Accurate determination of the strong coupling

ALPHA Collaboration

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In collaboration with:

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GENERALITAT VALENCIANA

The problem: Summary on a difficult multi-scae problem

$$O(Q) \overset{Q\to\infty}{\sim} \alpha_s(Q) + \sum_{n=2}^{N} c_n \alpha_s^n(Q) + \mathcal{O}(\alpha_s^{N+1}(Q)) + \mathcal{O}\left(\frac{\Lambda^p}{Q^p}\right) + \ldots$$

Difficulties in extracting α_s "solved" by using large Q

- Difficult to compute (NP physics is difficult!)
- Difficult to estimate (i.e. scale variation might fail)
- PT errors decrease very slowly (i.e. $\alpha(Q)$ runs logarithmically) $\Longrightarrow$ One needs large scales

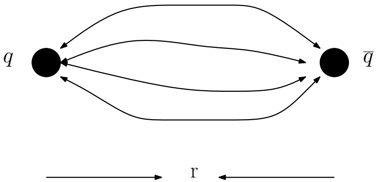
Problem in using large Q

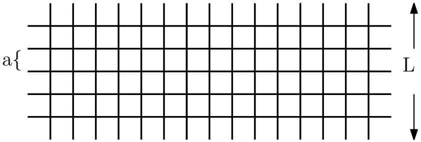
- Experimental sensitivity to α_s is small at large $Q \Longrightarrow$ precision requires a "low energy" scale
- In most lattice simulations $aQ \ll 1 \Longrightarrow Q \ll 4$ GeV

Real challenge: Have determinations with errors dominated by **statistics**

- Clear meaning of error bars!
- Keep high statistical precision. . . While removing all systematic related with PT at low energies

THE STRENGTH OF YM





$$a \ll \frac{1}{\mu_{\max}} < \frac{1}{\mu_{\min}} \ll L$$

Non-perturbative couplings

- This defines the "potential scheme"

$$\alpha_{qq}(Q) = \frac{3r^2}{4} F(r)\Big|_{Q=1/r} \overset{Q\to\infty}{\sim} \alpha_{\overline{\mathrm{MS}}}(Q) + \dots$$

- The **window problem:**

$$1/L \ll Q \ll 1/a$$

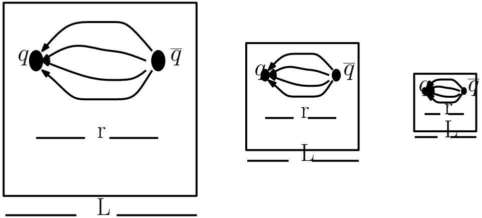
Convenient scales

i.e. the TRENTO scale

$$\alpha_{qq}(\mu_{\mathrm{TRENTO}}) = 12.34/(4\pi)$$

(**NOTE:** Many lattice scales are basically this!: $r_0, t_0, w_0, r_1, \dots$)

The solution: Finite volume renormalization schemes [Lüscher, Weisz, Wolff '91]



Fix QL = constant

- Coupling $\alpha(Q)$ depends on no other scale but L
- Small $L \Longrightarrow$ small $\alpha(L)$
- $aQ \ll 1$ easy: $L/a \sim 10 - 40$
- $1/L$ is a IR cutoff $\Rightarrow$ simulate directly $m_q = 0$

Step scaling function

- How much changes $\alpha(Q)$ if $Q \to Q/2$?

$$\sigma(u) = \alpha(Q/2)\Big|_{\alpha(Q)=u}$$

- Simply change $L/a \to 2L/a$!

