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Static and radiative cylindrically symmetric spacetimes

Workshop on Tensions in Cosmology,
SEPTEMBER 6-13, 2023, CORFU

Why cylindrical symmetry?

Einstein equation needs symmetry assumption to be solved

- next simplest geometry after spherical symmetry
- precursor of axial symmetry
  (essential for the omnipresent rotation in astrophysics)

Useful in discussing extended sources

- cosmic strings

B. Linet, The Static Metrics with Cylindrical Symmetry Describing a Model of Cosmic Strings, Gen. Rel. Grav. **17**, 1109-1115 (1985).

S. A. Hayward, Gravitational waves, black holes and cosmic strings in cylindrical symmetry, Class. Quantum Grav. **17**, 1749, https://doi.org/10.1088/0264-9381/17/8/302 (2000).

https://www.sciencephoto.com/media/572783/view/cosmic-string-and-galaxies-artwork

Why cylindrical symmetry?

- cosmic filaments of galaxies and dark matter extending across hundreds of millions of light years

P. Wang, N. I. Libeskind, E. Tempel, X. Kang and Q. Guo, Possible observational evidence for cosmic filament spin, Nature Astronomy **5**, 839–845 (2021), [arXiv:2106.05989 [astro-ph.GA]].

S. E. Hong, D. Jeong, H. S. Hwang, and J. Kim, Revealing the Local Cosmic Web from Galaxies by Deep Learning, Astrophys. J. **913**, 76 (2021).

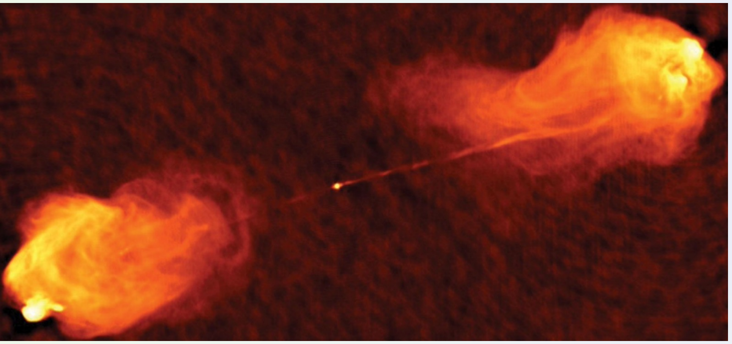
- jet topologies (magnetic field and charged particle distributions)

J. Ponstein, Instability of rotating cylindrical jets, Applied Scientific Research A **8**, 425–456 (1959).

A. Meli, et al., 3D PIC Simulations for relativistic jets with a toroidal magnetic field, Monthly Not. Royal Astron. Soc. **519**, 5410–5426 (2023), https://doi.org/10.1093/mnras/stac3474.

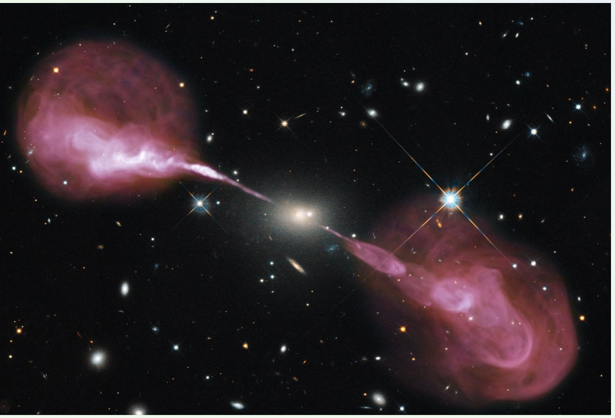
K.-I. Nishikawa, et al., Evolution of global relativistic jets: collimations and expansion with kKHI and the Weibel instability, Astrophys. J. **820**, 94 (2016), [arXiv:1511.03581 [astro-ph.HE]].

Dark matter: Scientists have mapped out dark matter filaments in the local universe (Image: GETTY STOCK)



Cygnus-A

https://cdn.britannica.com/72/118372-050-17CCC5EC/Cygnus-A-radio-waves-speed-of-light.jpg?fbclid=IwAR2Tp9rps_XO_z7EsCMcfOMnbfIZbAyq1q1m8FogFjmHXS9UwL5_UOFlgEU

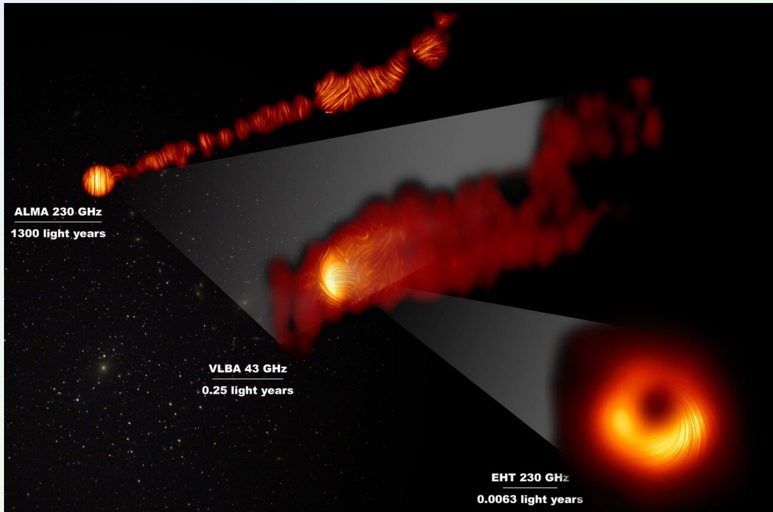


Hercules A

https://de.wikipedia.org/wiki/Aktiver_Galaxienkern?fbclid=IwAR1t79DBLADf-wyGkv20vu6su5wvSAqz8xQMigZVECjKyonui9PKTa6hWw0#/media/Datei:A_Multi-Wavelength_View_of_Radio_Galaxy_Hercules_A.jpg

