# Gradient flow scales and alpha_s from 2+1 flavor lattice QCD

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**alpha_s 2024**

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# Lattice setup

$$N_f = 2+1,\ \text{HISQ},\ m_s/m_l = 20$$

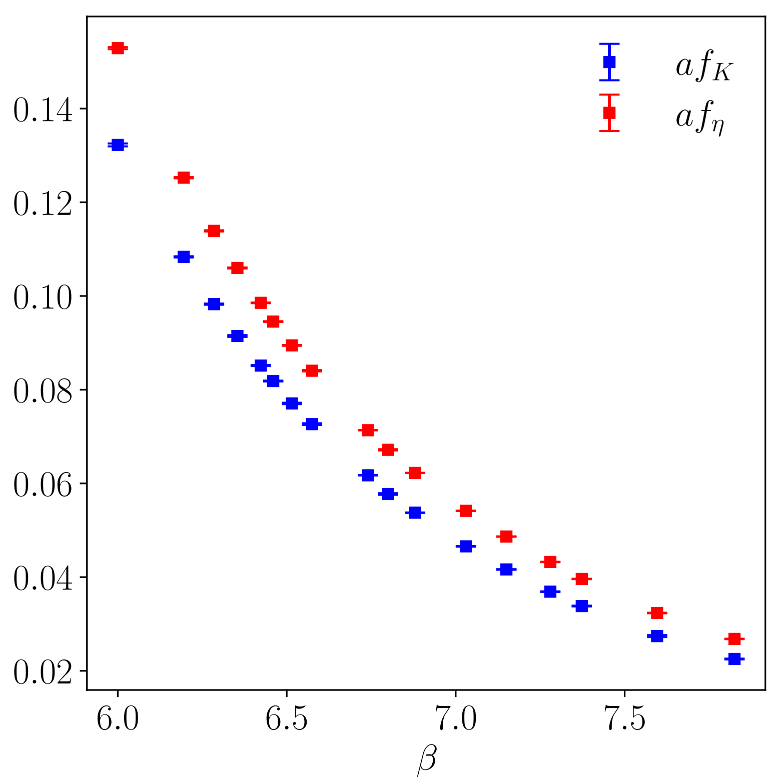
| $\beta$ | $am_s$ | $am_l$ | $N_\sigma$ | $N_\tau$ | # conf. |
|---|---|---|---|---|---|
| 7.030 | 0.0356 | 0.00178 | 48 | 48 | 900 |
| 7.150 | 0.0320 | 0.00160 | 48 | 64 | 395 |
| 7.280 | 0.0280 | 0.00142 | 48 | 64 | 398 |
| 7.373 | 0.0250 | 0.00125 | 48 | 64 | 554 |
| 7.596 | 0.0202 | 0.00101 | 64 | 64 | 577 |
| 7.825 | 0.0164 | 0.0082 | 64 | 64 | 471 |
| 8.000 | 0.01299 | 0.002598 | 64 | 64 | 1004 |
| 8.200 | 0.01071 | 0.002142 | 64 | 64 | 961 |
| 8.249 | 0.01011 | 0.002022 | 64 | 64 | 2241 |
| 8.400 | 0.00887 | 0.001774 | 64 | 64 | 1228 |

- $t_0$ and $w_0$ scale
- Lambda_msbar

# Relate indirect scales to (known) physical scales

## Kaon/eta_ss decay constant:

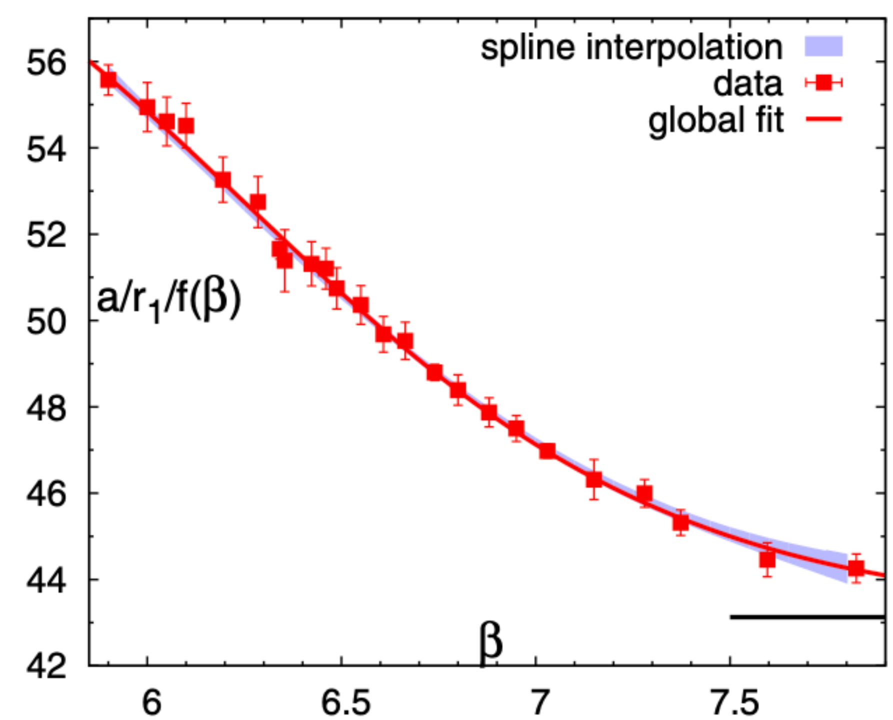
$$f_{\text{meson}}^2 M_{\text{meson}}^2 = -m_{\text{quark}}^{(r)} N_f \Sigma^{(r)}$$




HotQCD, PRD 90, 094503 (2014)

## Static quark potential:

$$r^2 \frac{\mathrm{d}V}{\mathrm{d}r}\bigg|_{r=r_1} = 1$$




HotQCD, PRD 90, 094503 (2014)

Build dimensionless objects:

$f_{\text{PS}}\sqrt{t_0}$ $f_{\text{PS}} w_0$

$\sqrt{t_0}/r_1$ $w_0/r_1$

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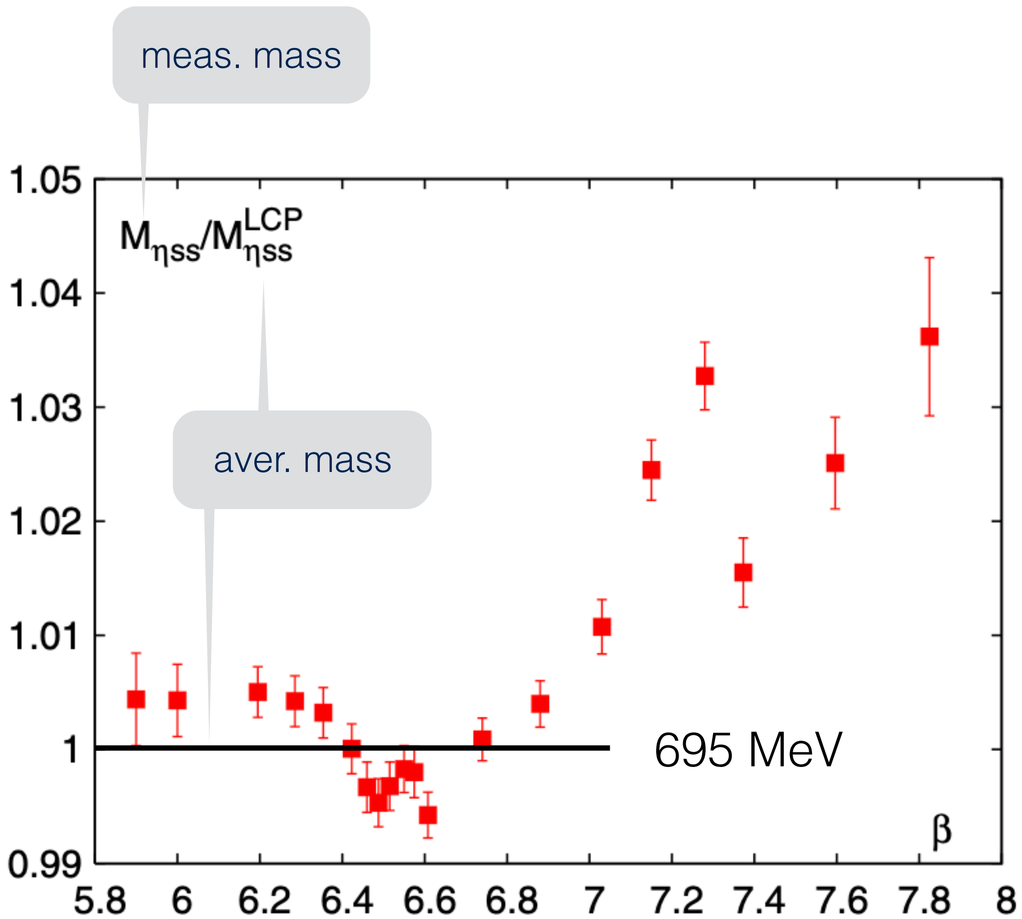
$\eta_{ss}$ : not the physical $\eta$ $(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)$ but a fictitious unmixed ps meson $\bar{s}\gamma_5 s$

# Mass-mismatch correction




HotQCD, PRD 90, 094503 (2014)

Measured eta mass deviates from the physical one:

$$m_\eta^{\text{phys.}} = 685.8\ \text{MeV}$$

Correct the mass discrepancy using LO ChPT:

$$f_K^{\text{meas./phys.}} = f_0 + s(m_s + m_l)^{\text{input/phys.}}$$

Physical quark mass from the LCP quark mass:

$$m_s^{\text{LCP}}(\beta) = m_s^{\text{input}}(\beta) \cdot \left(\frac{m_\eta^{\text{LCP}}}{m_\eta^{\text{input}}}\right)^2$$

LCP quark mass takes RG-inspired form: $r_1 m_s^{\text{LCP}} = r_1 m^{\text{RGI}} \left(\frac{20 b_0}{\beta}\right)^{4/9} \frac{1 + m_1 \frac{10}{\beta} f^2(\beta) + m_2 (\frac{10}{\beta})^2 f^2(\beta) + m_3 \frac{10}{\beta} f^4(\beta)}{1 + dm_1 \frac{10}{\beta} f^2(\beta)}$