We need dedicated simulations of the **femto-universe**

Example: massless running in $N_\mathrm{f} = 3$ QCD [ALPHA '17]

Gradient flow scheme [P. Fritzsch, AR '13; Phys.Rev.D 95 (2017)]

- Determine lattice version of SSF

$$\Sigma(u, L/a) = \alpha(Q/2)\Big|_{\alpha(Q)=u,\mathrm{fixed}L/a}$$

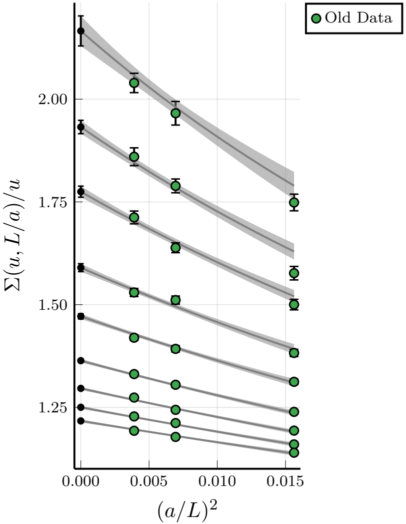
Use $8 \to 16, 12 \to 24, 16 \to 32$ at fixed (g_0, am_0)

- Continuum limit

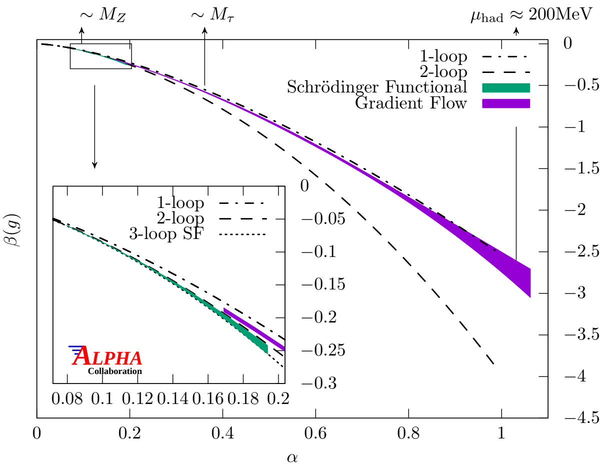
$$\sigma(u) = \lim_{a/L\to 0} \Sigma(u, L/a)\,.$$

- Use to determine β-function

$$\log(2) = \int_u^{\sigma(u)} \frac{\mathrm{d}x}{\beta(x)}\,.$$



Example: massless running in $N_{\rm f} = 3$ QCD [ALPHA '17]



- We can determine ratio of scales

$$\left.\begin{array}{ll}\alpha_{\rm GF}(\mu_{\rm had}) & = 0.9 \\ \alpha_{\rm GF}(\mu_{\rm dec}) & = 0.314\end{array}\right\} \Longrightarrow \frac{\mu_{\rm had}}{\mu_{\rm dec}} = 3.988(45).$$

- We can match schemes

$$\alpha_{\rm SF}(\mu_0) = 0.16 \Longrightarrow \alpha_{\rm GF}(\mu_0/2) = 0.2127(5)$$

- Using PT for $Q \in [100\,\text{GeV}, \infty]$ we get α_s

$$\frac{\Lambda^{(3)}_{\overline{\rm MS}}}{\mu_{\rm had}} = .$$

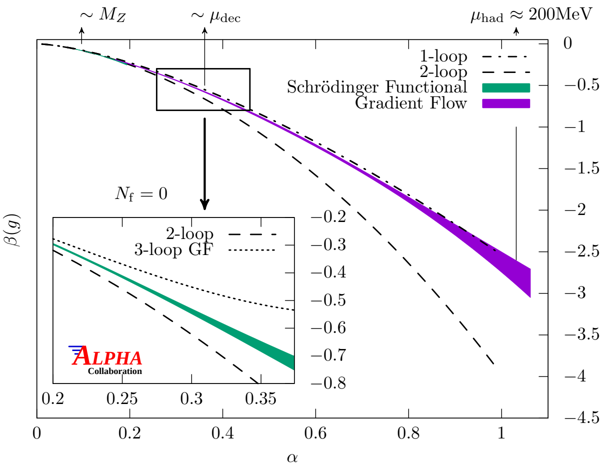
- Most of the uncertainty:

NP running from μ_0 − 100 GeV

How to improve?

1. Better ways to reduce statistical uncertainties than using more machines?
2. α_s relying only on one computation/method?