M87

https://www.eso.org/public/images/eso2105b/



The cylindrically symmetric vacuum in GR

- The existence of spherically symmetric GWs in vacuum is forbidden by the Jebsen–Birkhoff unicity theorem, leading to either the static Schwarzschild or the homogeneous Kantowski-Sachs spacetime

- In contrast, the cylindrically symmetric vacuum is not unique, including
  - The static Levi-Civita spacetime
  - Einstein–Rosen waves:
    - standing and approximate progressive waves
    - solitonic waves
    - impulsive waves

- The Einstein-Rosen waves:

$$\begin{aligned} ds^2 &= e^{2(K-U)}\left(-dt^2+dr^2\right)+e^{-2U}r^2 d\varphi^2+e^{2U}dz^2 \ , \\ U &= J_0(r)\cos t \ , \\ K &= \frac{r^2}{2}\left[J_0^2(r)+J_1^2(r)\right]-rJ_0(r)J_1(r)\cos^2 t \ . \end{aligned}$$

D. Kramer, Exact gravitational wave solution without diffraction, Class. Quantum Grav. **16**, L75 (1999), DOI 10.1088/0264-9381/16/11/101.

A. Einstein and N. Rosen, On gravitational waves, J. Franklin Inst. **223**, 43 (1937).
R. F. Penna, Einstein-Rosen Waves and the Geroch Group, J. Math. Phys. **62**, 082503 (2021), [arXiv:2106.13252 [gr-qc]].

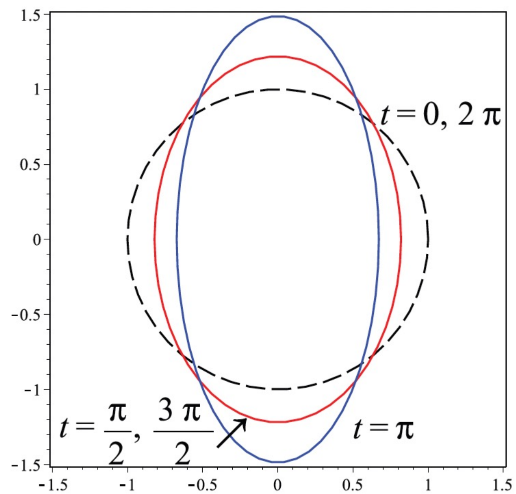


Figure 5. Einstein–Rosen waves. Deformation of a ring of particles initially at $\rho = 4$ induced by the travelling wave at different coordinate times $t = [0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi]$ and the same parameter choice as in figure 4.

D. Bini, A. Geralico and W. Plastino, Cylindrical gravitational waves: C-energy, super-energy and associated dynamical effects, Class. Quantum Grav. **36**, 095012 (2019), [arXiv:1812.07938 [gr-qc]].

Quantization of cylindrically symmetric gravitational waves

Canonical quantization of cylindrically symmetric gravitational waves

earliest example of the midisuperspace approach (1971)

K. Kuchař, Canonical Quantization of Cylindrical Gravitational Waves, Phys. Rev. D **4**, 955

much richer structure than previous minisuperspace quantizations:
- of Friedmann universe by DeWitt
- of mixmaster universe by Misner

Later made more accurate by Torre and Varadarajan
Another approach developed by Ashtekar and Pierri

Detailed review in J. F. Barbero G. and E. J. S. Villaseñor, Quantization of Midisuperspace Models, Living Rev. Relativ. **13**, 6 (2010). https://doi.org/10.12942/lrr-2010-6

Compromising between:
the simplicity induced by degrees of freedom frozen by symmetry assumptions
the full complexity of the gravitational degrees of freedom

—> ideal testbed for comparing quantization approaches

Gravitational waves on curved background

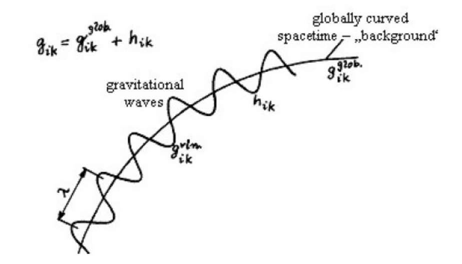
Need to go beyond the "perturbations on flat spacetime" view

What can be called a GW on strongly curved background?

$$\tilde{g}_{ab} = g_{ab} + \varepsilon h_{ab}$$

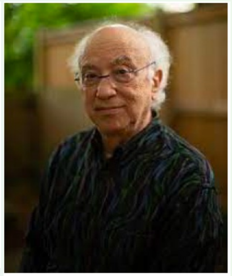
GW wavelength $\lambda \ll L$ characteristic curvature radius

This always holds in Minkowski, but is the high-frequency / geometrical optics approximation otherwise



R. A. Isaacson, Gravitational Radiation in the Limit of High Frequency. I. The Linear Approximation and Geometrical Optics, Phys. Rev. **166**, 1263 (1968).

R. A. Isaacson, Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Effective Stress Tensor, Phys. Rev. **166**, 1272 (1968).