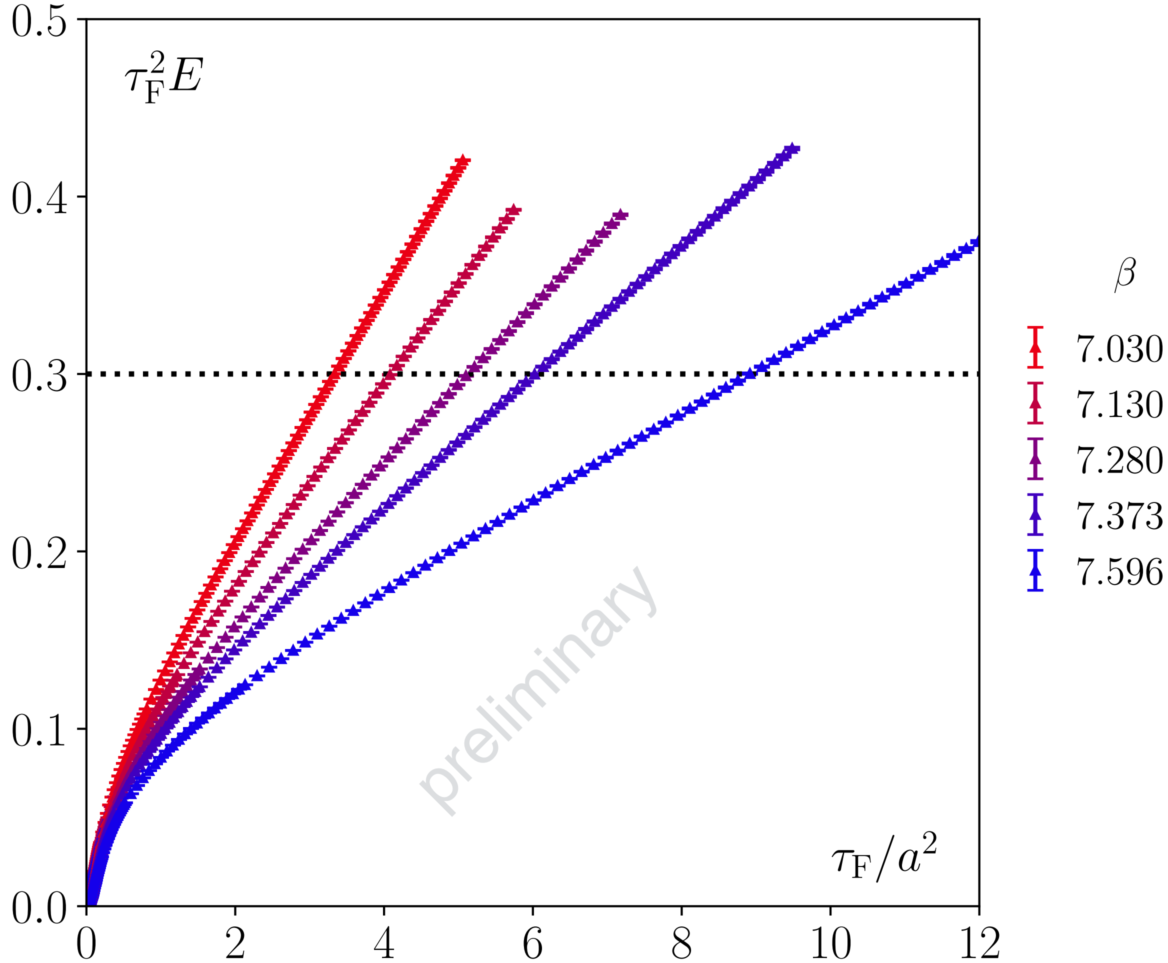
Physical decay const.:

$$f_\eta^{\text{phys}}(\beta) = f_\eta^{\text{meas}}(\beta) + 2s \cdot \left( m_s^{\text{LCP}}(\beta) \left(\frac{m_\eta^{\text{phys}}}{m_\eta^{\text{LCP}}}\right)^2 - m_s^{\text{input}}(\beta) \right)$$

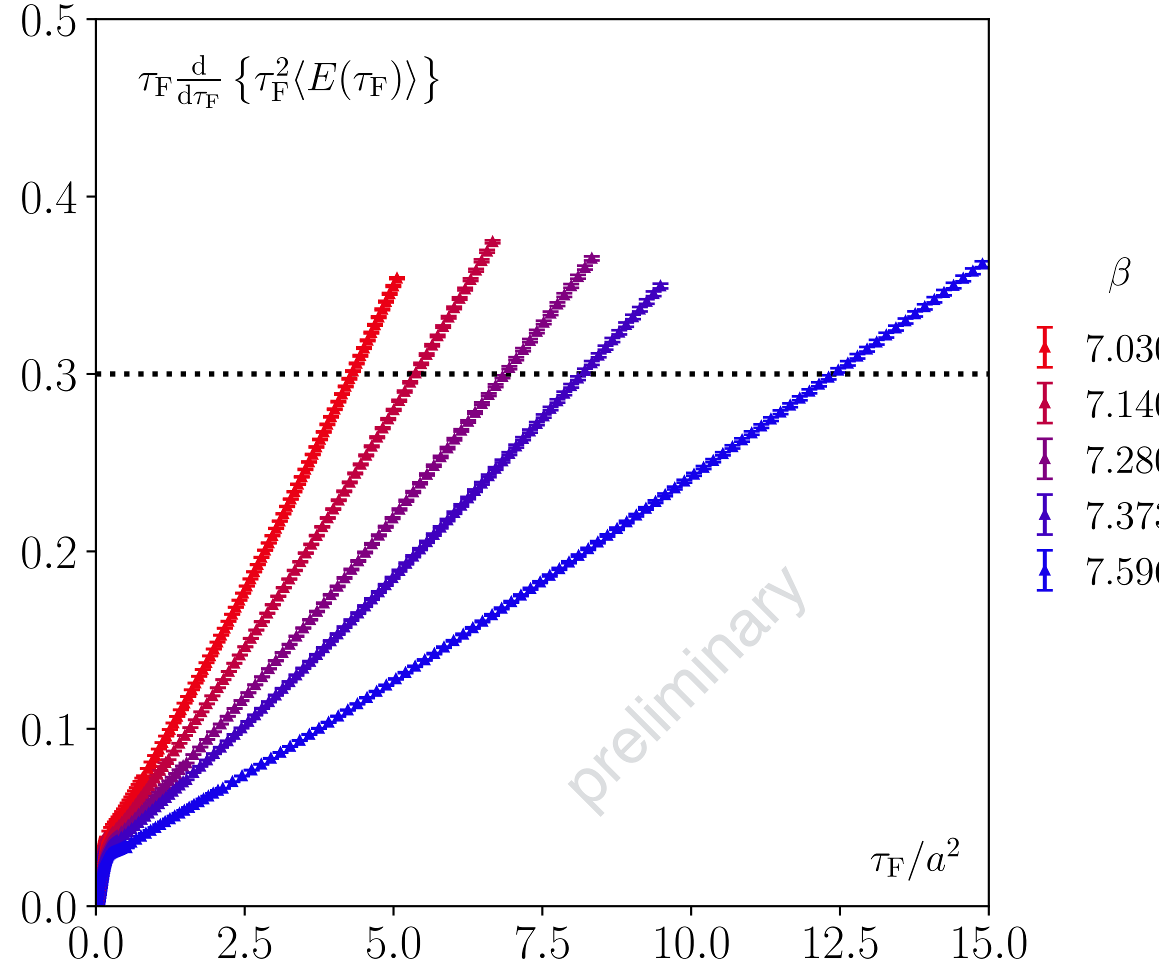
$$f_K^{\text{phys}}(\beta) = f_K^{\text{meas}}(\beta) + s \cdot \left( m_s^{\text{LCP}}(\beta) \left(\frac{m_\eta^{\text{phys}}}{m_\eta^{\text{LCP}}}\right)^2 \cdot \frac{28.3}{27.3} - m_s^{\text{input}}(\beta) \cdot \frac{21}{20} \right)$$

# Scale condition

$$\tau_{\mathrm{F}}^2 \langle E(\tau_{\mathrm{F}}) \rangle \Big|_{\tau_{\mathrm{F}}=t_0} = 0.3.$$

