0.09735 (51)	0.00088	0.09340 (114)
$\mathcal{G}_V(s) = 1 - \frac{1.35}{s}$	1	-0.00095	0.04447 (34)	0.00098	0.04529 (100)
$\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	4	-0.00970	0.01049 (20)	0.00128	0.01896 (77)
3	0	0.00611	0.09735 (51)	0.00429	0.09340 (114)
$\mathcal{G}_V(s) = 1 - \frac{2}{s}$	1	0.01081	0.04447 (34)	0.00500	0.04529 (100)
$\hat{s}_0 = 1.55\ \mathrm{GeV}^2$	4	0.07952	0.01049 (20)	0.00620	0.01896 (77)
4	0	-0.00511	0.08944 (52)	0.00042	0.09340 (114)
$\mathcal{G}_V(s) = 1$	1	-0.00575	0.03926 (40)	0.00051	0.04529 (100)
$\hat{s}_0 = 2\ \mathrm{GeV}^2$	4	-0.00725	0.01224 (27)	0.00081	0.01896 (77)