Using heavy quarks and decoupling



- Match non-perturbatively $N_f = 3$ QCD with $N_f = 0$ using **three** heavy quarks $M \gg \Lambda$

$$\alpha_{\mathrm{GF}}(\mu_{\mathrm{dec}}) = 0.314$$

$$\alpha_{\mathrm{GF}}(\mu_{\mathrm{dec}}, M) \underset{M\to\infty}{\sim} \alpha^{(0)}_{\mathrm{GF}}(\mu_{\mathrm{dec}})$$

- Get $\Lambda^{(0)}_{\overline{\mathrm{MS}}}$ in units of μ_{dec}

$$\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} = \dots$$

- Using PT to cross the **three** quark thresholds

$$\frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} = \lim_{M\to\infty} \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}} \times P(M/\Lambda)} .$$

- Most of the uncertainty:

Pure gauge

3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

Alice uses fundamental theory

$$S_{\mathrm{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{3} \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\mathrm{eff}}[A_\mu] = -\frac{1}{2g_{\mathrm{eff}}^2} \int \mathrm{d}^4x \,\{\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})\} + \frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x \, \mathcal{L}_k^{(6)} + \ldots$$

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Decoupling

- Dimensionless "low energy quantities" $\sqrt{t_0}/r_0, w_0/\sqrt{8t_0}, r_0/w_0, \ldots$ from effective theory

$$\frac{\mu_1^{\text{fund}}(M)}{\mu_2^{\text{fund}}(M)} = \frac{\mu_1^{\text{eff}}}{\mu_2^{\text{eff}}} + \mathcal{O}\left(\frac{\mu^2}{M^2}\right)$$

Renormalization in 3M: Bob determines the strong coupling

$$\frac{\Lambda_s}{\mu} = \left[b_0 \bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}.$$

- Determine non-perturbatively the β-function in the effective ($N_f = 0$) theory.
- Integral up to $\bar{g}^{(0)}(\mu'_{\rm dec}) =$ value gives (exactly known):

$$\frac{\Lambda^{(0)}_{\overline{\rm MS}}}{\mu'_{\rm dec}} = \frac{\Lambda^{(0)}_s}{\mu'_{\rm dec}} \times \frac{\Lambda^{(0)}_{\overline{\rm MS}}}{\Lambda^{(0)}_s}$$

- Match across quark threshold to convert to $\Lambda^{(3)}$ (using perturbation theory)

$$\frac{\Lambda^{(3)}_{\overline{\rm MS}}}{\mu'_{\rm dec}} = \frac{\Lambda^{(0)}_{\overline{\rm MS}}}{\mu'_{\rm dec}} \times \frac{1}{P(\Lambda/M)}.$$

But what is $\mu'_{\rm dec}$? What about Λ/M ?

Bob **needs** input from the "real" world (and help from Alice!)

Bob and Alice determination of the strong coupling

- Alice fixes one scale in a massive scheme:

$$\alpha(\mu_{\mathrm{dec}}, M) = \text{value}\,.$$

- Decoupling: since $M \gg \Lambda$

$$\alpha(\mu_{\mathrm{dec}}, M) = \alpha^{(0)}(\mu'_{\mathrm{dec}}) + \mathcal{O}(1/M^2) \Longleftrightarrow \mu_{\mathrm{dec}} = \mu'_{\mathrm{dec}} + \mathcal{O}(1/M^2)$$

- Therefore

$$\frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu'_{\mathrm{dec}}} \times \frac{1}{P\left(\Lambda^{(3)}_{\overline{\mathrm{MS}}}/M\right)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu^2_{\mathrm{dec}}}{M^2}\right)$$

is an implicit equation for $\Lambda^{(3)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}}$ (**note:** $\Lambda^{(3)}_{\overline{\mathrm{MS}}}/M = \Lambda^{(3)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}} \times \mu_{\mathrm{dec}}/M$)

- We need:
  - Running in pure gauge: $\Lambda^{(0)}/\mu'_{\mathrm{dec}}$
  - A scale in a world with degenerate massive quarks: μ_{dec} (determined by Alice!)

We do not live in $3M$, but we can simulate it!