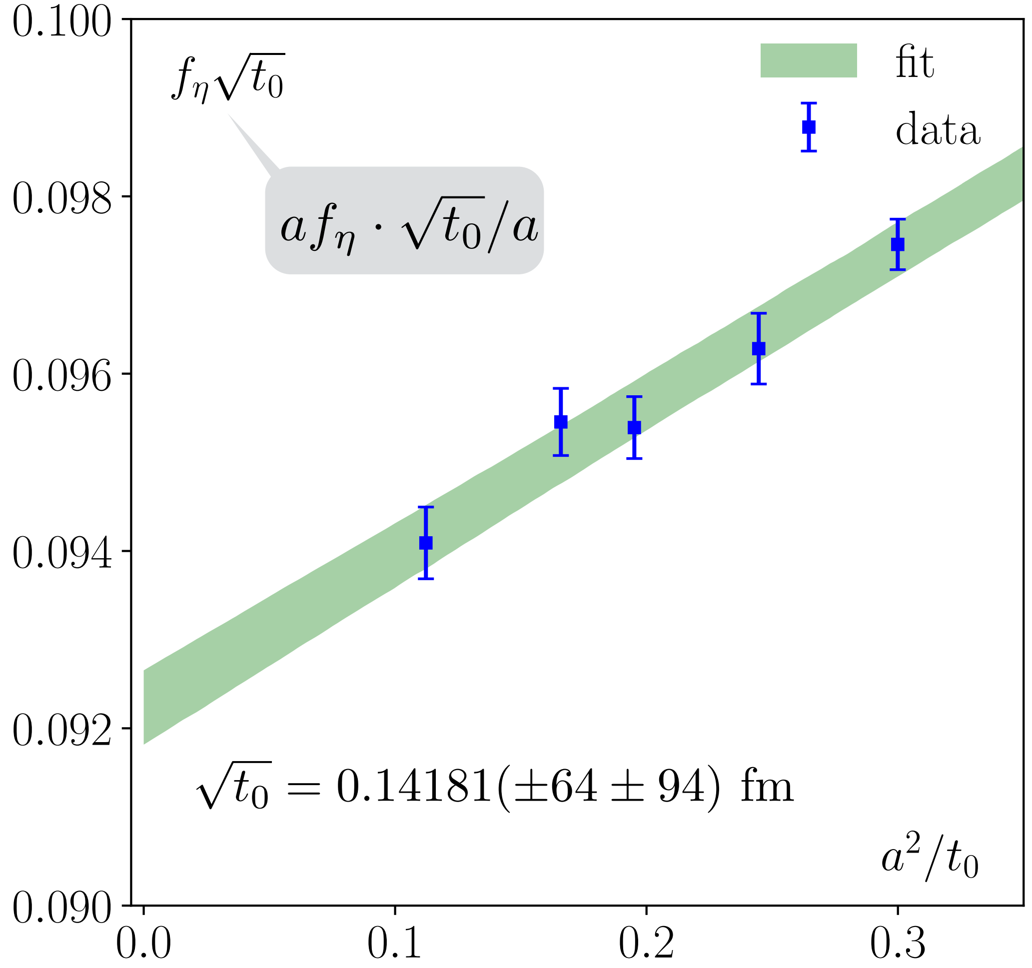


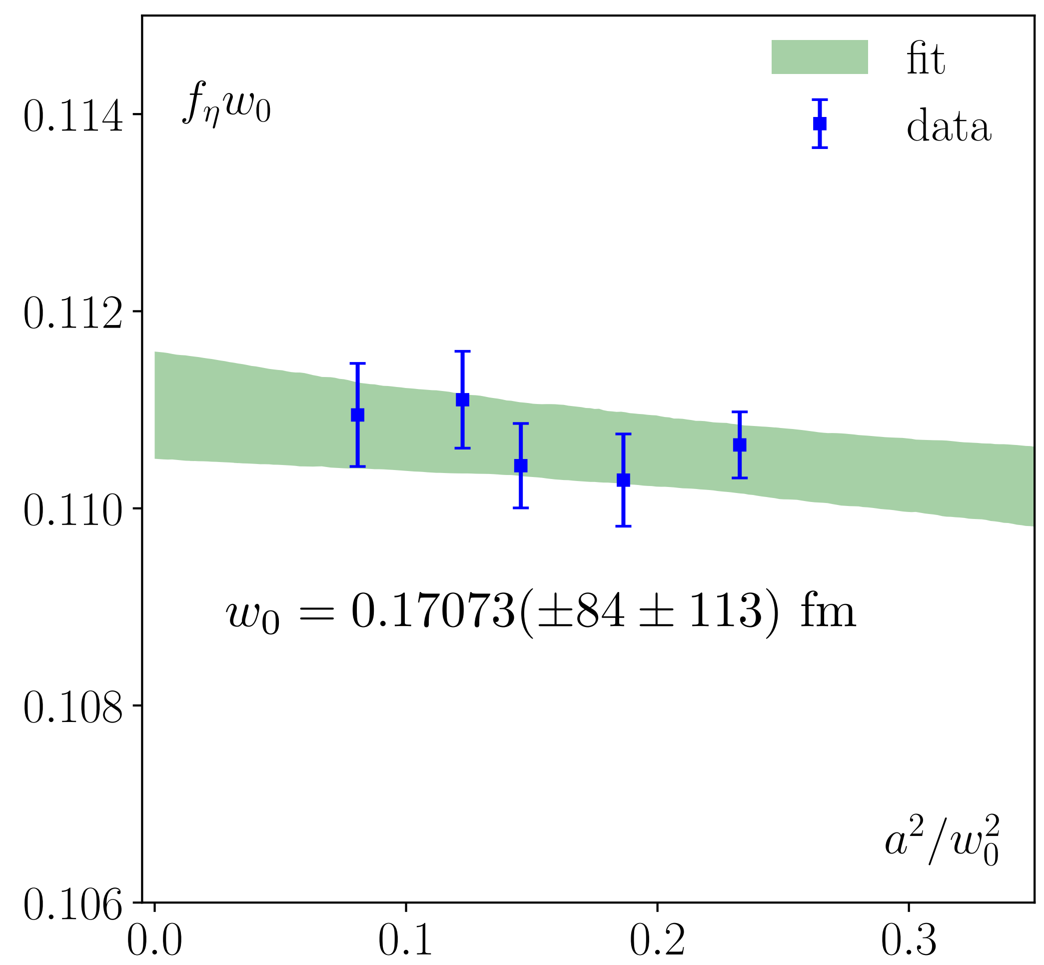

$$\tau_{\mathrm{F}} \frac{\mathrm{d}}{\mathrm{d}\tau_{\mathrm{F}}} \left\{ \tau_{\mathrm{F}}^2 \langle E(\tau_{\mathrm{F}}) \rangle \right\} \Big|_{\tau_{\mathrm{F}}=w_0^2} = 0.3$$




- Precise determination on the lattice
- Zeuthen flow: improved cutoff effects

# Flow scales via eta decay constant

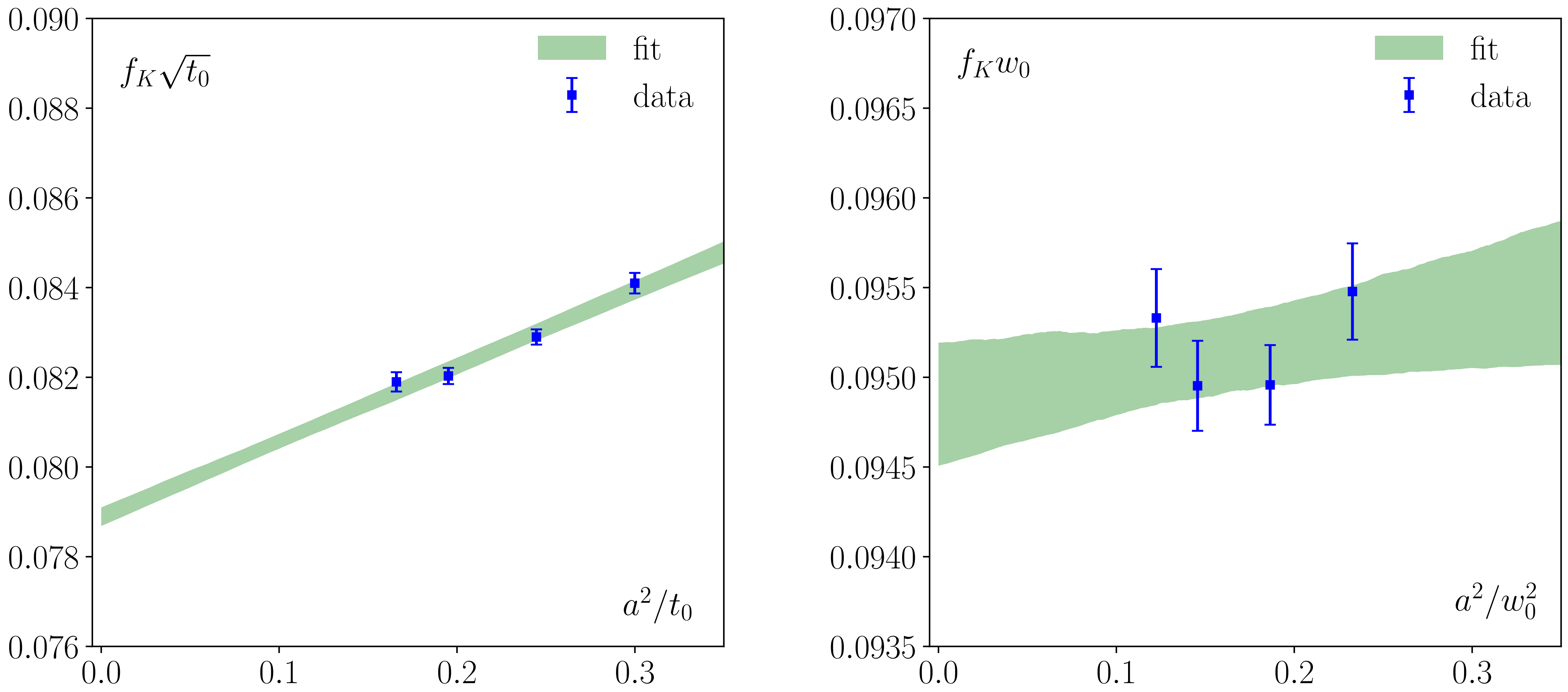



Physical scales at sub% level: $f_\eta = 128.34(85)$ MeV Davies et.al., PRD 81, 034506 (2010)

