On the moments method

based on L.Chimirri, N.Husung, RS: arXiv: 2211.15750
thesis L. Chimirri: https://edoc.hu-berlin.de/handle/18452/28451

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Integrated correlation functions

- Heavy quark moments for the determination of α_s, in particular:

  time slice correlator:

$$G(x_0, M) = \int \mathrm{d}^3\mathbf{x}\, \langle P^{\mathrm{RGI}}(x)\overline{P}^{\mathrm{RGI}}(0)\rangle\,, \quad P^{\mathrm{RGI}} = Z^{\mathrm{RGI}}\bar{c}\gamma_5 c'$$

  4th moment:

$$M_4(M) = \int_{-\infty}^{\infty} \mathrm{d}t\, t^4\, G(t, M) \qquad M = M_c = M_c' = \text{RGI mass}$$

[Bochkharev, DeForcrand]

  dimensionless, normalized

$$R_4(M) = \frac{M^2 M_4(M)}{M^2 M_4(M)\big|_{g=0}} = 1 + \sum_{k=1}^{3} c_k\, \alpha_{\overline{\mathrm{MS}}}^k(m_\star) + \text{unknown}$$

- large mass: perturbative, determine $\alpha_{\overline{\mathrm{MS}}} \rightarrow \Lambda_{\overline{\mathrm{MS}}}$ [HPQCD+Karlsruhe group, …]

Integrated correlation functions

- but: window problem (large scale needs very small lattice spacing)
- and log-enhanced discretisation errors

from small t :
$$\int_0^\epsilon \mathrm{d}t\, t^4\, G(t,M) \sim \int_0^\epsilon \mathrm{d}t\, t\, [\bar{g}^2(1/t)]^\eta \to a \sum_t \ldots$$

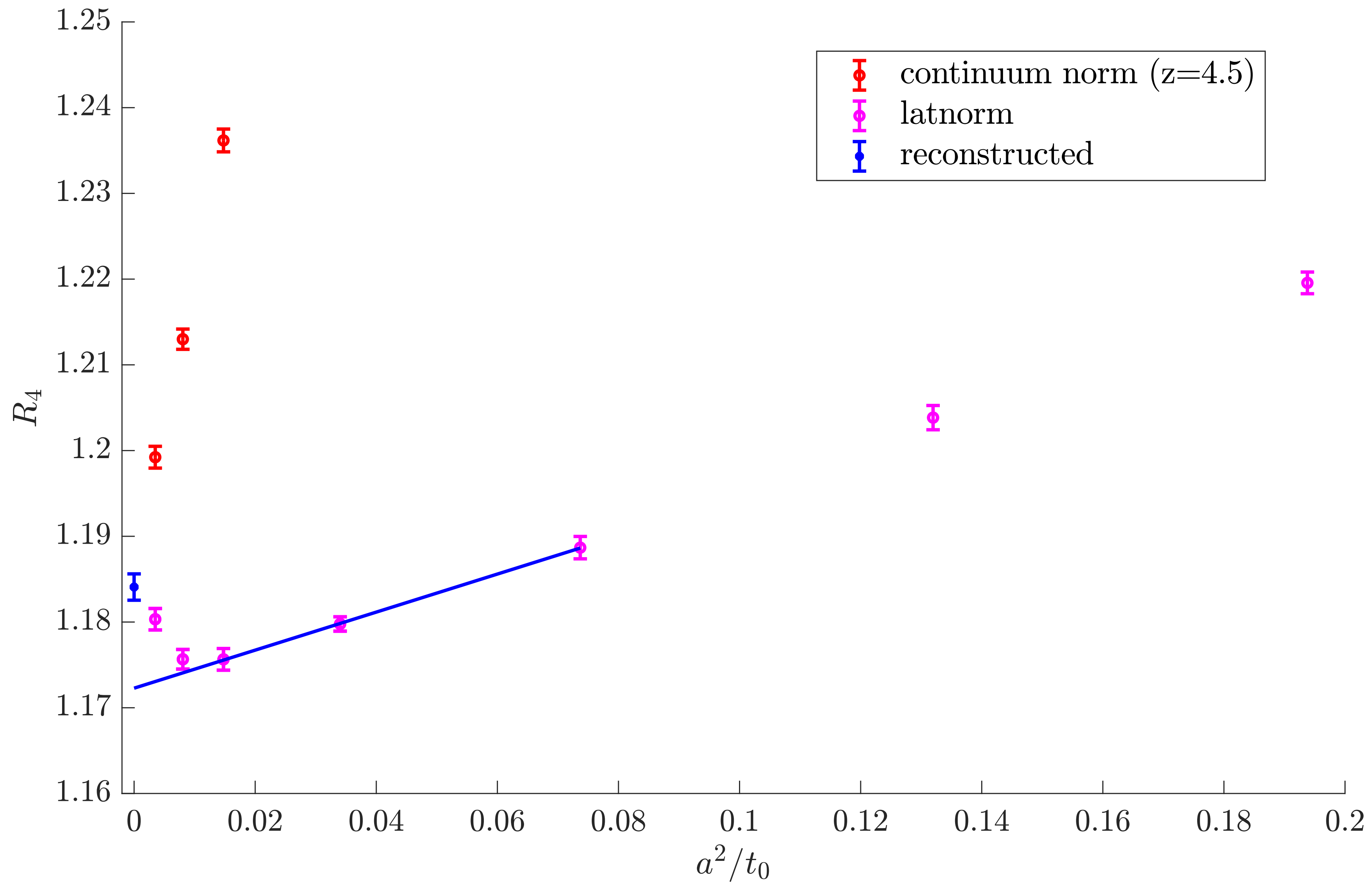
Numerical results from

- quenched -> no numerical estimate of $\alpha_s(M_z)$
- 2fm x 5fm
- open BC (no topology freezing)
- tmQCD at maximal twist + NP clover
- lattice spacings

$$a = 0.01\ \mathrm{fm} \times 2^{n/2}, n = 0 \ldots 6 : 0.01\ \mathrm{fm}\ \ldots\ 0.08\ \mathrm{fm}$$

[Husung, Krah, Koren, S. 2018]