Richard A. Isaacson

Identifying the expansion parameter in the metric with the parameter characterizing the geometrical optics approximation leads to the order shift of certain contributions to the Riemann-tensor:

$$\varepsilon = \frac{\lambda}{L} << 1$$

- Leading order: Einstein equation for the background geometry + backreaction in the form of an effective (Isaacson) energy-momentum tensor, quadratic in the GW
- Next order: (Lichnerowitz) wave equation for the GW

Isaacson showed that in the WKB approximation of perturbations, for superposed GWs with spherical symmetry (in an averaged sense) the passing through GW transforms Minkowski into Vaidya !

Gravitational waves on curved background

Main ingredients:

- Geometrical optics approximation

- WKB approximation: slowly changing amplitude, fast changing phase

Gregor Wentzel

Hendrik Antony Kramers

Léon Brillouin

$$\psi_{\mu\nu} = A_{\mu\nu} e^{ik_\alpha x^\alpha}$$

$O(1)$ (amplitude) $O(\epsilon^{-1})$ (phase)

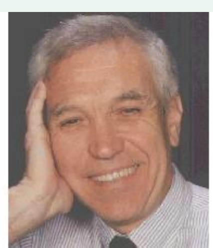
→ Lichnerowitz-wave equation becomes wave eq. on curved background

- Brill-Hartle averaging scheme:

Whenever the regions of interest are large enough to contain many wavelengths, employ spacetime averaged quantities
(similar to averaging leading to the Maxwell equations in media)
→ spectacular simplification in the second order terms, responsible for the backreaction, which appear as new source terms for the background

Jim Hartle.

Dieter Brill



Backreaction of weak, cylindrically symmetric GWs

work in progress with Attila Fóris

Coords. (u, ρ, φ, z)

Polarizations:

$$\mathfrak{e}^{\times}_{ab} = \frac{\rho}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathfrak{e}^{+}_{ab} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \left(a^{1-c}\rho^{\frac{1}{1-c}}\right)^2 & 0 \\ 0 & 0 & 0 & -\rho^2\left(a^{1-c}\rho^{\frac{1}{1-c}}\right)^{-2} \end{pmatrix}$$

Example: + polarization, the WKB-type perturbation is

$$h^{+}_{ab} = \frac{A_+(u)}{\rho^{1/2}} \mathfrak{e}^{+}_{ab} \cos\phi(u)$$

Leading to

$$\nabla_c h^{+}_{ab} \nabla_c h^{ab}_{+} = \frac{A_+^2(u)}{\rho} \begin{pmatrix} s_+^2 & \frac{s_+}{\rho}\cos\phi(u) & 0 & 0 \\ \frac{s_+}{\rho}\cos\phi(u) & \left[\frac{8}{(c-1)^2} + \frac{8}{c-1} + \frac{5}{2}\right]\frac{1}{\rho^2}\cos^2\phi(u) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $s_+ = \left[\frac{\dot{A}_+}{A_+}\cos\phi(u) - \sin\phi(u)\,\dot{\phi}\right] \approx -\dot{\phi}\sin\phi(u) = \mathcal{O}\left(\varepsilon^{-1}\right)$, last expression due to WKB

Therefore

$$\nabla_c h^{+}_{ab} \nabla_c h^{ab}_{+} = \begin{pmatrix} \frac{A_+^2(u)}{\rho}\dot{\phi}^2\sin^2\phi(u) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{A_+^2(u)}{\rho}\sin^2\phi(u)\,k_c k_d\,.$$

After Brill-Hartle averaging, obtain the Isaacson energy-momentum tensor:

$$T_{cd} = \frac{\varepsilon^2}{32\pi}\left\langle \nabla_c h^{+}_{ab} \nabla_c h^{ab}_{+} \right\rangle = \frac{\varepsilon^2 A_+^2(u)}{64\pi\rho} k_c k_d$$

→ The generated source is null dust, leading to the radiating Rao solution :

J. Krishna Rao, Radiating Levi-Civita metric. J. Phys. A Gen. Phys. **4**, 17 (1971). https://doi.org/10.1088/0305-4470/4/1/004.

Static limit of the Einstein-Rosen GWs

Cylindrically symmetric metric, with vorticity-free Killing vectors and orthogonally transitive group action: the Einstein-Rosen (canonical) metric form

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{-2U} r^2 d\varphi^2 + e^{2U} dz^2$$

K and U functions of the coordinates (t, r)

Strangely, it has two types of Minkowski limits: $K = 0 = U$ or $K = U = \ln r$

Static Levi-Civita spacetime for: $U = \sigma \ln r, \quad K = \sigma^2 \ln r,$

$$ds^2 = r^{2\sigma(\sigma-1)}(-dt^2 + dr^2) + r^{2(1-\sigma)} d\varphi^2 + r^{2\sigma} dz^2,$$

Axially symmetric, static Weyl form:

$$\sigma = \frac{2\lambda}{2\lambda - 1}$$

$$ds^2 = -\hat{r}^{4\lambda} d\hat{t}^2 + \hat{r}^{4\lambda(2\lambda-1)}(d\hat{r}^2 + d\hat{z}^2) + \hat{r}^{2(1-2\lambda)} d\hat{\varphi}^2,$$

The constant λ can be interpreted as the mass density of a cylinder on the axis

Interpretation of the Levi-Civita spacetime

The cylindrically symmetric, static field generated by a string with constant mass density λ and negligible pressure

... but only if λ is small !

Indeed, check the Minkowski limit :

	flat	flat	flat
λ	0	$1/2$	$\pm\infty$
σ	0	$\pm\infty$	1

$\longrightarrow$ 4 different flat limits ? Why?

Let's understand this !