- Discretization errors linear in $a^2$
- Good control for the lattice effects

# Flow scales via kaon decay constant



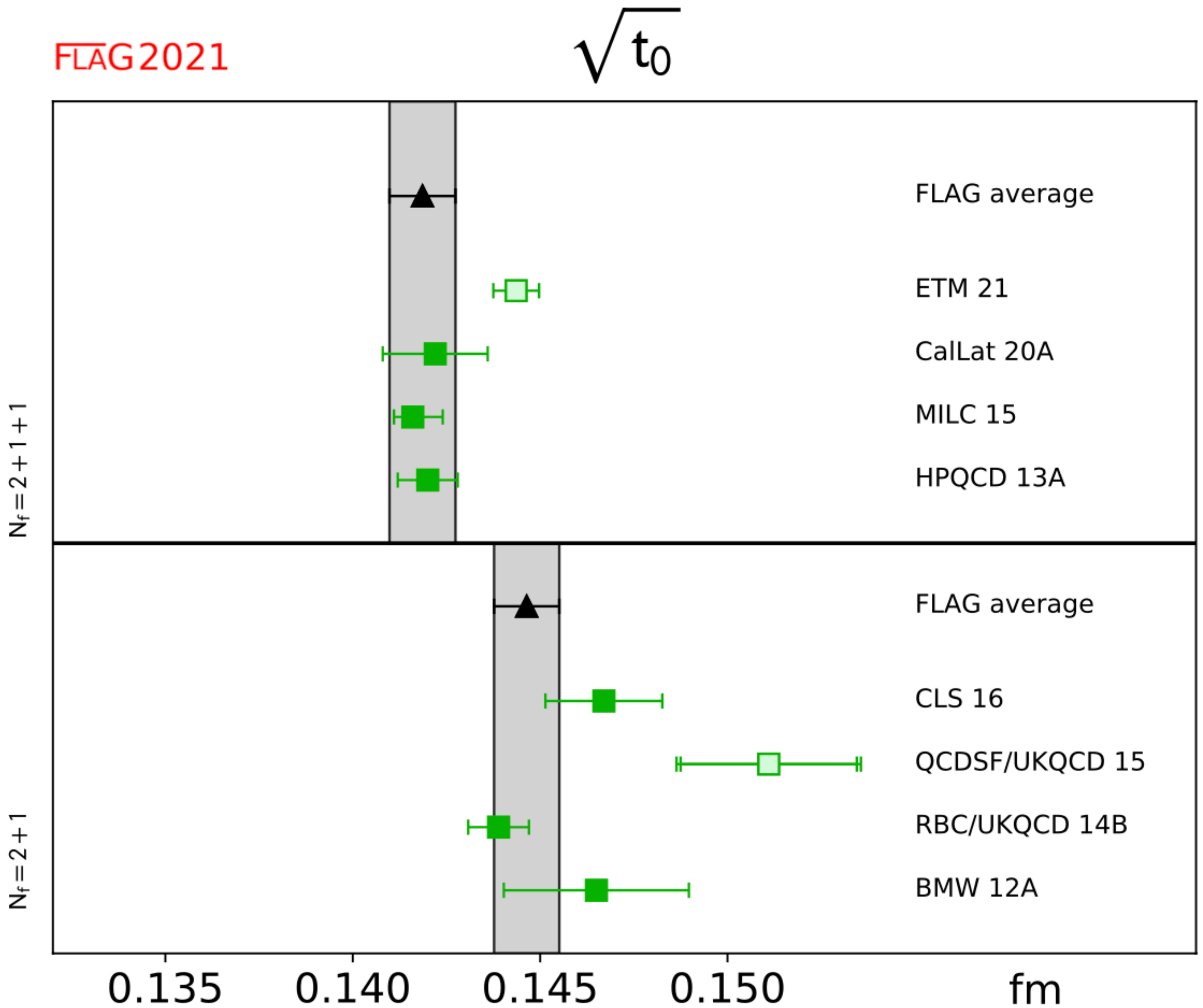

Physical scales at sub% level: $f_K = 110.10(64)$ MeV HotQCD, PRD 90, 094503 (2014)

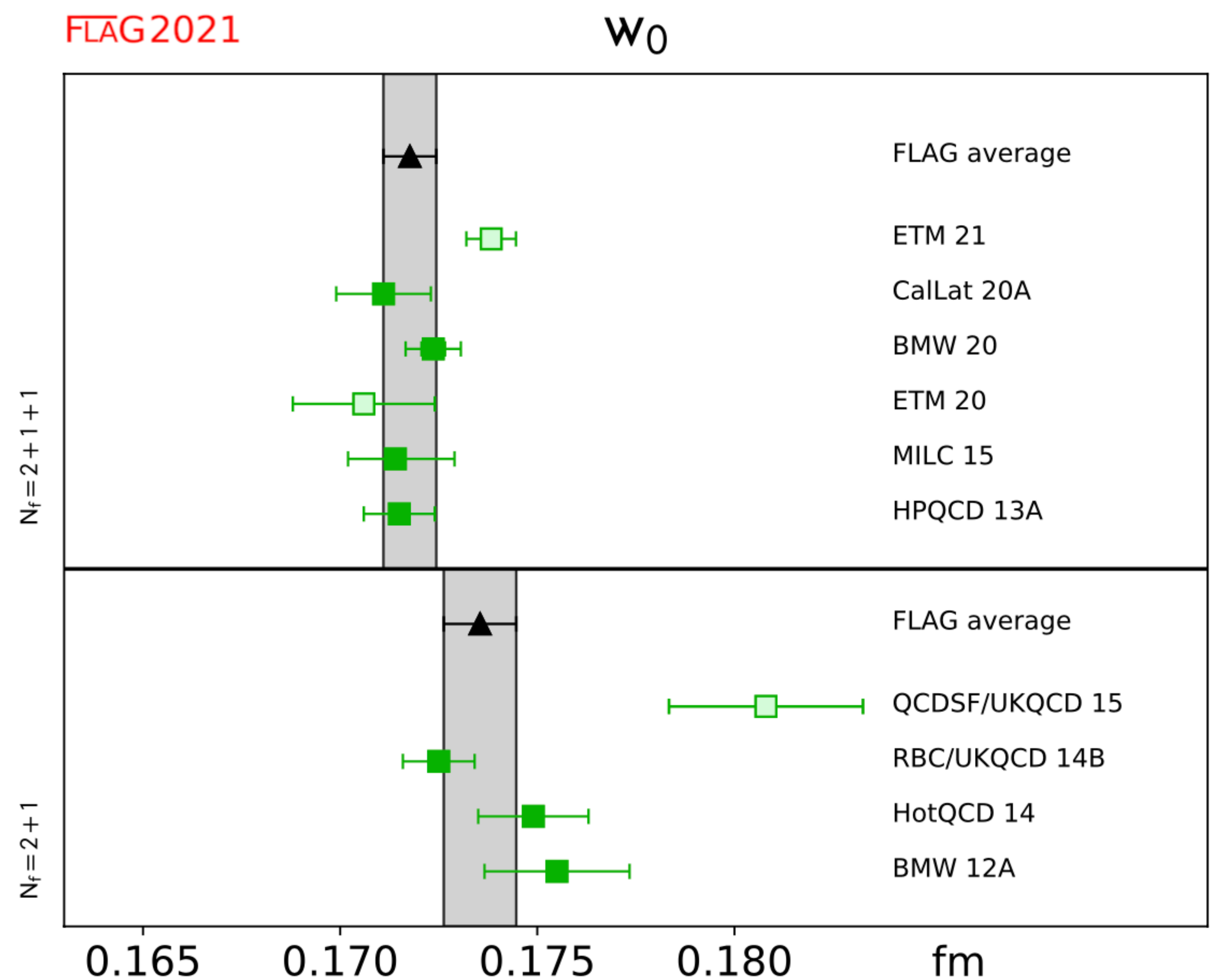
via f_eta: $\sqrt{t_0} = 0.14181(\pm 64 \pm 94)$ fm $w_0 = 0.17073(\pm 84 \pm 113)$ fm

via f_K: $\sqrt{t_0} = 0.14157(\pm 37 \pm 82)$ fm $w_0 = 0.17021(\pm 62 \pm 99)$ fm