The problem

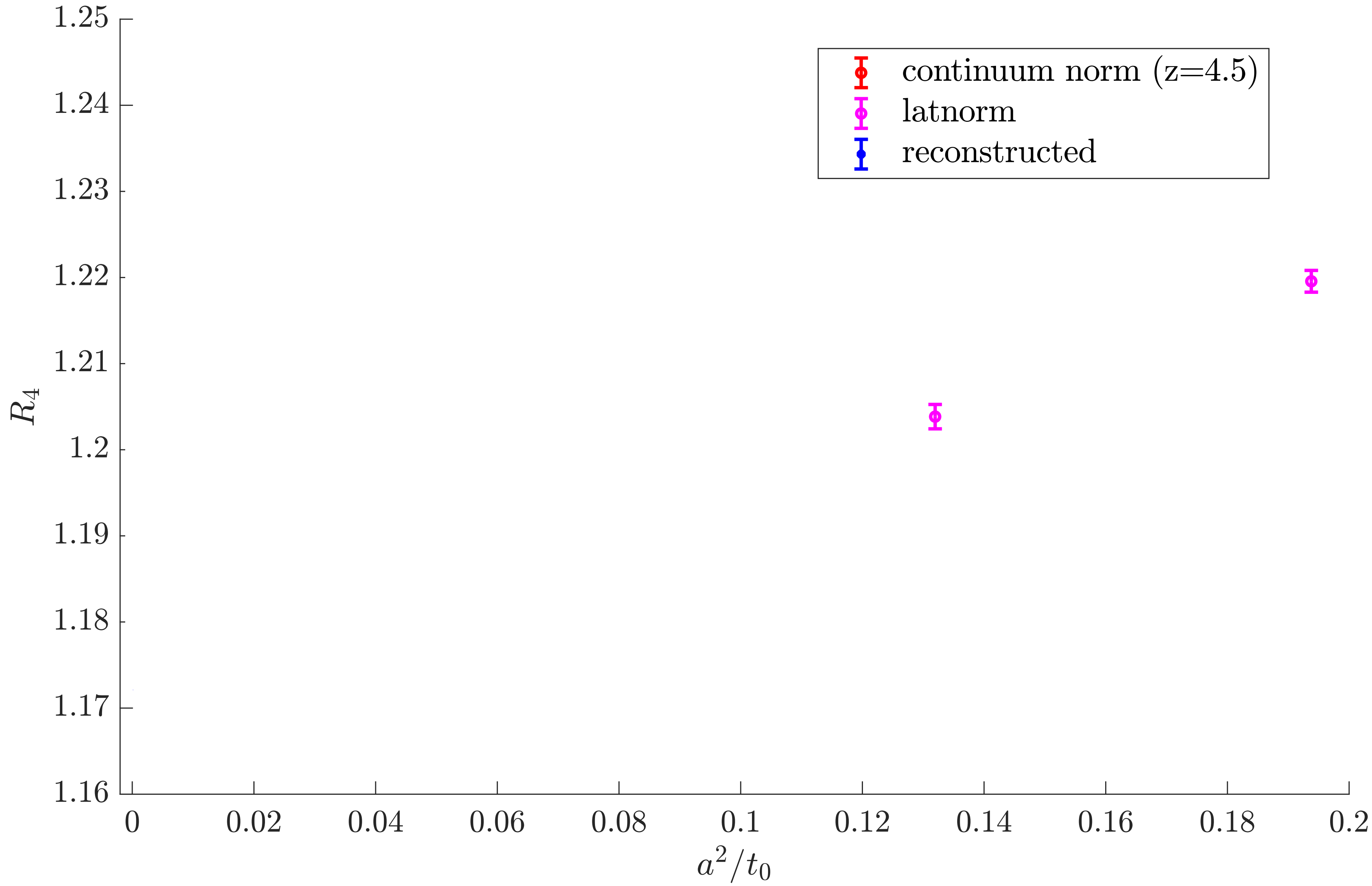


1.25
1.24
1.23
1.22
1.21
1.2
1.19
1.18
1.17
1.16
R_4
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
a^2/t_0
continuum norm (z=4.5)
latnorm
reconstructed

The problem

lattice normalized:

$$R_4^{\rm latnorm}(M) = \frac{M^2 M_4(M)}{M^2 M_4(M)\Big|_{g=0}^{aM\neq 0}}$$



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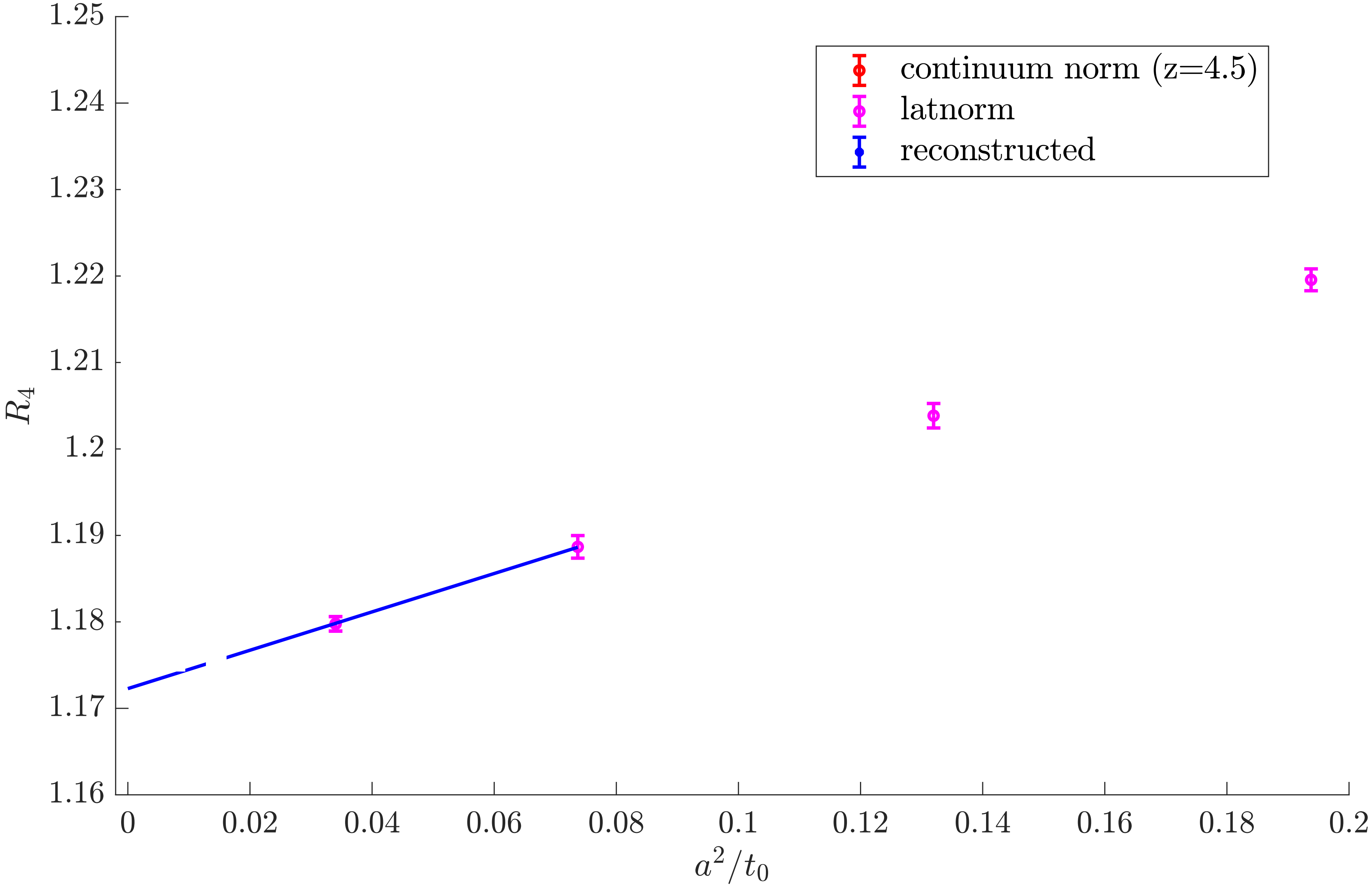