OUR SETUP: CHOICES OPTIMIZED TO BE ABLE TO SIMULATE HEAVY QUARKS

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_{\mathrm{dec}}} = \frac{\Lambda_{\overline{\mathrm{MS}}}^{(0)}}{\mu'_{\mathrm{dec}}} \times \frac{1}{P\left(z\Lambda_{\overline{\mathrm{MS}}}^{(3)}/\mu_{\mathrm{dec}}\right)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\mathrm{dec}}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\mathrm{dec}}^2}{M^2}\right)$$

- Work in finite volume $T \times L^3$ with Dirichlet bcs. in time (SF). ($\mu \sim 1/L$): "Only" two scales.
- Use Gradient Flow couplings

$$\bar{g}^2(\mu) = \mathcal{N}^{-1}(c, a/L)\, t^2 \langle E(t) \rangle \Big|_{\mu^{-1}=\sqrt{8t}=cL}.$$

- Fix $\bar{g}^2(\mu_{\mathrm{dec}})|\Big|_{N_\mathrm{f}=3, M=0, T=L} = 3.95$. This defines μ_{dec}
- Small volume $\Longrightarrow$ We can simulate heavy quarks (i.e. $a \sim 30 - 50\ \mathrm{GeV}^{-1}$)
- Matching condition ($\{N_\mathrm{f} = 3, M\} \leftrightarrow \{N_\mathrm{f} = 0\}$) between massive scheme and effective theory

$$\bar{g}^2(\mu_{\mathrm{dec}}(M))\Big|_{N_\mathrm{f}=3, M, T=2L} = \bar{g}^2(\mu_{\mathrm{dec}})\Big|_{N_\mathrm{f}=0, T=2L}.$$

  Matching: QCD in a finite volume!
- Convenient variable: $z = M/\mu_{\mathrm{dec}}$

Our setup: Choices optimized to be able to simulate heavy quarks

$$\frac{\Lambda^{(3)}_{\overline{\mathrm{MS}}}}{\mu_{\mathrm{dec}}} = \frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\mu'_{\mathrm{dec}}} \times \frac{1}{P\left(z\Lambda^{(3)}_{\overline{\mathrm{MS}}}/\mu_{\mathrm{dec}}\right)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\mathrm{dec}}}{M}\right) + \mathcal{O}\left(\frac{\mu^2_{\mathrm{dec}}}{M^2}\right)$$

We only need to fill in a table!

$M/\mu_{\mathrm{dec}}(M)$	$\bar{g}_z^2$	$\Lambda^{(0)}/\mu_{\mathrm{ref}}$	$\Lambda^{(3)}_{\mathrm{eff}}$
1.972	-	-	-
4	-	-	-
6	-	-	-
8	-	-	-
10	-	-	-
12	-	-	-

- Difficult continuum extrapolations to determine $\bar{g}_z^2 = \bar{g}^2(\mu_{\mathrm{dec}}(M))\Big|_{N_{\mathrm{f}}=3,M,T=2L}$
- Use combined Heavy-Quark / Symanzik effective theories.

The continuum extrapolation of massive couplings

Quadratic dependence on lattice spacing (a) via $a\mu_{\rm dec}$ and aM

For large enough masses, effective theory applies:

$$\bar{g}^2(z_i, a) = C_i + p_1[\alpha_(a^{-1})]^{\hat{\Gamma}}(a\mu_{\rm dec})^2 + p_2[\alpha_(a^{-1})]^{\hat{\Gamma}'}(aM_i)^2 \,.$$

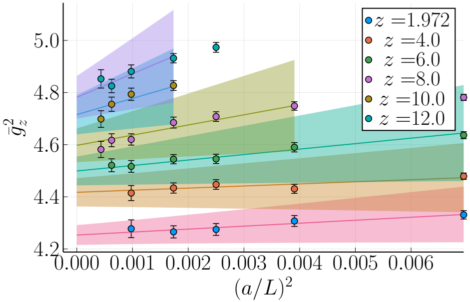
- Continuum values (our target quantity)
- Mass independent cutoff effects
- Mass dependent cutoff effects
- Loop corrections in effective theory: $-1 \leq \hat{\Gamma} \leq 1$ and $-1/9 \leq \Gamma' \leq 1$