B. Racskó, L. Á. Gergely, *Geometrical and physical interpretation of the Levi-Civita spacetime in terms of the Komar mass density,*
Eur. Phys. J. Plus **138**, 439 (2023)

Komar mass density

Komar superpotential :
$$\mathbf{U}_\xi = *\mathrm{d}\boldsymbol{\xi} = \frac{1}{2}\left(\nabla^i \xi^j - \nabla^j \xi^i\right)\sqrt{\mathfrak{g}}(\mathrm{d}^2 x)_{ij}$$

Current :
$$\mathbf{S}_\xi = \mathrm{d}\mathbf{U}_\xi = \nabla_j\left(\nabla^i \xi^j - \nabla^j \xi^i\right)\sqrt{\mathfrak{g}}(\mathrm{d}^3 x)_i$$

If the 1-form ξ is Killing + vacuum + Einstein eqs.

→ conserved Komar charge $Q_\xi(\mathcal{S}) = \int_\mathcal{S} \mathbf{U}_\xi$

If ξ timelike , $m_K = -\frac{1}{8\pi} Q_\xi(\mathcal{S}) = -\frac{1}{8\pi}\int_\mathcal{S} \mathbf{U}_\xi$ Komar mass

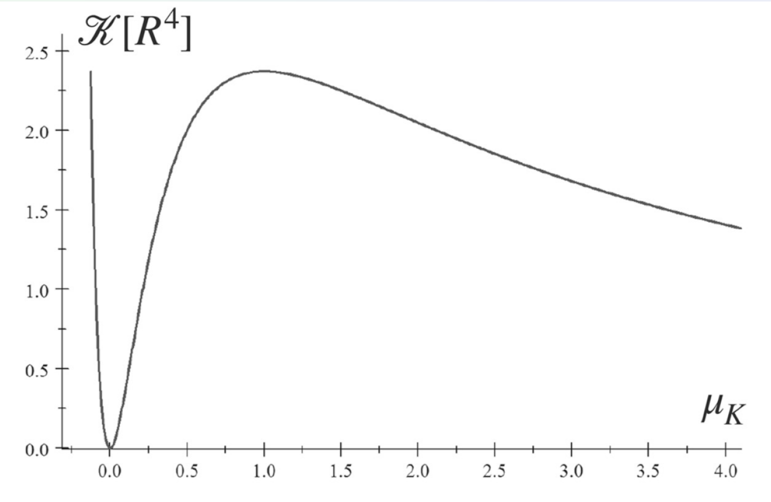
- On a technical level S should be closed 2-surface, so the axis cannot be contained
- Still possible to define a Komar mass density : $\mu_K = \frac{\sigma(\sigma - 1)}{2} = \frac{\lambda}{(1-2\lambda)^2}$
(compactify, calculate, then decompactify)
- For small values also has the interpretation of mass density on the axis

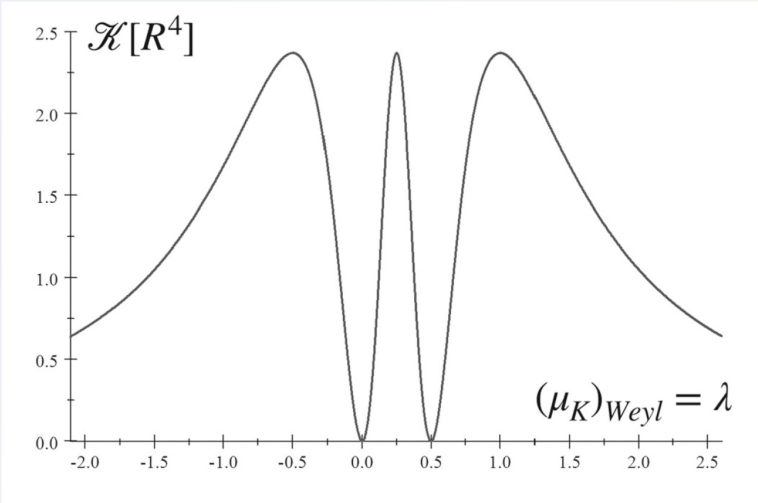
Get rid of a double parameter coverage

Parametrize the metric with the Komar mass density

$$ds^2 = r^{4\mu_K}\left(-dt^2 + dr^2\right) + r^{1+\sqrt{1+8\mu_K}}d\psi^2 + r^{1-\sqrt{1+8\mu_K}}dZ^2$$

Kretchmann scalar : with the Komar mass density   with the original parameter





→ the double coverage of the parameter space disappeared

But still two types of flat limits : at $\mu_K = 0$ (Minkowski limit)

$\mu_K \to \infty$ (Rindler limit)

Curvature invariants

Curvature of vacuum in 4D characterized by 4 scalar invariants :

$$J_1 = A_{ij}{}^{ij}, \quad J_2 = B_{ij}{}^{ij},$$
$$J_3 = A_{ij}{}^{kl} A_{kl}{}^{ij} - \frac{J_1^2}{2},$$
$$J_4 = A_{ij}{}^{kl} B_{kl}{}^{ij} - \frac{5 J_1 J_2}{12},$$

$$A_{ijkl} = C_{ij}{}^{mn} C_{mnkl}, \quad B_{ijkl} = C_{ij}{}^{mn} A_{mnkl}.$$

$$J_1 = \frac{64\mu_K^2(1+2\mu_K)}{r^{4(1+2\mu_K)}} = \mathcal{K},$$
$$J_2 = -\frac{768\mu_K^4}{r^{6(1+2\mu_K)}} = -\frac{3\mu_K}{2}\left(\frac{\mathcal{K}}{1+2\mu_K}\right)^{3/2},$$
$$J_3 = -\frac{1024\mu_K^4(1+2\mu_K)^2}{r^{8(1+2\mu_K)}} = -\frac{\mathcal{K}^2}{4},$$
$$J_4 = 0.$$

→ Curvature fully characterized by the Kretchmann scalar $\mathcal{K} = \frac{64\mu_K^2}{(1+2\mu_K)^3} R^{-4}$

expressed in term of the proper radial distance $R = \int_0^r r^{2\mu_K}\,\mathrm{d}r = \frac{r^{1+2\mu_K}}{1+2\mu_K}.$

Metric singular on the axis and no horizon !