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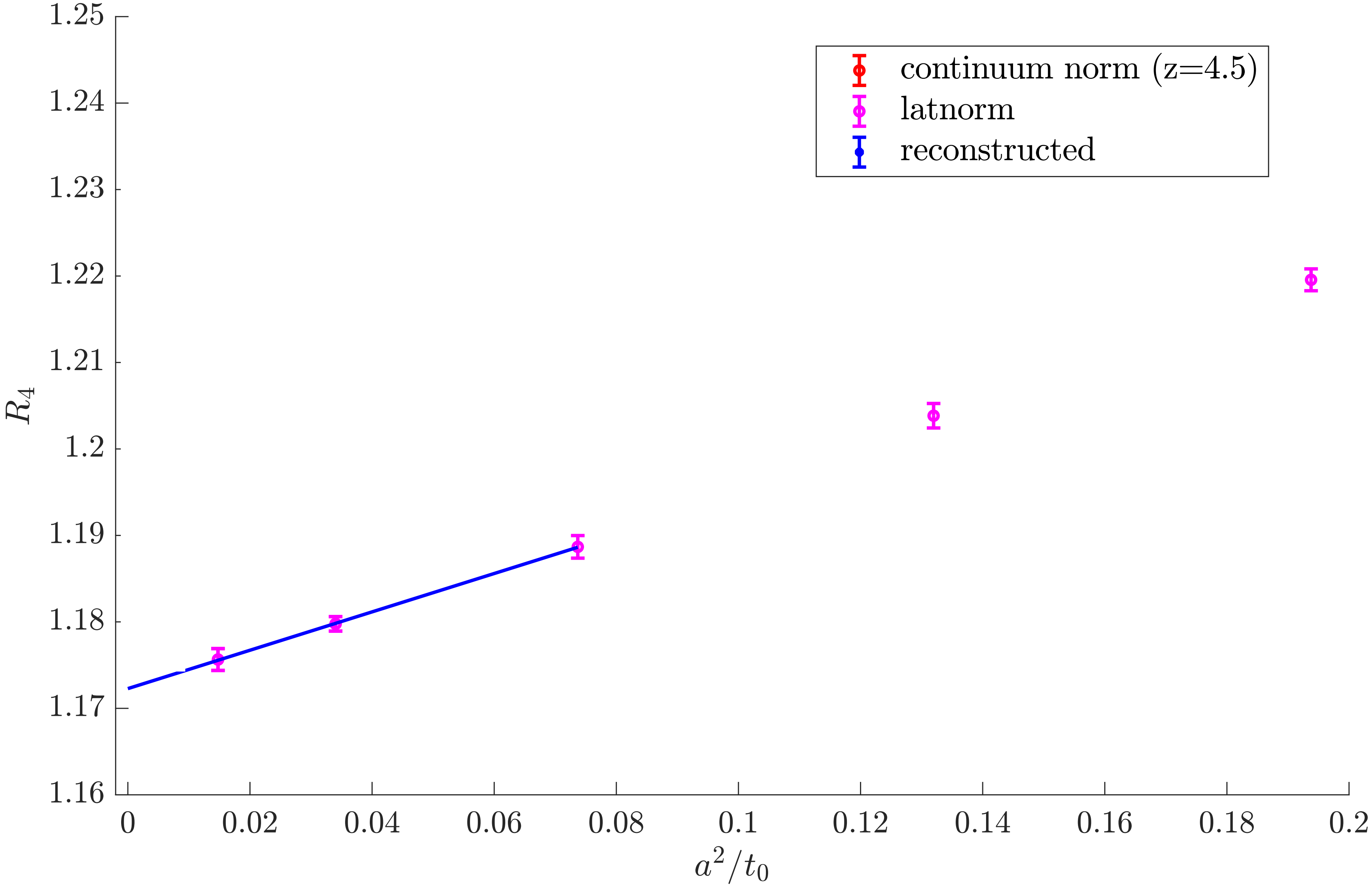

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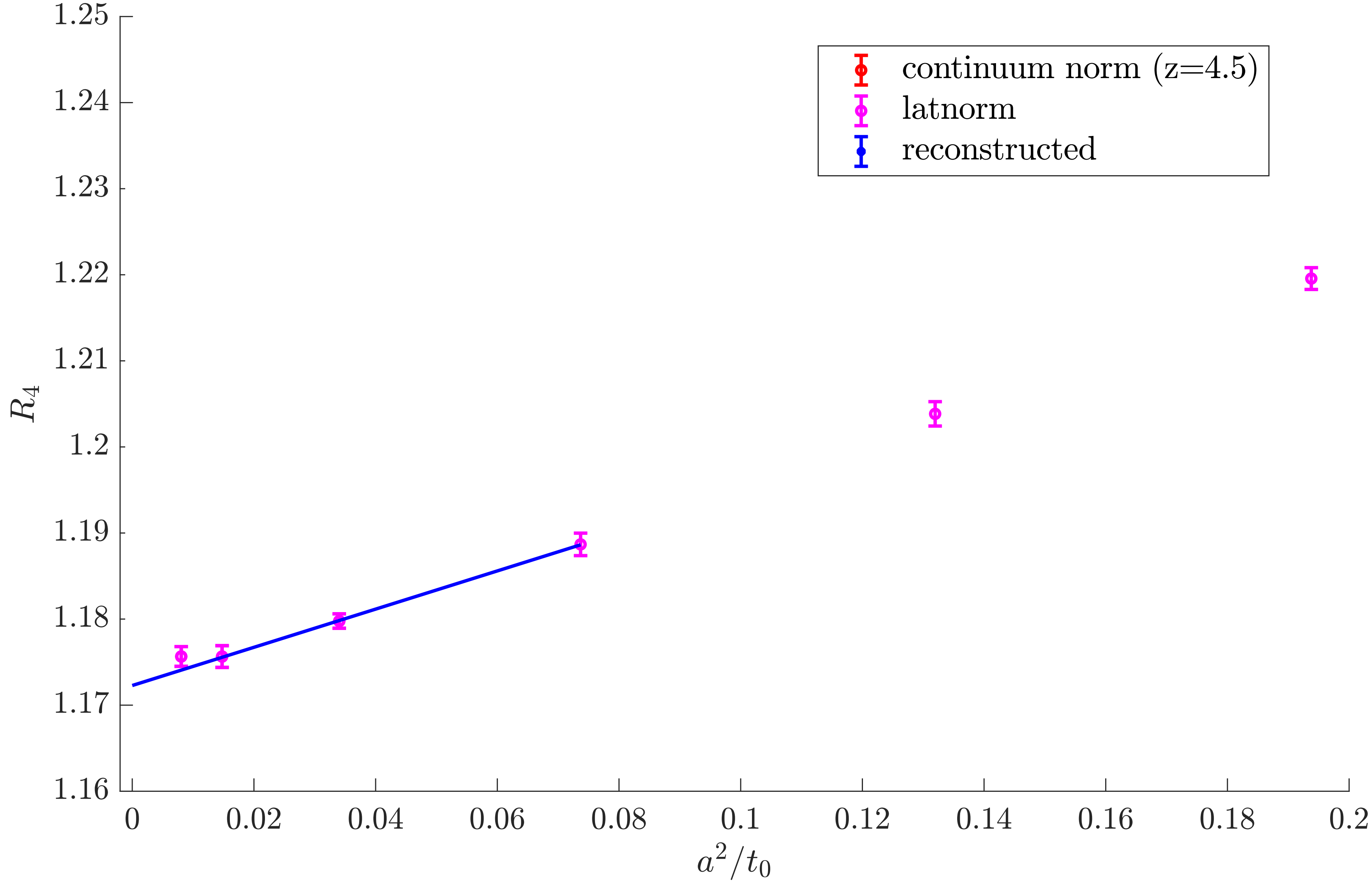