- Consistent with results obtained via eta decay constant

# Compare flow scales with literature



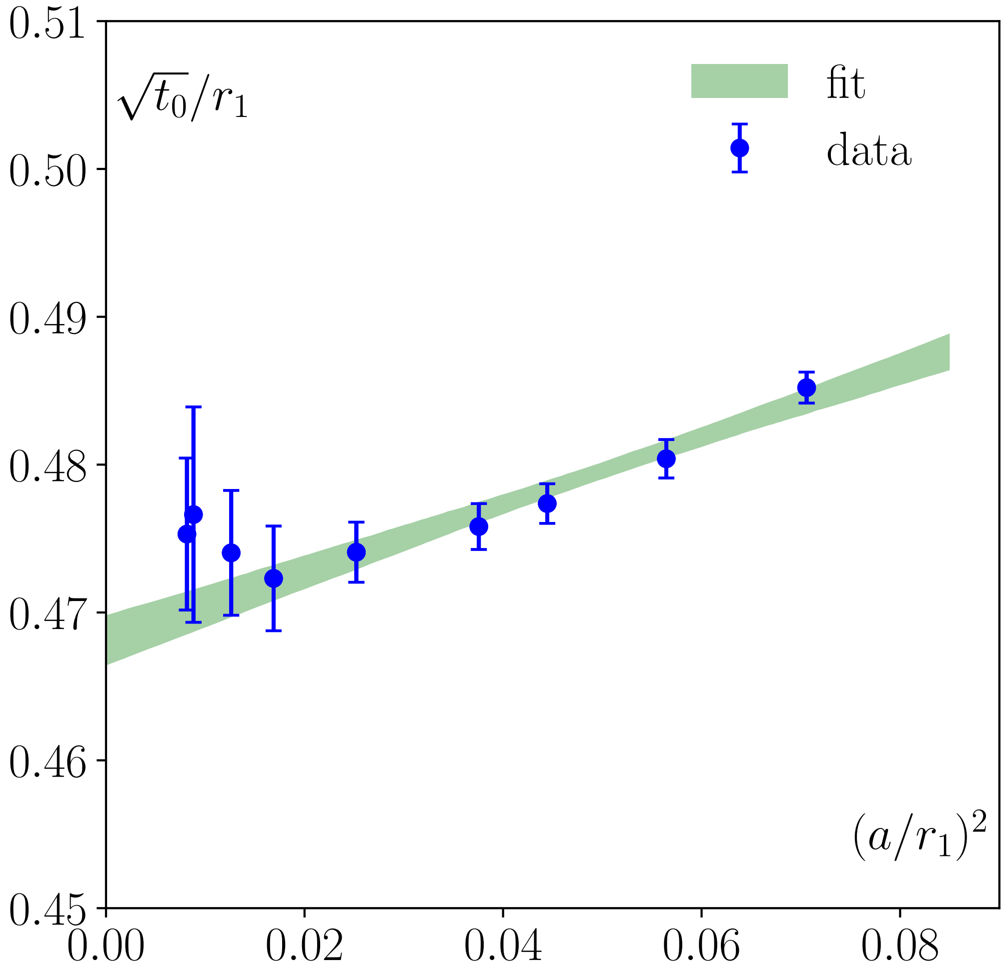




$$\sqrt{t_0} = 0.14166(134)\ \mathrm{fm}$$

$$w_0 = 0.17042(178)\ \mathrm{fm}$$

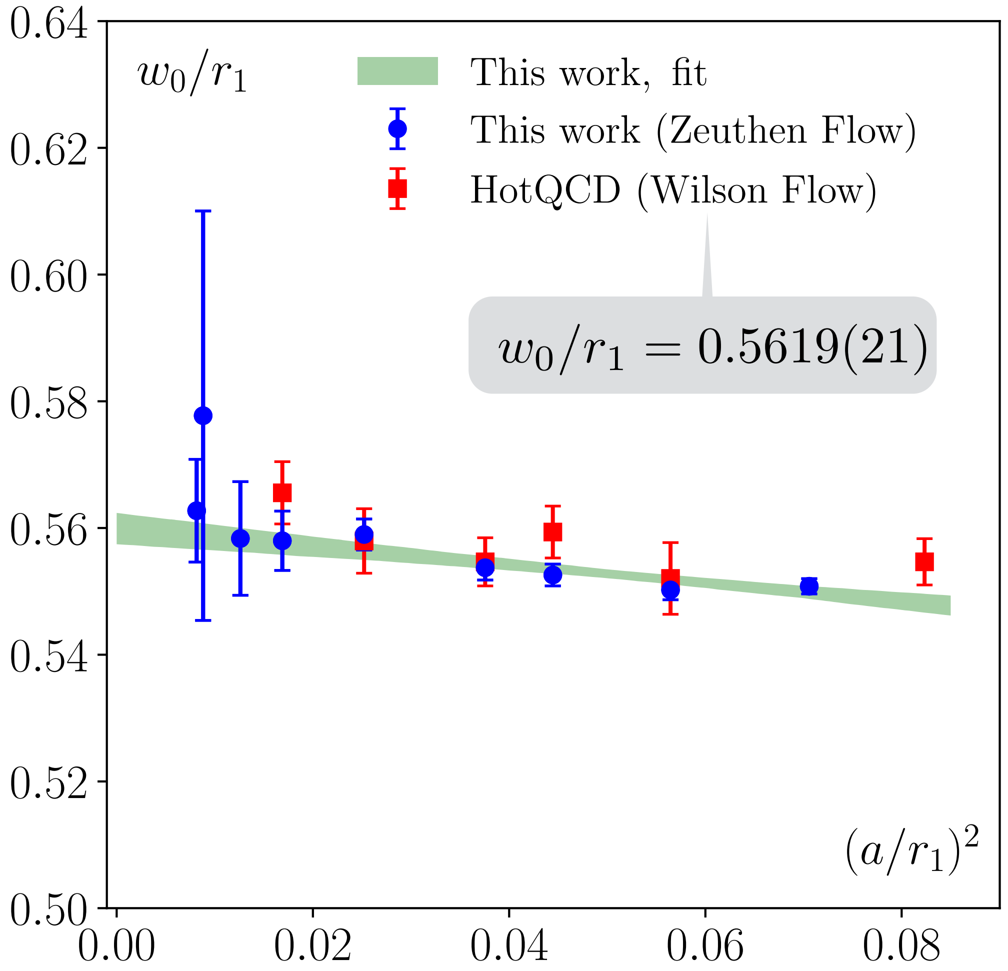
- Both flow scales are on the lower side of FLAG2021

# Potential scale $r_1$ via flow scales




$$\sqrt{t_0}/r_1 = 0.4681(17)$$

$$r_1 = 0.3026(31)\ \text{fm}$$



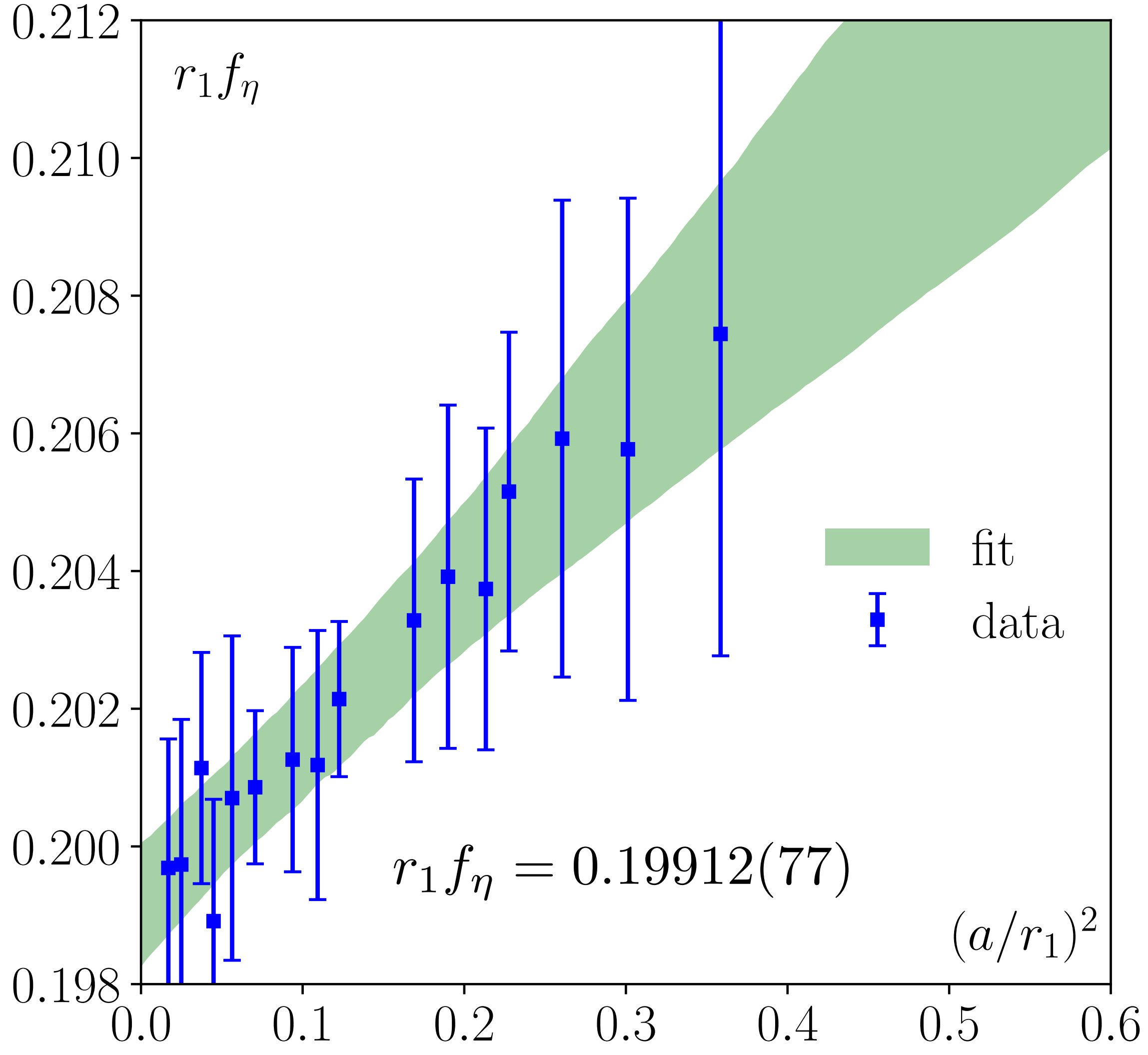

$$w_0/r_1 = 0.5599(25)$$

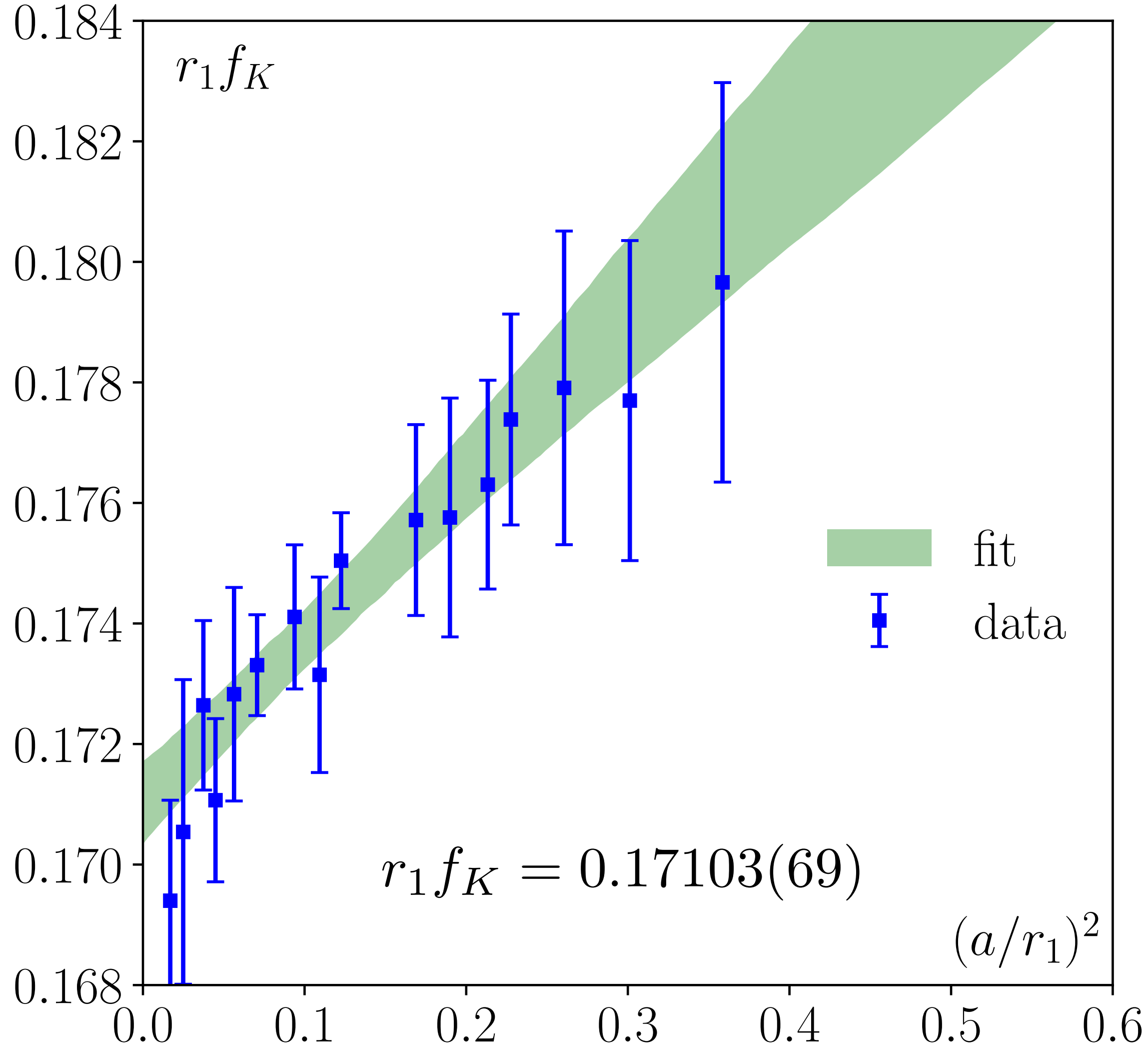
$$r_1 = 0.3044(34)\ \text{fm}$$

- Discretization errors linear in $a^2$
- Consistent $r_1$ from different flow scales
- Zeuthen flow and Wilson flow give consistent $w_0/r_1$