Radial null geodesics satisfy both the Tipler and Królak singularity conditions

→ Curvature singularity is strong (and naked)

Kasner form and Rindler limit

Kasner form :

$$ds^2 = -R^{2p_0}dT^2 + dR^2 + R^{2p_+}\,d\chi^2 + R^{2p_-}d\zeta^2,$$

Written in the coordinates :

$$\chi = (1+2\mu_K)^{p_+}\psi, \qquad \zeta = (1+2\mu_K)^{p_-}Z,$$

Powers :

$$p_0 = \frac{2\mu_K}{1+2\mu_K}, \quad p_\pm = \frac{1 \pm \sqrt{1+8\mu_K}}{2(1+2\mu_K)}.$$

$$p_0 + p_+ + p_- = 1$$
$$p_0^2 + p_+^2 + p_-^2 = 1,$$

Rindler limit : $\mu_K \to \infty$ $p_0 \to 1,\ p_\pm \to 0$ $(1+2\mu_K)^{2p_\pm} \to 1$

→

$$ds^2_{\mu_K\to\infty} = -R^2dT^2 + d\,R^2 + dZ^2 + d\psi^2.$$

Rindler metric with topology $S^1 \times \mathbb{R}^3$

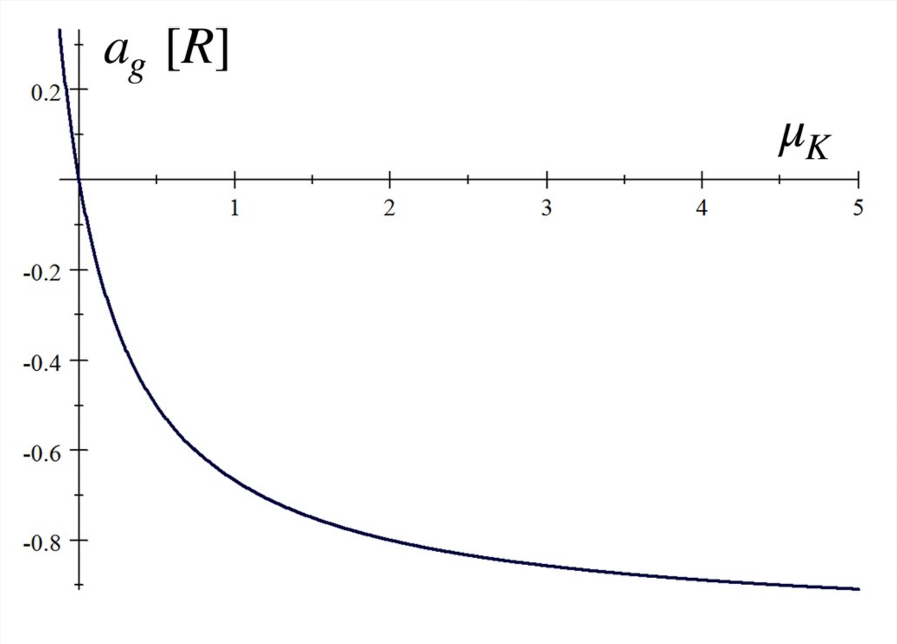
Represents flat spacetime
perceived by a uniformly accelerated observer
with acceleration R^{-1} along R

Newtonian gravity

Stationary observer at fixed proper distance R from the axis Z

4-velocity : $u^a = \left(R^{-p_0}, 0, 0, 0\right)$, 4-acceleration : $a^a \equiv u^b \nabla_b u^a = \frac{2\mu_K}{1+2\mu_K} R^{-1} \delta^a_R,$

Gravitational acceleration
(defined in Newtonian sense through the Equivalence Principle):



$$a_g = -\frac{2\mu_K}{1+2\mu_K} R^{-1},$$

→ Newtonian gravity increases monotonically with Komar mass density
Asymptotes to R^{-1}

Einsteinian gravity

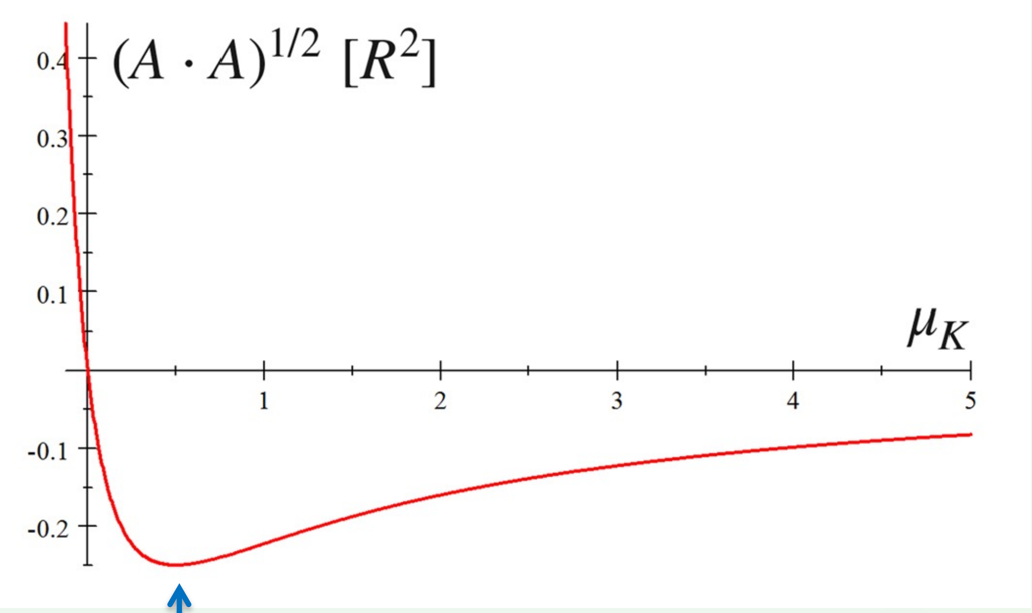
Geodesic congruence $U^a \equiv (\partial/\partial\tau)^a$, with proper time τ given by $d\tau = R^{p_0} dT$
thus $U^a = R^{-p_0} \delta^a_T$