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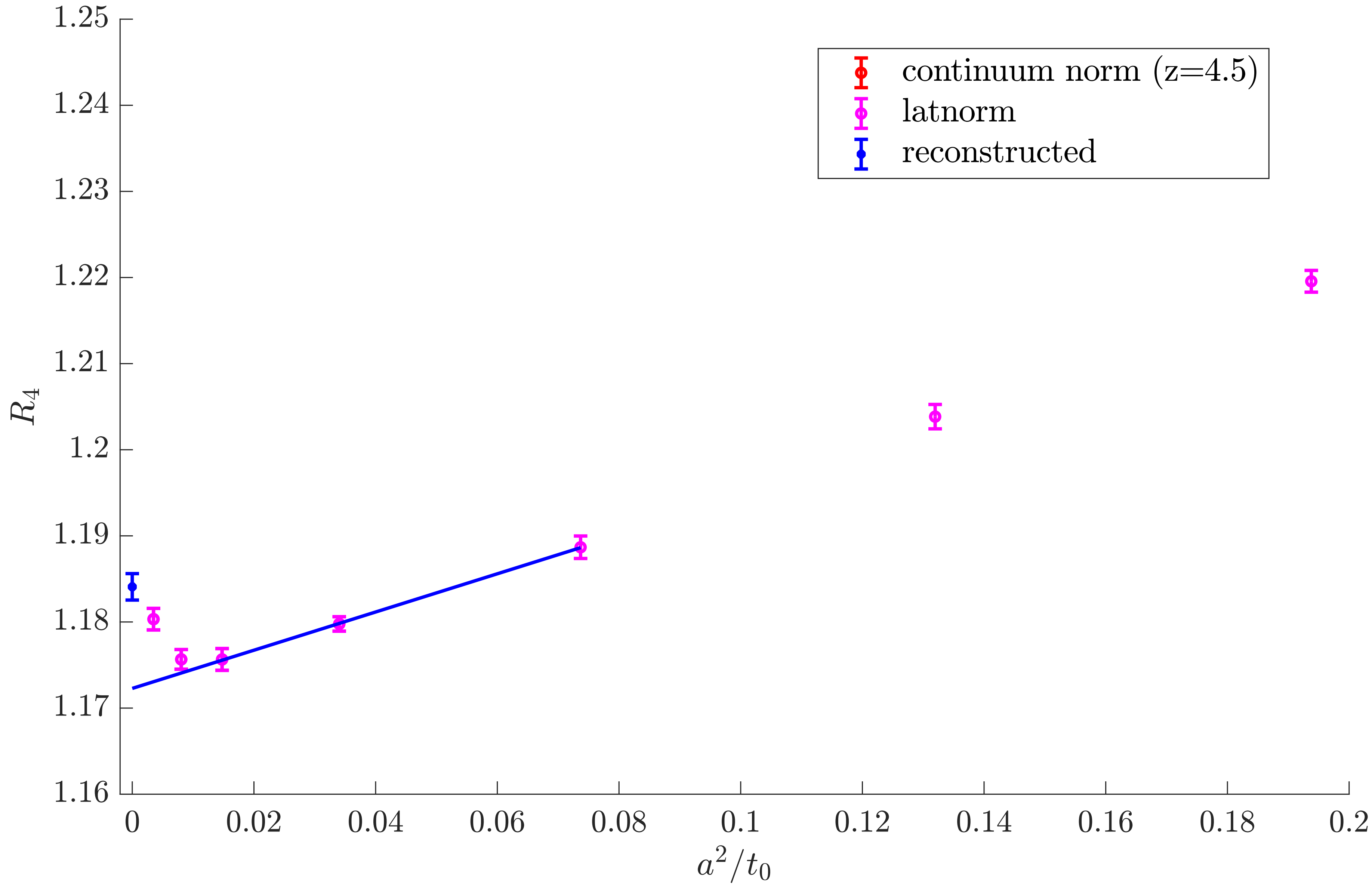

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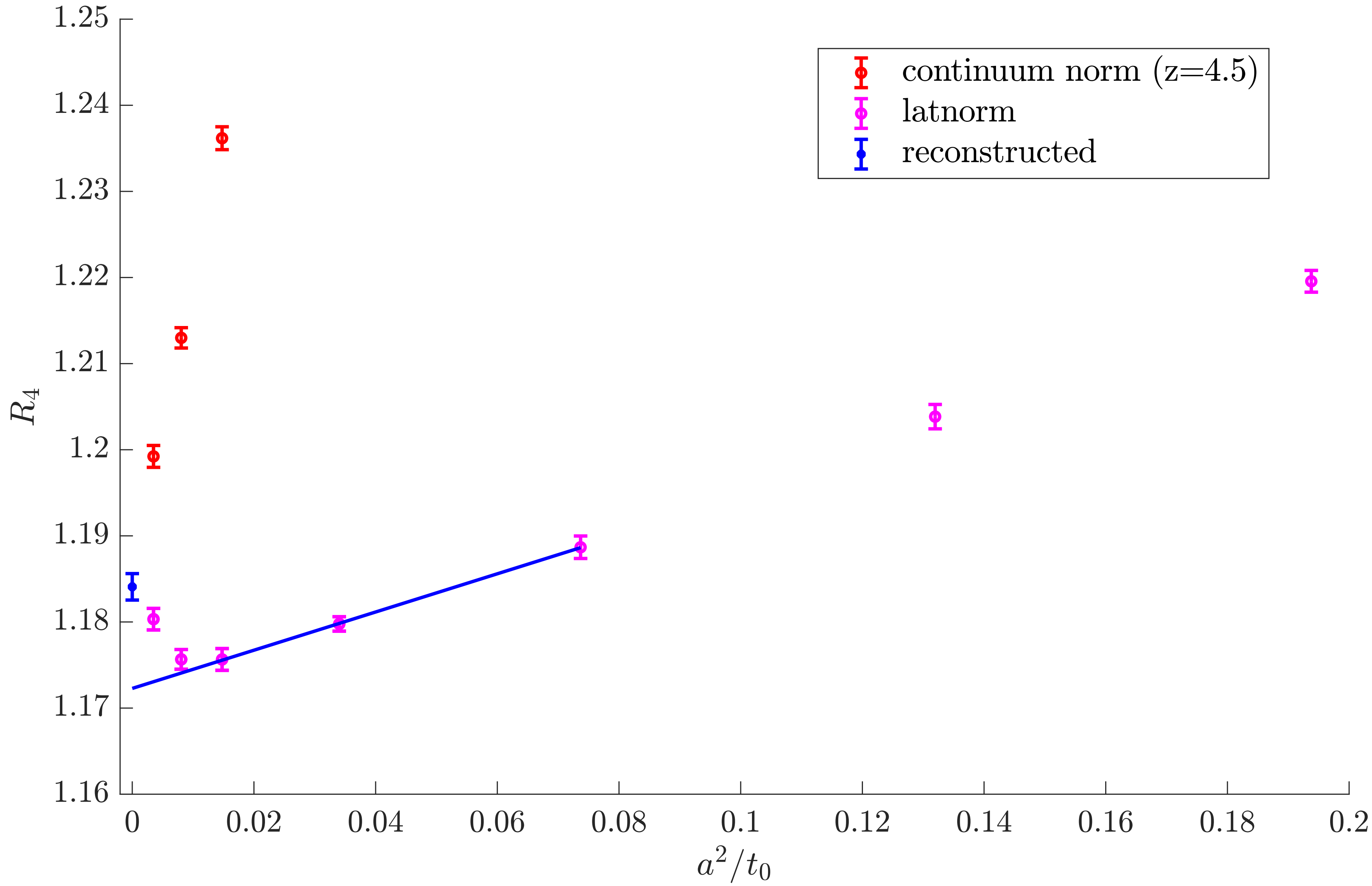