# Potential scale $r_1$ revisited via decay constants

Data from HotQCD, PRD 90, 094503 (2014)




via f_eta: $r_1 = 0.3062(23)$ fm

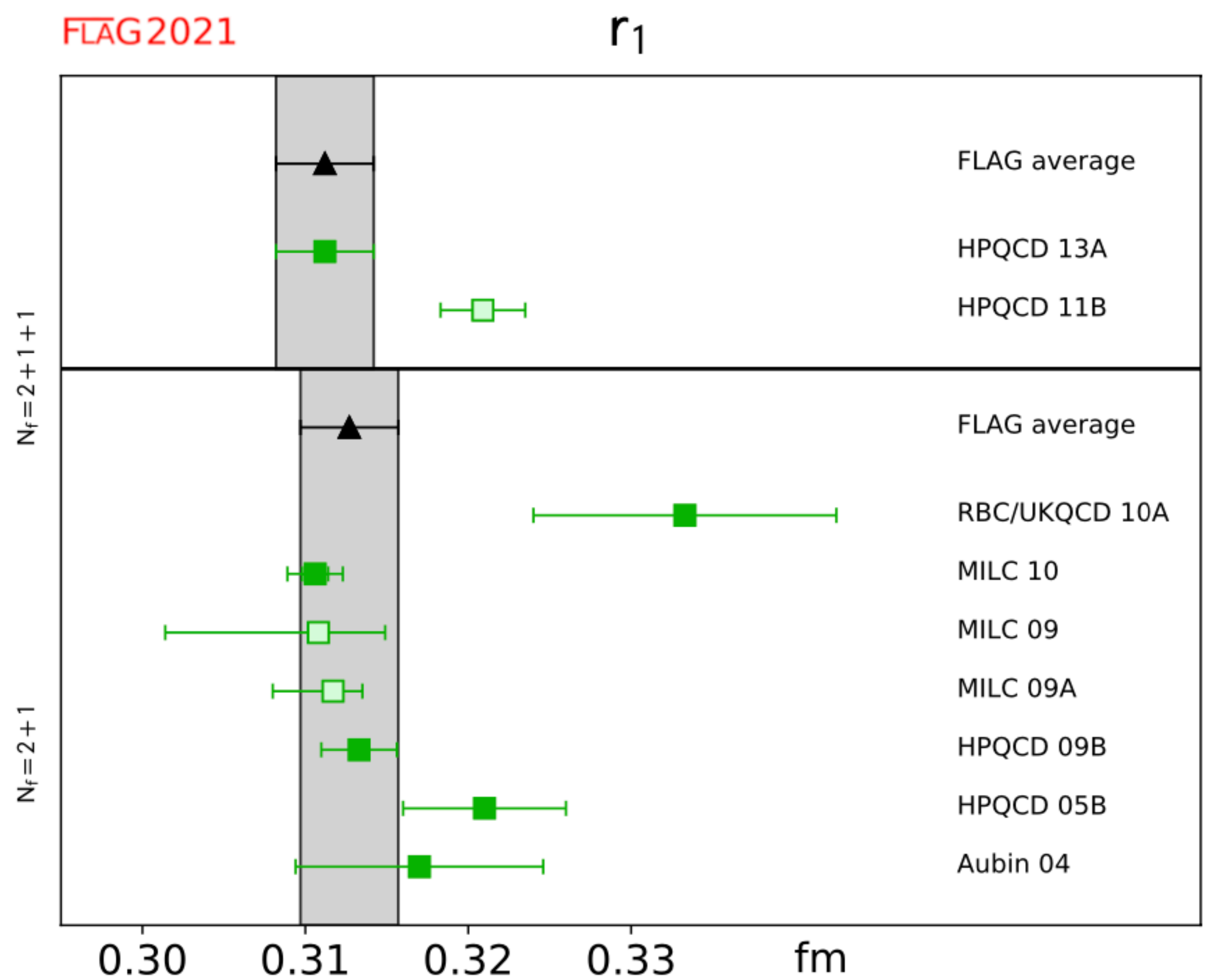
via f_K: $r_1 = 0.3065(22)$ fm

via $t_0$: $r_1 = 0.3026(31)$ fm

via $w_0$: $r_1 = 0.3044(34)$ fm

- Consistent $r_1$ from eta decay const. and kaon decay const.
- Consistent $r_1$ from decay consts. and flow scales

# Compare $r_1$ scale with literature




This work: $r_1 = 0.3052(35)$ fm

TUMQCD, PRD 107, 074503 $r_1 = 0.3037 \pm 0.0025$ fm

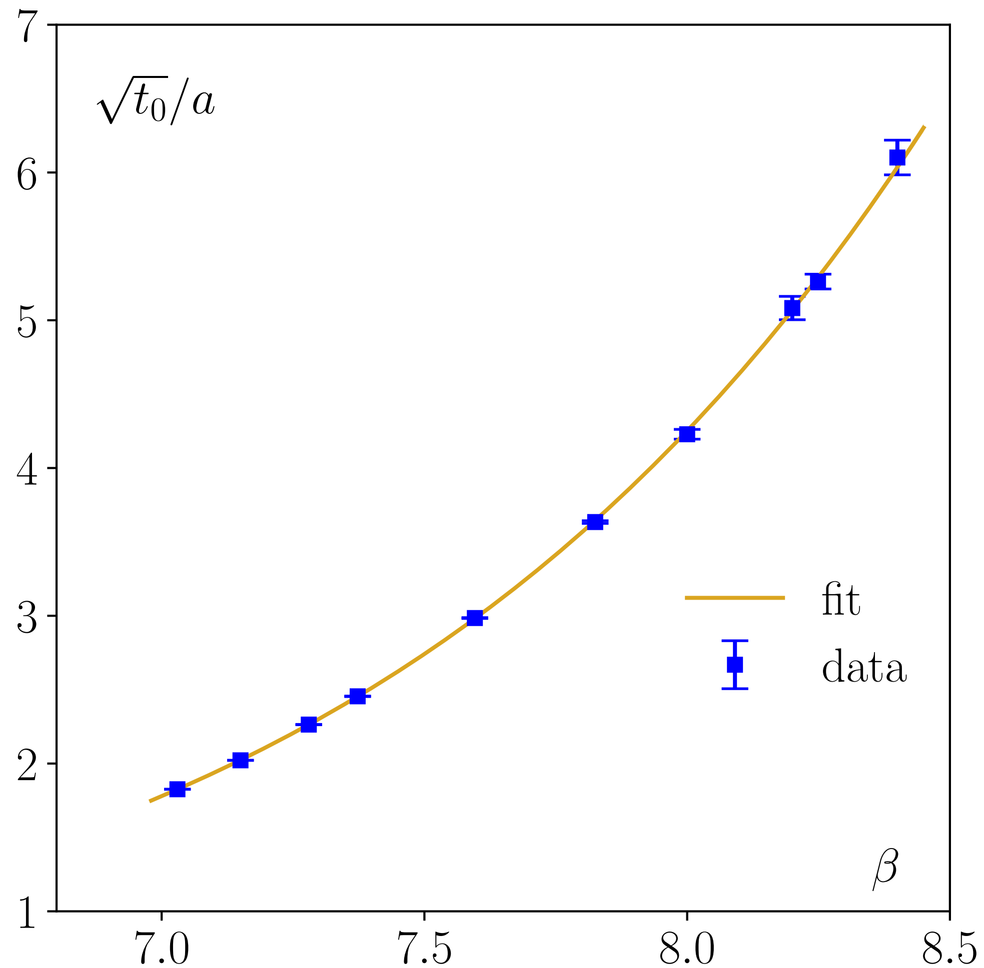
$N_f = 2 + 1 + 1$

- Slightly smaller than FLAG lower limit but consistent with TUMQCD '23 estimate

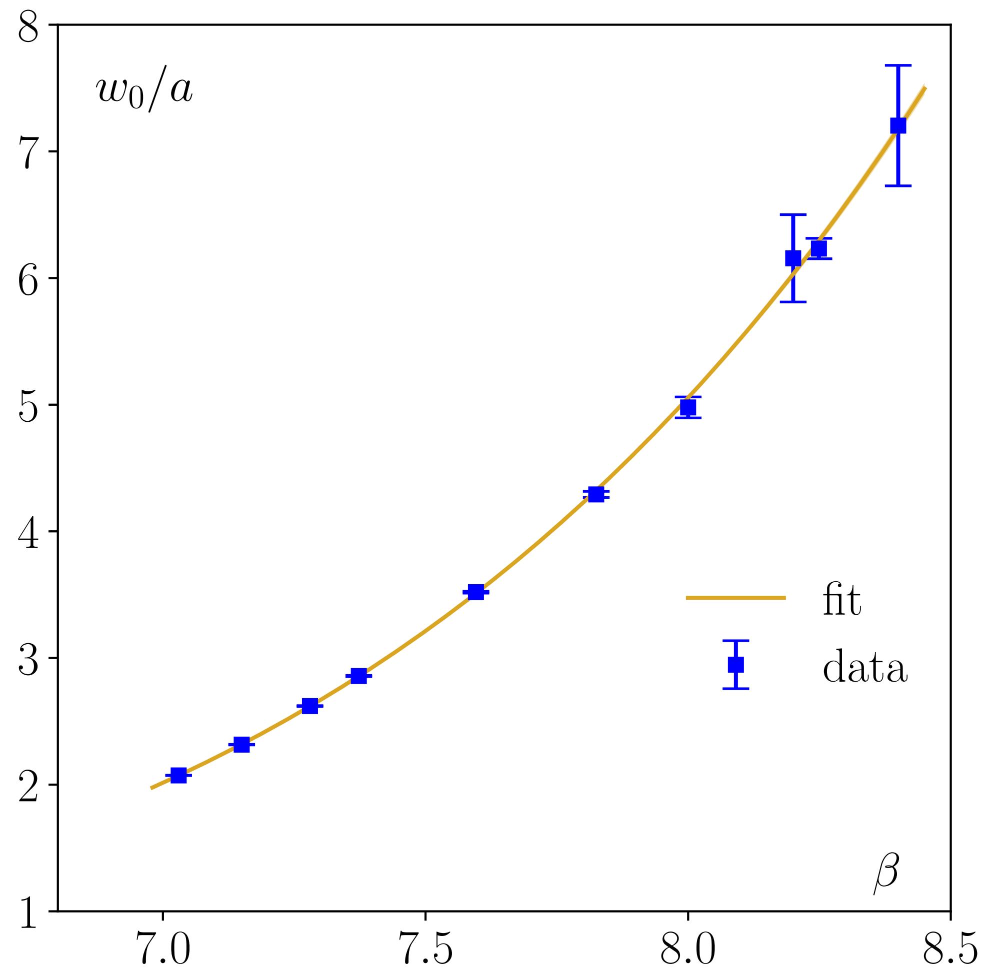
# Scale setting via flow scales

Allton-type form: $\ln\left(\frac{\sqrt{t_0}}{a}\right) = \frac{1 + d_2(10/\beta)f^2(\beta)}{c_0 f(\beta) + c_2(10/\beta)f^3(\beta)}$ $\qquad f(\beta) = \left(\frac{10b_0}{\beta}\right)^{-b_1/(2b_0^2)} \exp(-\beta/(20b_0))$