Deviation of nearby geodesics $X^a \equiv (\partial/\partial R)^a = \delta^a_R$.

Acceleration $D^2 X^a / d\tau^2 = A^a = R^a{}_{bcd} U^b U^c X^d = R^{-2p_0} R^a{}_{TTR} = p_0(1 - p_0) R^{-2} \delta^a_R$

Its magnitude (negative of tidal acceleration)

$$\left(g_{ab} A^a A^b\right)^{1/2} = \frac{2|\mu_K|}{(1 + 2\mu_K)^2} R^{-2}$$



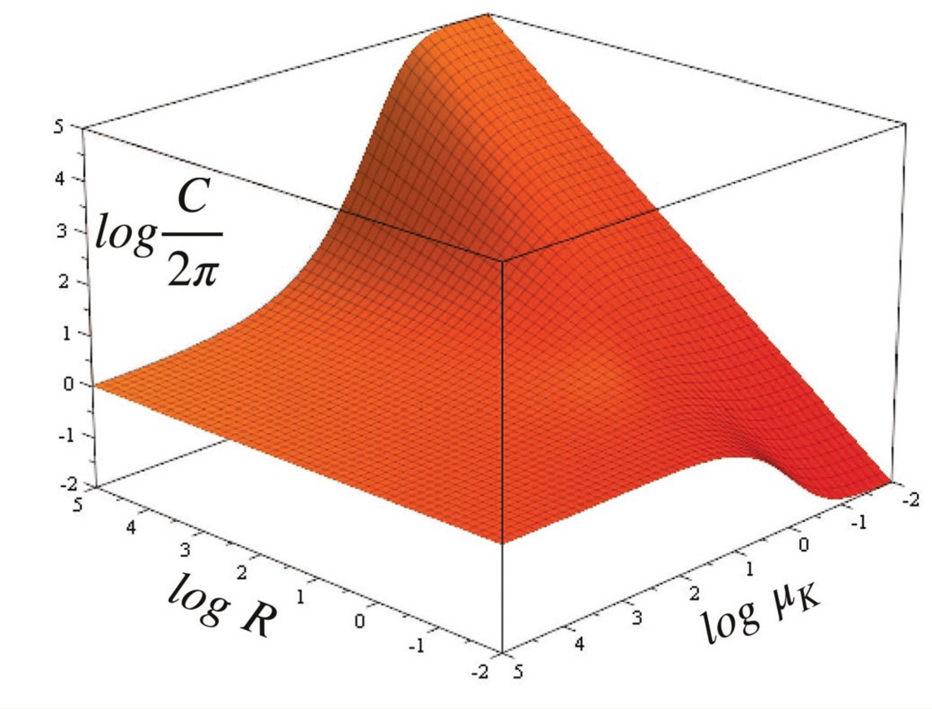
$\mu_K = 1/2$

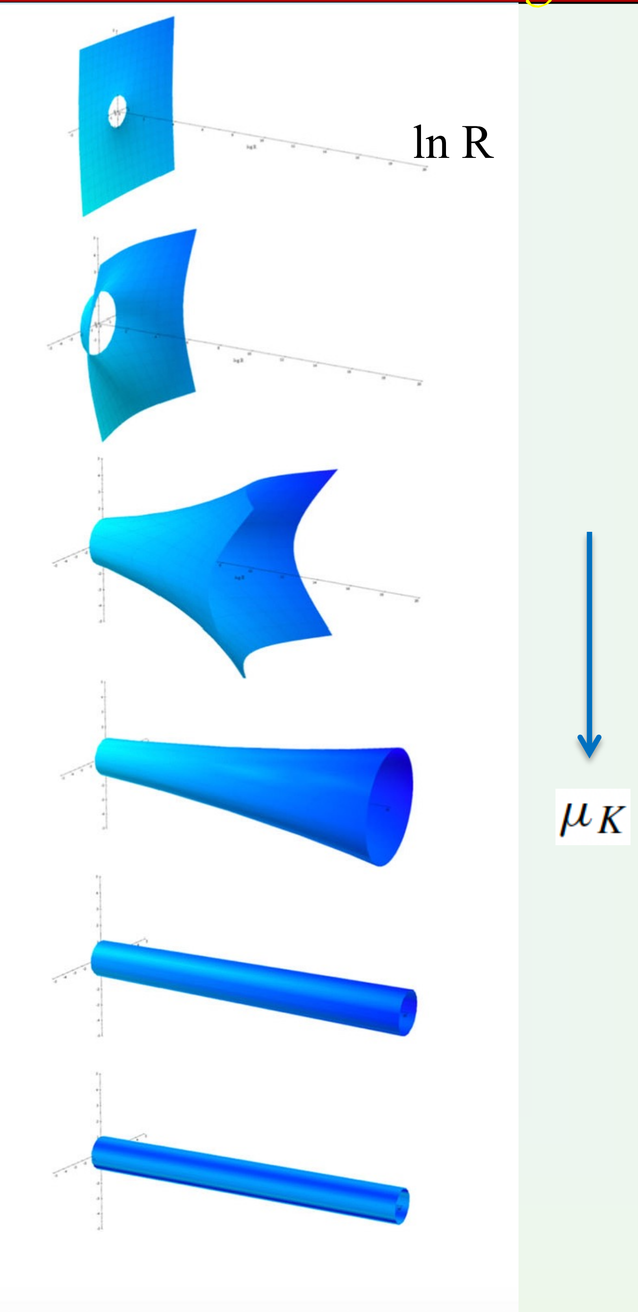
- The tidal accelereration (Einsteinian gravity) reaches a maximum, after which decreases with the Komar mass density
- Some part of the Newtonian gravity is pure acceleration field, which increases and dominates at large μ_K