Tree level (free theory)

- on the lattice (Symanzik expansion for $t \gg a$)

$$G(t,M,a) = a^3 \sum_{\mathbf{x}} \langle P(x)\overline{P}(0)\rangle = [G(t,0,0) + k_L \frac{a^2}{t^5}]\,[1 + \mathrm{O}(tM)] + \mathrm{O}(\frac{a^4}{t^4})$$

$$M_4(M,a) = a \sum_t t^4\, G(t,M)$$

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- short distance contribution to discretisation errors ΔI with ($w(t) = 1/2$ at end points (trapezoidal))

$$\Delta I(t_1,t_2) = 2a \sum_{t=t_1}^{t_2} w(t)\, t^4\, G(t,M,a) - 2\int_{t_1}^{t_2} \mathrm{d}t\ t^4\, G(t,M,0)\,, \quad t_1 M \ll 1,\, t_2 M \ll 1\,.$$

for $t_2 > t_1 \gg a$: (Symanzik expansion) and $t_1 M \ll 1,\, t_2 M \ll 1\,.$

$$\Delta I(t_1,t_2) = k_L\, a^2 \int_{t_1}^{t_2} \mathrm{d}t\, t^{-1} + \ldots = k_L\, a^2 \log(t_2/t_1) + \ldots = k_L a^2\,[\log(t_2/a) - \log(t_1/a)] + \ldots$$

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- now $\Delta I(0,t) = \Delta I(0,t_1) + \Delta I(t_1, t)$ does not depend on t_1 =>

$$\Delta I(0,t) = \Delta I(0,t_1) + \Delta I(t_1, t) = \underbrace{[\Delta I(0,t_1) - a^2 k_L \log(t_1/a)]}_{=ka^2} + k_L a^2 \log(t/a)$$

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- just dimensional reasoning

$$[\Delta I(0,t_1) - a^2 k_L \log(t_1/a)] = k a^2$$

made it easy to get the general form

Interacting theory: what changes?

- anomalous dimensions

$$G(t,0,0) \sim \frac{1}{t^3}[\bar{g}^2(1/t)]^{-2\hat{\gamma}_P}, \quad \Delta G \sim \frac{a^d}{t^{3+d}}[\bar{g}^2(1/t)]^{-2\hat{\gamma}_P - \hat{\Gamma}_i^{(d)}}$$

with a sum over dimensions $d = [\mathcal{O}_i^{(d)}] - 4$ and numbering i of the operators $\mathcal{O}_i^{(d)}$ of Symanzik EFT

- dimensional reasoning becomes

$$\Delta I(0,t_1) + a^2 F(\bar{g}^2(1/t_1)) = a^2 K(a\Lambda)$$

and all terms of any power a^n in the expansion of G contribute to $K(a\Lambda)$

- in the free theory we could do $\int_a^t s^{-1} \mathrm{d}s$ to get the a dependence

with the AD's this gives an infinite sum over d, i. Seems impossible.

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with a sum over dimensions $d = [\mathcal{O} \ldots$ …ring i of the

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dimensional reasoning becomes unpredictive

(Λ_{QCD} and a)

- dimensio… …s

$$\ldots a^2 F(\bar{g}^2(1/t_1)) = a^2\, K(a\Lambda)$$

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back to the specific problem

- Tree-level normalised

$$R_4^{\text{latnorm}}(M) = \frac{M^2 M_4(M)}{M^2 M_4(M)\Big|_{g=0}^{aM\neq 0}}$$

- denominator: $M^2 a^2 \log(Ma)$
- numerator: suppression of short distance behavior by anomalous dimension
- log-effect left over, dominantly from the denominator
- but not dividing by tree-level lattice, yields very large discretisation effects

Solutions

- An integral of the considered type

  (correlator diverges like $\sim t^{-k}$ weight function $\sim t^{k-1}$ suppresses the divergence only to $\sim t$)

  can't be computed well on the lattice as such

- Solutions

  - develop theory for a-expansion of integrated functions … not yet available

  - Instead: Regulate the short distance part

    — Explicit example with full numerical demonstration for α_s from heavy quark moment

Regulated $M_4(M) \to \rho(M_1, M_2)$

- The problematic short distance region is mass-independent.
  —> combine two masses to eliminate it.

$$\rho(M_1, M_2) \propto M_1^2[M_4(M_1) - M_4(M_2)]\,, \quad r = M_1/M_2 > 1\,.$$

$$\rho(M_1, M_2) = \frac{2\pi^2}{3}\,\frac{\overline{M}_4(M_1) - r^2\overline{M}_4(M_2)}{1 - r^2}\,, \quad \overline{M}_4(M) = M^2 M_4(M)$$

integrand shifted to larger t, short distance suppressed

$$\rho(M_1, M_2) \propto \int_{-\infty}^{\infty} \mathrm{d}t\, t^4\, \underbrace{[G(t, M_1) - G(t, M_2)]}_{t^{-3}[t^2(M_1^2 - M_2^2) + \mathrm{O}(t^4)]}$$

- no log-enhancement and generically smaller a-effects
- PT from R_4 : $\rho(M_1, M_2) = 1 + c_1\alpha(m_{2\star}) + \ldots$

same c_1 as in R_4.

(chosen ren. scale: smaller mass dominates, integrand shifted to larger t —> choose M_2)