Allton, NPBP. Suppl. 53, 867 (1997)




$c_0 = 88.14(93),\ c_2 = 8057092(1828213)$
$d_2 = 81230(18771),\ \chi^2/\text{dof} = 1.9$




$c_0 = 74.72(73),\ c_2 = 1902823(435837),$
$d_2 = 19557(4967),\ \chi^2/\text{dof} = 1.7$

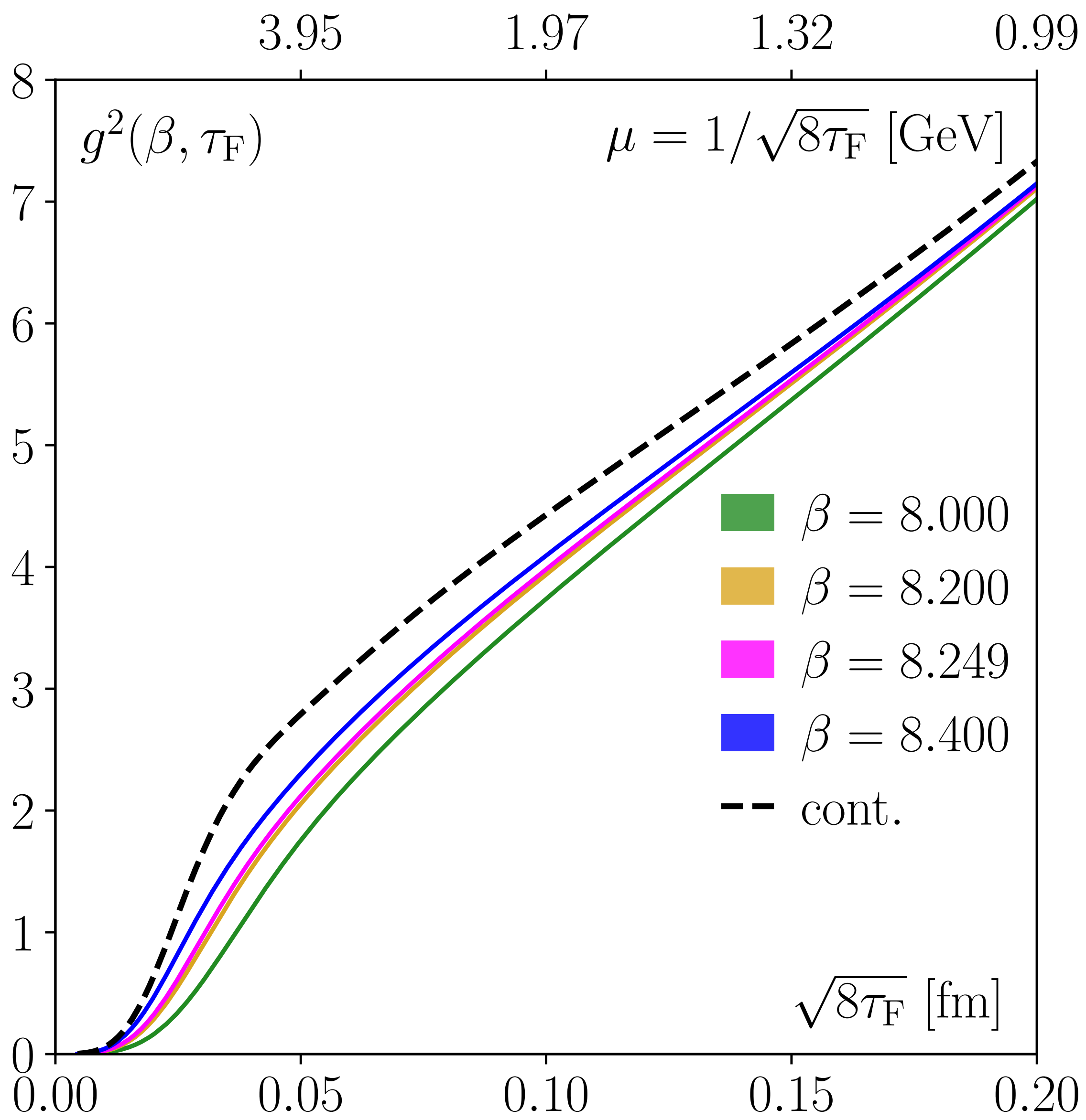
- Good description of lattice data up to fine lattice

# Coupling constant from gradient flow

NNLO matching from flow scheme to MSbar scheme

$$\alpha_{\rm flow}(\mu_F) \equiv \frac{4\pi}{3}\tau_F^2\langle E\rangle_{\tau_F} \rightarrow \alpha_{\rm flow} = \alpha_s\left(1 + k_1\alpha_s + k_2\alpha_s^2\right) \quad \text{for } \bar{\mu}^2 = \mu_F^2$$

Harlander and Neumann, JHEP 06, 161 (2016)

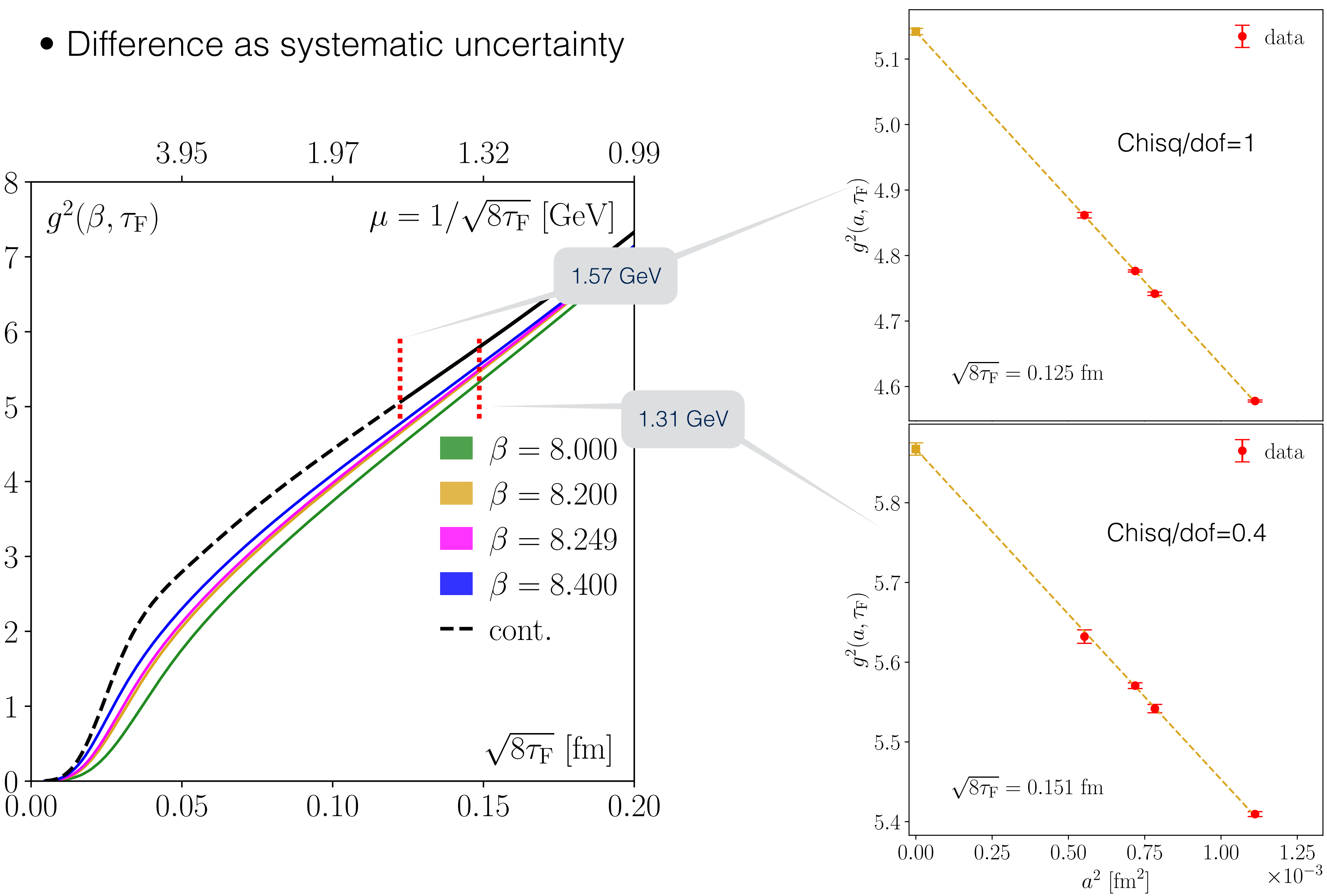



Where to do the conversion?