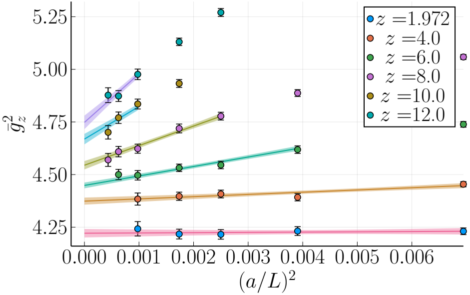
Additional assumptions about $\mathcal{O}(aM)$ effects

Partial knowledge based on PT: **Propagate difference between last known orders as additional uncertainty**

- Schrödinger functional boundaries: Small (negligible to our level of precision). Explicit computation.
- Quark mass improvement: $b_m, b_A, b_P, \ldots$. **Very** small effect.
- Improved bare coupling: b_g. Large effect at large masses (comparable to statistical uncertainties). Decreases as $aM \to 0$.

The continuum extrapolation of massive couplings





Previous determination [ALPHA '23]

- Most error: estimate of $b_g - b_g^{1-\text{loop}}$
- This is a systematic!
- Still: Error in $\bar{g}^2(\mu, M)$ **subdominant**

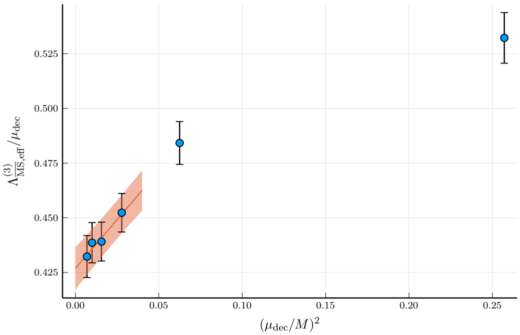
NP determination of b_g [ALPHA '24]

- Precise of continuum values
- **Subdominant** effect in α_s
- But removes largest systematic!

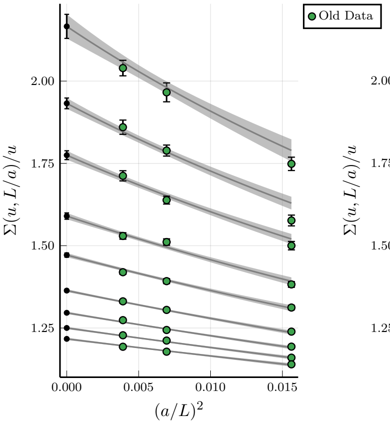
The $M \to \infty$ limit

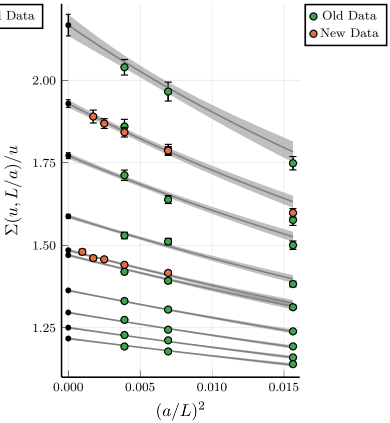
$$\Lambda_{\text{eff}}^{(3)} = \Lambda^{(3)} + \frac{B}{z^2}[\alpha(m_\star)]^{\Gamma_m}$$

- Approach $M \to \infty$ with various Γ_m
- Discard $z = 1.972, 4$



Improved running at low scales





$$\frac{\mu_{\rm had}}{\mu_{\rm dec}} = 3.988(45) \Longrightarrow \frac{\mu_{\rm had}}{\mu_{\rm dec}} = 3.998(30)\,.$$