Continuum limit for $\rho(M_1, M_2)$

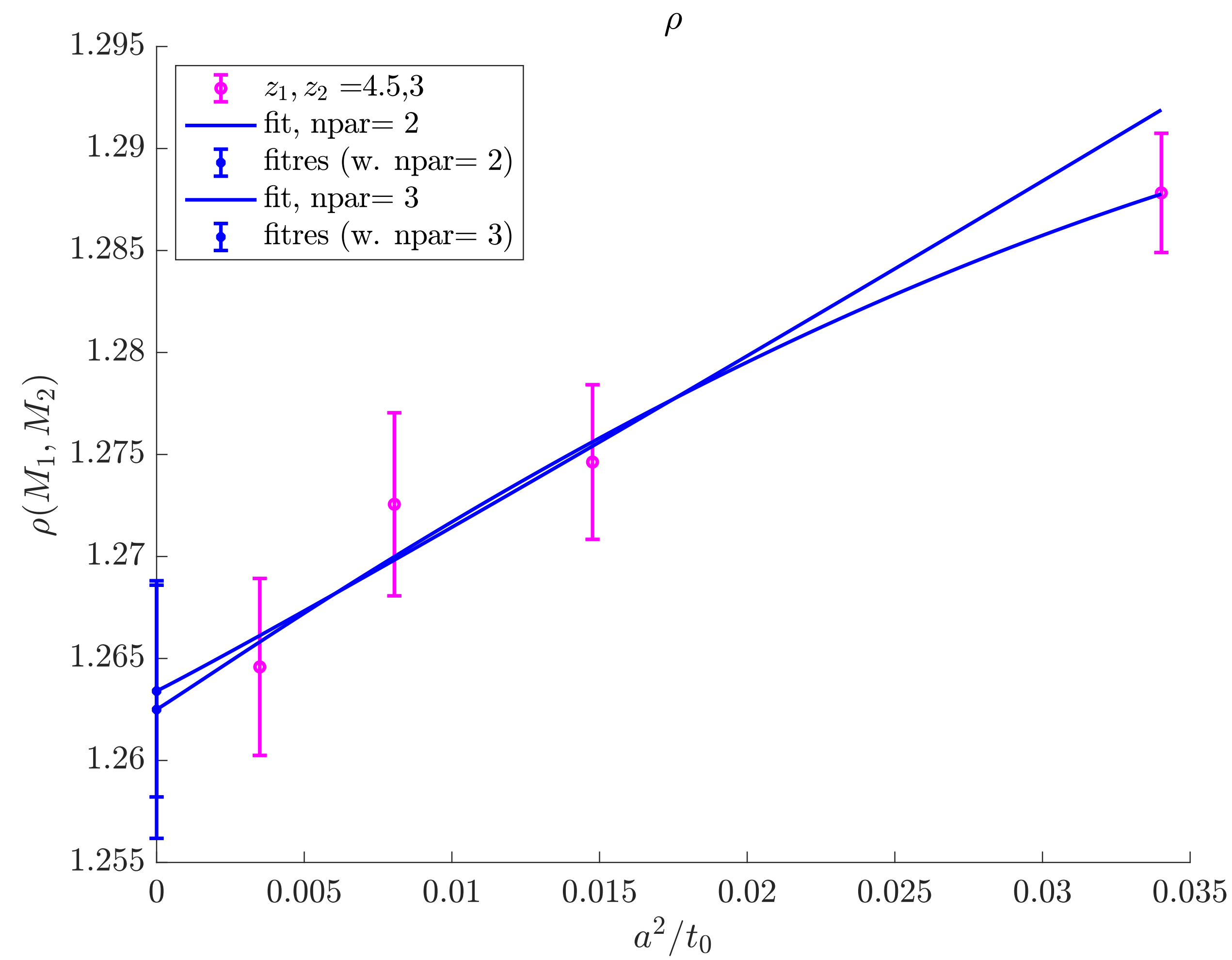
- dimensionless variable: $z = M\sqrt{8t_0}$
- best consider $\rho(rM_2, M_2)$ with $r =$ fixed
- we choose $r = 1.5$ with one exception $r = 1.33...$
- expl. $z_1 = 4.5,\ z_2 = 3$

fits

$$\rho = \rho_0 + \rho_2\, a^2[2b_0\bar{g}^2(1/a)]^{0.273}$$

$$[\ +\rho_4 a^4\ ]$$

0.273 from quenched contribution of d=6 SymEFT Lagrangian
[N. Husung 2022]



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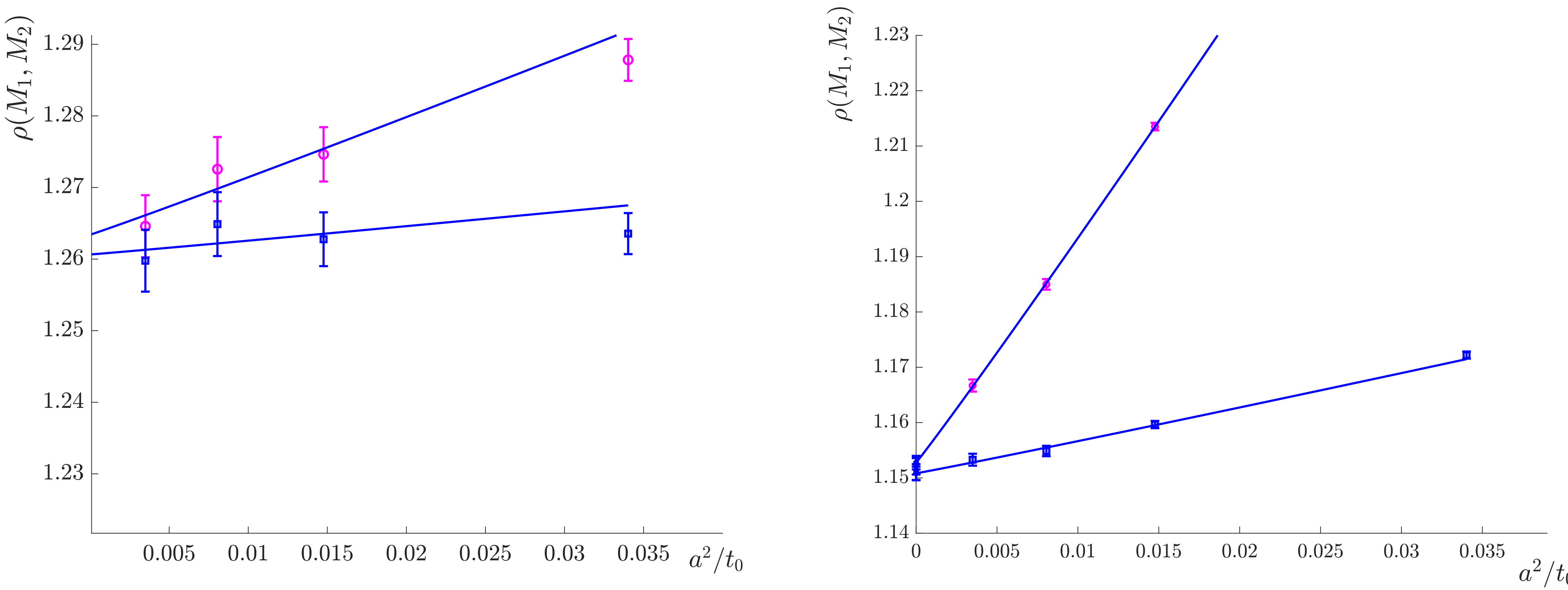


Figure 2: Continuum limit extrapolations of ρ and its TL improved version, $\rho(M_1, M_2)^{\text{Latnorm}}$. Masses, specified in units $z_i = M_i\sqrt{8t_0}$, are $z_1 = 4.5,\ z_2 = 3$ (left) and $z_1 = 13.5,\ z_2 = 9$ (right).