Topology change by increasing Komar mass density

The circumference $C=2\pi R$ of the circles

- increases with the radius at small Komar mass density
- Stays constant at its large values (eventually a cylindrical topology appears = Rindler limit)





Interpretation of Levi-Civita in terms of Kasner parameters

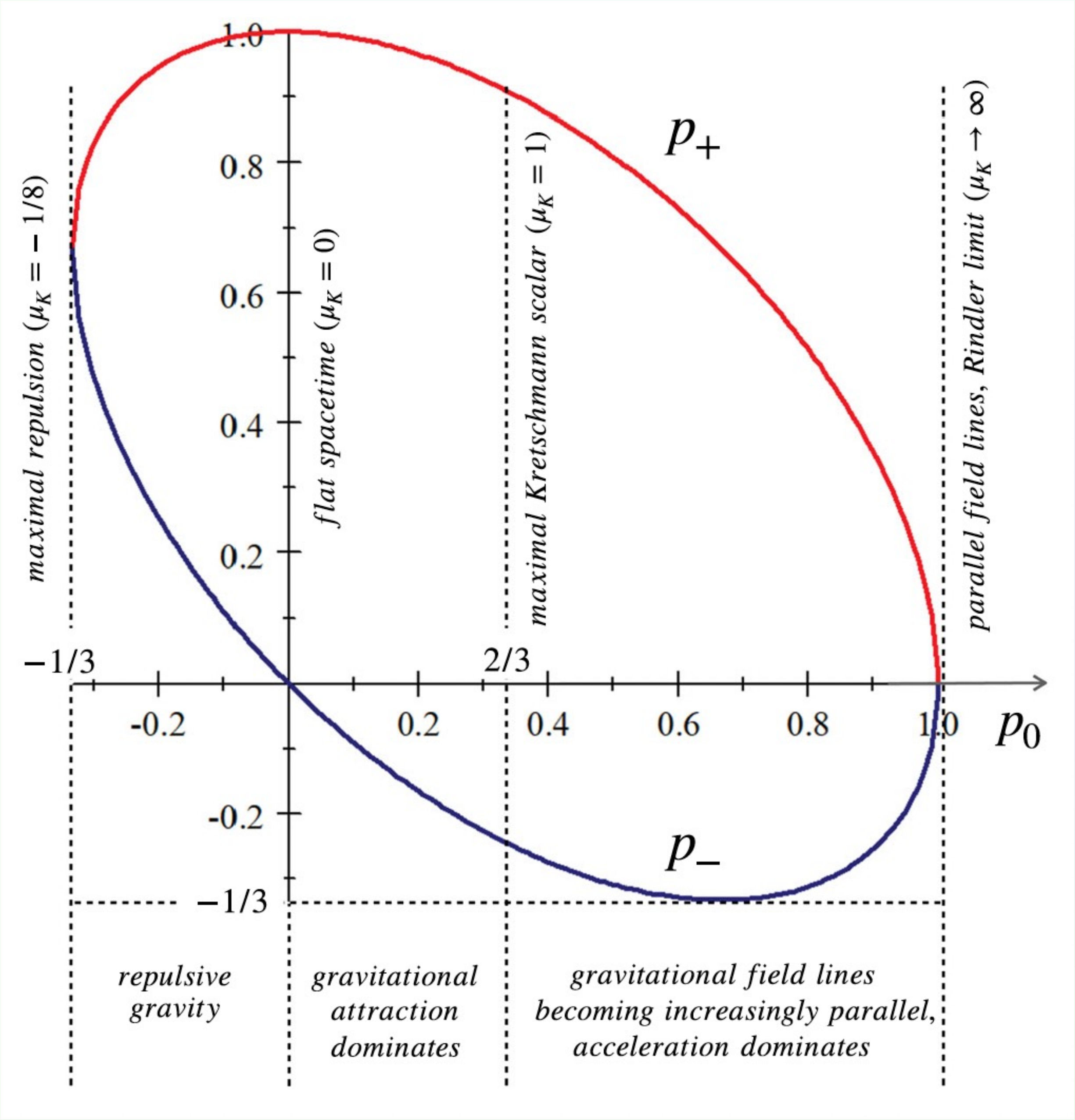


Fig. 8 The various regimes of the Levi-Civita metric in terms of the Kasner parameters p_+ (red curve), p_- (blue curve) and p_0 (horizontal axis). The Komar mass density increases from left to right, spanning to the regime of negative gravity $\mu_K \in [-1/8, 0)$, no gravity $\mu_K = 0$, gravitational attraction dominated regime $\mu_K \in (0, 1)$, maximal Kretschmann scalar (the metric of Ref. [28]) $\mu_K = 1$, increasingly parallel field lines transforming gravity into an acceleration field $\mu_K \in (1, \infty)$, and perfectly parallel field lines, the Rindler limit $\mu_K \to \infty$