Conversion to physical units

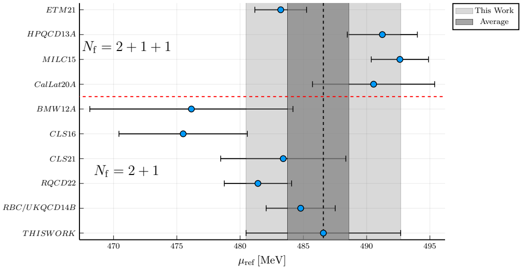
Determination of μ_{had}

- Use as intermediate scale $\mu_{\mathrm{ref}} = 1/\sqrt{t_0^\star}$ in fm
- Use all scales entering FLAG average
- Use conservative average

$$\frac{1}{\sqrt{t_0^\star}} = 486.6(6.1)\,, \quad \mathrm{MeV}$$

$$\sqrt{t_0^\star} \times \mu_{\mathrm{had}} = 0.4119(31)\,,$$

$$\mu_{\mathrm{had}} = 200.4(2.9) \quad \mathrm{MeV}$$



Serious discrepancies between individual computation

- **Subdominant** in α_s but crucial for LQCD (i.e. $g-2$)
- Needs serious discussion/clarification

Final results for α_s

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = \frac{1}{\sqrt{t_0^\star}} \times (\sqrt{t_0^\star} \times \mu_{\mathrm{dec}}) \times \frac{\Lambda_{\overline{\mathrm{MS}}}^{(3)}}{\mu_{\mathrm{dec}}}$$

$$\begin{aligned}
\text{QCD:} \quad & \sqrt{t_0^\star} \times \Lambda_{\overline{\mathrm{MS}}}^{(3)} = 0.712(22) \\
\text{Decoupling:} \quad & \sqrt{t_0^\star} \times \Lambda_{\overline{\mathrm{MS}}}^{(3)} = 0.703(18) \\
\text{Average:} \quad & \sqrt{t_0^\star} \times \Lambda_{\overline{\mathrm{MS}}}^{(3)} = 0.706(15)\,.
\end{aligned}$$

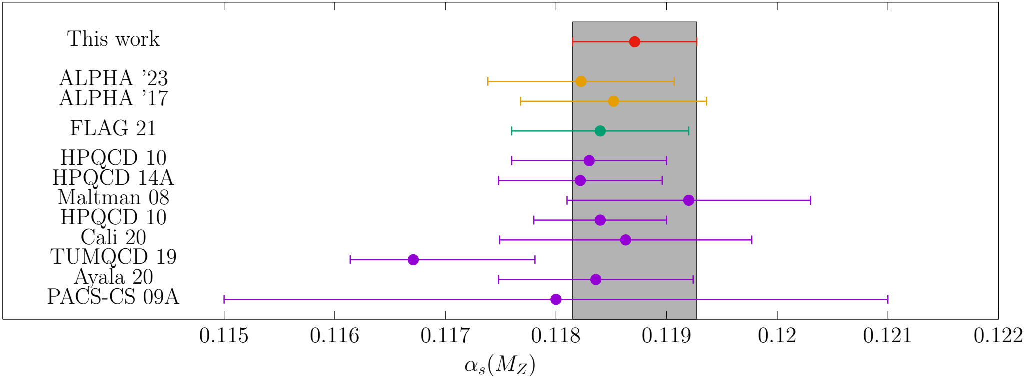
- Result in physical units (PRELIMINARY)

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = 343.7(83)$$

- Crossing c, b thresholds

$$\alpha_s(m_Z) = 0.11871(56)\,, \quad [0.47\%]\,.$$

Final results for α_s



Conclusions and discussion

Precise and accurate determination of α_s

- Non perturbative running from 200 MeV to m_z
- "High" energy running (2 GeV - m_Z) done in two independent ways:
  - QCD
  - Pure gauge

  Decoupling used to match theories
- Average result conservative and precise

$$\alpha_s(m_Z) = 0.11871(56)\,, \quad [0.47\%]\,.$$

- Most error from non-perturbative running 4 GeV - m_Z
- Errors dominated by statistics: meaningful σ!
- Improvements?
  - Better pure gauge running
  - Better scale
  - Isospin breaking corrections if one wants serious improvement

Conclusions and discussion

MANY THANKS