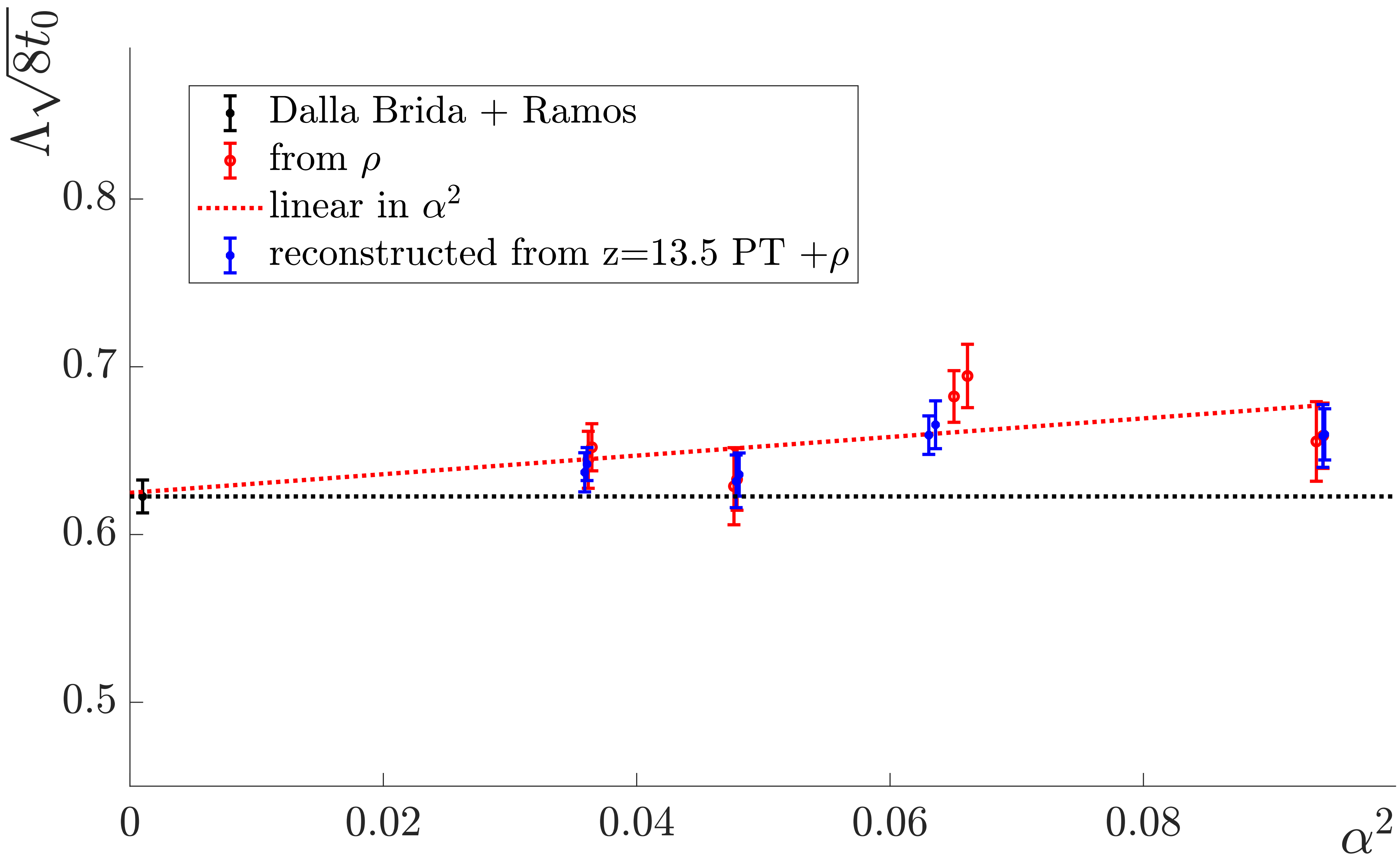
Reconstruct R_4

- from $R_4^{\mathrm{PT}}(M_{\mathrm{ref}})$ at large M_{ref} computed perturbatively to general M

$$R_4(M) = (1 - r^{-2})\,\rho(M_{\mathrm{ref}}, M) + \underbrace{r^{-2}}_{\frac{M_2^2}{M_{\mathrm{ref}}^2} \ll 1} R_4^{\mathrm{PT}}(M_{\mathrm{ref}})$$

- perturbative contribution is power suppressed for large M_{ref}

Directly showing $\Lambda_{\overline{MS}}$



moments method

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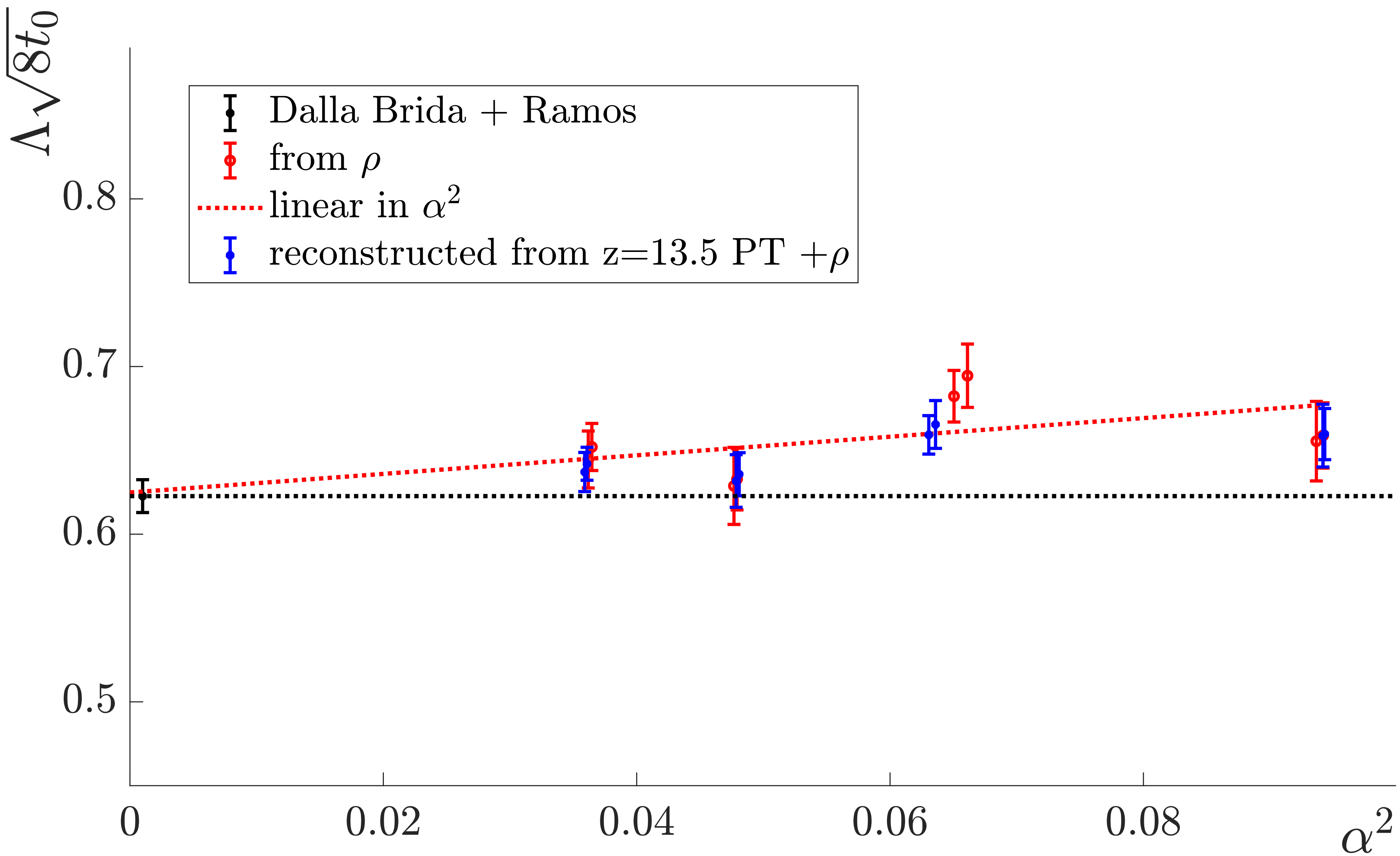
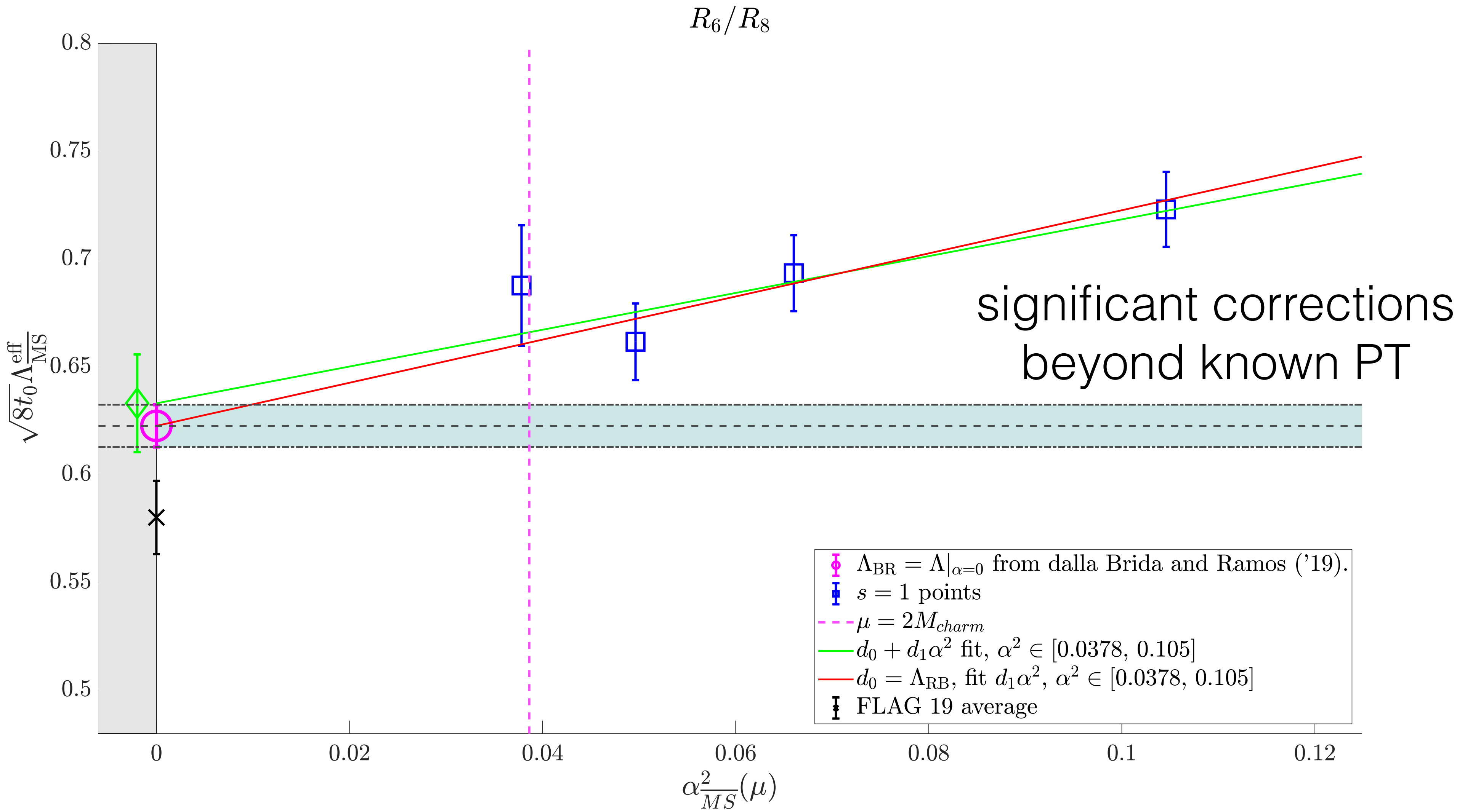