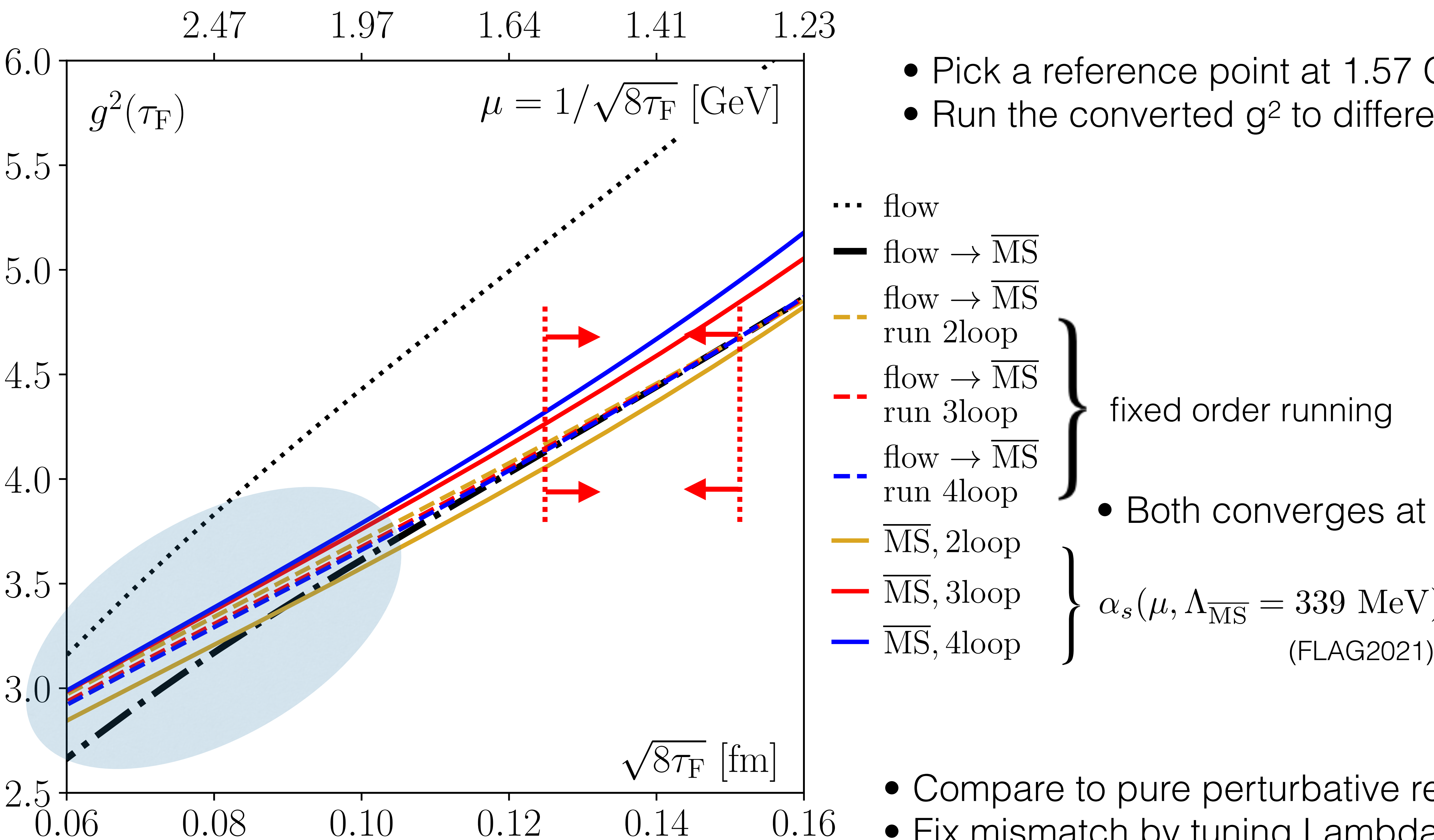
- Lattice artifacts should be under control
  - ➡ chisq/dof of cont. extrap. should be ~1
- Perturbative effects should be under control
  - ➡ Running scale should be large

# Coupling constant from gradient flow

- Difference as systematic uncertainty



# Compare to FLAG

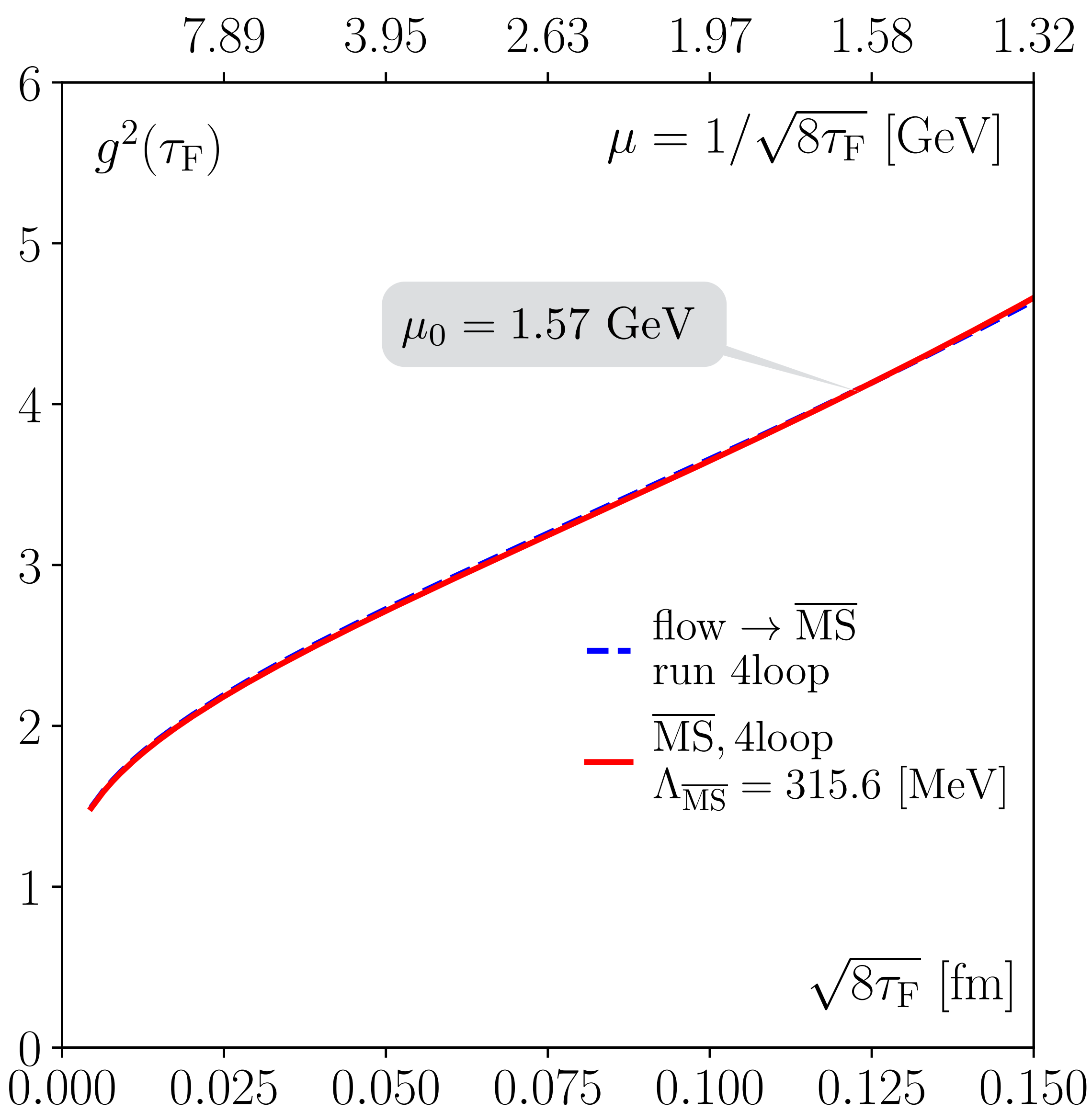



- Pick a reference point at 1.57 GeV
- Run the converted $g^2$ to different scales

- Both converges at 4th order

- Compare to pure perturbative results
- Fix mismatch by tuning Lambda_MSbar

# Solve for Lambda_MSbar/alpha_s




Error budget:

- Statistical uncertainty from (Gaussian) bootstrap
- Lattice cutoff effects by excluding the coarsest lattice at beta=8.000
- Perturbative uncertainty by adding a N3LO term with coefficient $k_3$=+-2$k_2$
- Matching range uncertainty by two scales 1.31 GeV and 1.57 GeV

$$\Lambda_{\overline{\text{MS}}} = 315.6^{+0.4+1.5+43.2+1.1}_{-0.4-1.5-26.5-1.1}\ \text{MeV}$$

Uncertainty of matching dominates

- Consistent with FLAG lattice results:

| FLAG | This work |
|---|---|
| $\Lambda^{(3)}_{\overline{\text{MS}}} = 339(12)$ MeV | $\Lambda_{\overline{\text{MS}}} = 315.6^{+46.2}_{-29.5}$ MeV |
| $\alpha^{(5)}_{\overline{\text{MS}}}(M_Z) = 0.1184(8)$ | $\alpha^{(5)}_{\overline{\text{MS}}}(M_Z) = 0.1166^{+31}_{-21}$ |

# Summary

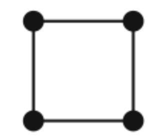
- $t_0$ and $w_0$ scale are determined on 2+1-flavor HISQ lattices
  - ✴ On the lower side of FLAG2021
- $r_1$ is determined in various ways
  - ✴ Self-consistent and consistent with TUMQCD ’23 estimate
  - ✴ slightly smaller than FLAG lower bound
- Lambda_MSbar is determined using gradient flow via matching
  - ✴ Perturbative matching dominates the uncertainty
  - ✴ Consistent with FLAG2021 within errors

# Backup: Wilson flow v.s. Zeuthen flow

Wilson flow diffusion equation:

Wilson action

$$a^2[\partial_t V_\mu(t,x)]V_\mu(t,x)^\dagger = -g_0^2 \partial_{x,\mu} S_{\mathrm{W}}[V]$$



**(a)** $\mathcal{S}_0$

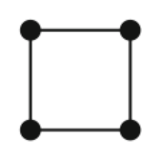
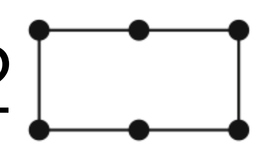
Flowed gauge links

Ramos, Saint, EPJC 76 (2016) 1, 15

Luescher-Weisz action

Zeuthen flow diffusion equation:

$$a^2\left(\partial_t V_\mu(t,x)\right)V_\mu(t,x)^\dagger = -g_0^2\left(1+\frac{a^2}{12}\nabla_\mu^*\nabla_\mu\right)\partial_{x,\mu} S_{\mathrm{LW}}[V]$$

5/3  - 1/12 

**(a)** $\mathcal{S}_0$ **(b)** $\mathcal{S}_1$

LW action eliminates O($a^2$) cutoff effects present in the Wilson action