- Nice consistency, but despite tiny lattice spacing not very precise

Results from R_6/R_8

- R_6/R_8 : log-enhancement only at O($a^n, n > 2$)
- PhD Thesis by Leonardo Chimirri



(a) *Fit for $\Lambda_{\overline{\text{MS}}}$ as a function of $\alpha^2_{\overline{\text{MS}}}$, compared with a fit constrained to pass through Λ_{BR} of [47], extracted from R_6/R_8.*

Results from R_8/R_{10}

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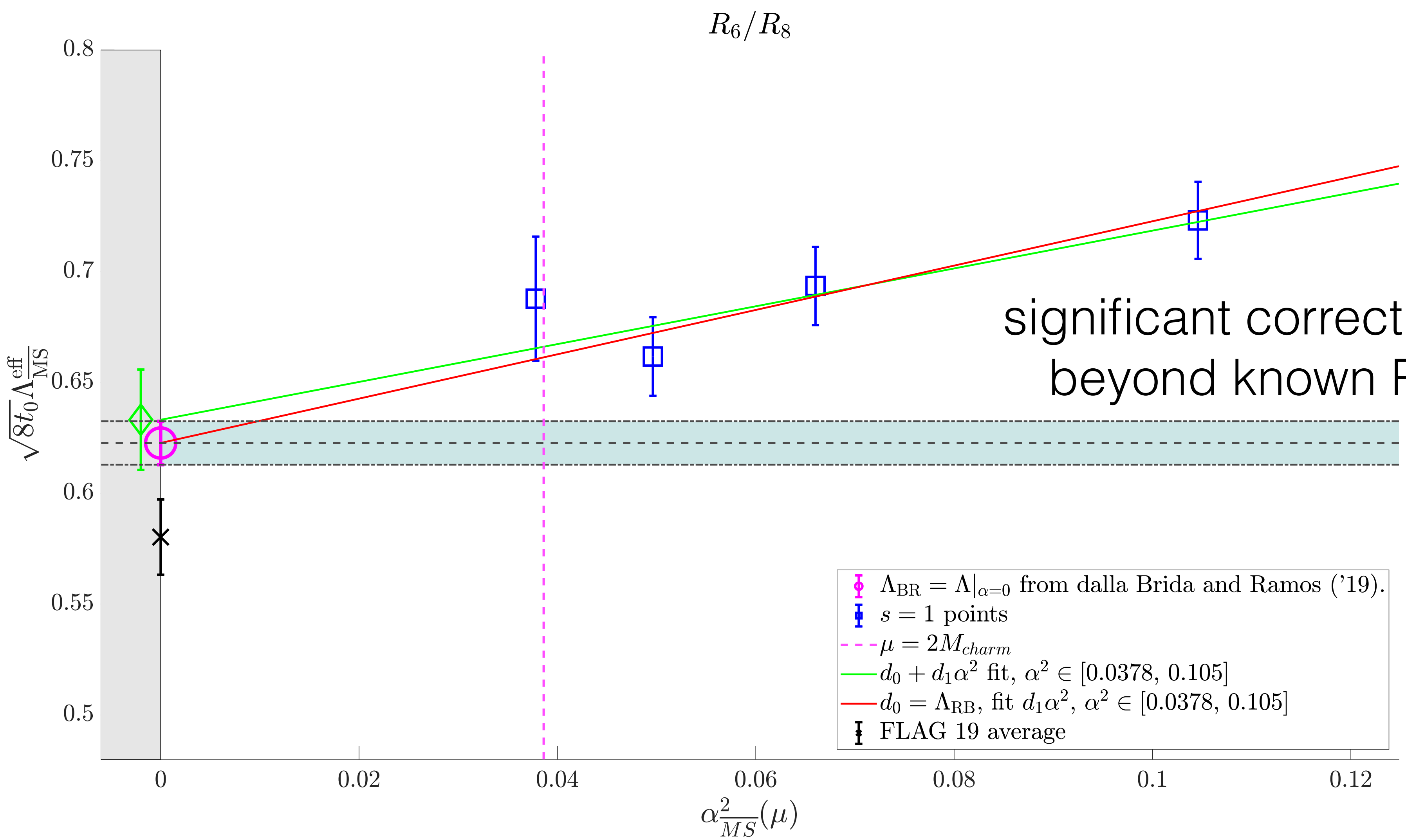
- PhD Thesis by Leonardo Chimirri



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Conclusions

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- for tree-level this is easily proven
- for the 4-th moment they show up at very small a
- for higher moments the problem is moved to higher powers of a
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