Einstein-Rosen waves in scalar-tensor theories

- Ground state of the Einstein-Rosen waves in GR, the Levi-Civita solution understood ✔
- What are its generalisations to scalar-tensor theories ?
- Even better: what are the generalizations of Einstein-Rosen waves in scalar-tensor theories?
- For Brans-Dicke that has already been explored ✔

L. Akyar, A. Delice, On generalized Einstein-Rosen waves in Brans-Dicke theory. Eur. Phys. J. Plus **129**, 226 (2014).

- Einstein-Rosen waves in Brans-Dicke theory are very similar to those in GR, they describe linearly polarized GWs
- Particular solution for : $\phi = \phi_0 r^{1-k}$
- Wave equation on flat spacetime in cylindrical coordinates : $U_{tt} - U_{rr} - \frac{U_r}{r} = 0,$
  (same as in GR)

Partial derivatives of the other metric function (GR limit for k=1)

$$K_t = 2rU_tU_r + (1-k)\,U_t,$$

$$K_r = r\left(U_t^2 + U_r^2\right) + \frac{r\phi_r}{\phi}U_r + \frac{(1-k)}{2r}\left[(1-k)\,\omega - 2k\right].$$

Scalar-tensor field equations

For a more generic system

$$L = F(\phi)X + \lambda X\Box\phi + G(\phi)R.$$

Variation gives

$$E^{(2)}_{\mu\nu} = -\frac{1}{2}FXg_{\mu\nu} - \frac{1}{2}F\phi_{;\mu}\phi_{;\nu}$$

$$E^{(3)}_{\mu\nu} = -\frac{1}{2}\lambda\phi_{;\mu}\phi_{;\nu}\Box\phi - \frac{1}{2}\lambda\phi^{;\kappa}_{;\lambda}\phi^{\lambda}\phi_{;\kappa}g_{\mu\nu} + \lambda\phi^{;\kappa}\phi_{;\kappa(\mu}\phi_{;\nu)}$$

$$E^{(4)}_{\mu\nu} = GG_{\mu\nu} + G_{\phi}\left(\Box\phi g_{\mu\nu} - \phi_{;\mu\nu}\right) + G_{\phi\phi}\left(\phi^{;\kappa}\phi_{;\kappa}g_{\mu\nu} - \phi_{;\mu}\phi_{;\nu}\right)$$

$$E^{(2)}_{\phi} = F\Box\phi - F_{\phi}X$$

$$E^{(3)}_{\phi} = \lambda\left(\Box\phi\right)^2 - \lambda\phi_{;\mu\nu}\phi^{;\mu\nu} - \lambda R_{\mu\nu}\phi^{;\mu}\phi^{;\nu}$$

$$E^{(4)}_{\phi} = G_{\phi}R,$$

Specify to generalized Brans-Dicke

$$G(\phi) = \phi,$$

$$F(\phi) = \frac{2\omega(\phi)}{\phi},$$

$$\lambda = 0.$$

$$E^{(2)}_{\mu\nu} = -\frac{\omega}{\phi}\left(\phi_{;\mu}\phi_{;\nu} + Xg_{\mu\nu}\right),$$

$$E^{(3)}_{\mu\nu} = 0,$$

$$E^{(4)}_{\mu\nu} = \phi G_{\mu\nu} + \Box\phi g_{\mu\nu} - \phi_{;\mu\nu},$$

$$E^{(2)}_{\phi} = \frac{2\omega}{\phi}\Box\phi - \frac{2\omega_{\phi}}{\phi}X + \frac{2\omega}{\phi^2}X,$$

$$E^{(3)}_{\phi} = 0,$$

$$E^{(4)}_{\phi} = R.$$

Tensorial eq.

$$R_{\mu\nu} = \frac{\phi_{;\mu\nu}}{\phi} + \frac{\omega}{\phi^2}\phi_{;\mu}\phi_{;\nu} + \frac{\omega_{\phi}}{3+2\omega}\frac{X}{\phi}g_{\mu\nu}.$$

Scalar eq.

$$\Box\phi = \frac{2\omega_{\phi}}{3+2\omega}X$$

assuming

$$3 + 2\omega \neq 0$$

Specify to cylindrical symmetry

- Three metric functions K, U, W

$$ds^2 = e^{2(K-U)}\left(-dt^2 + dr^2\right) + e^{2U}dz^2 + e^{-2U}W^2 d\varphi^2,$$

- Introduce $\Omega := W\phi$ to obtain from the field equations $\Omega_{tt} - \Omega_{rr} = 0.$

Generic solution $\Omega(t,r) = \Psi(t-r) + \Xi(t+r)$

u v

Ansatz : $\phi = \phi(\Omega)$ and $W = W(\Omega)$

Scalar eq. : (ODE) $0 = \phi'' + \dfrac{\phi'}{\Omega} + \left(\dfrac{f(\phi)-1}{\phi}\right)\phi'^2.$ with $f(\phi) = \phi\dfrac{\omega_\phi(\phi)}{3+2\omega(\phi)}$

Implicit solution in terms of quadratures : $b\Omega^a = \exp\int d\phi \exp\left(\int d\phi \dfrac{f(\phi)-1}{\phi}\right)$

The generalised Einstein-Rosen wave

- Generalised Einstein-Rosen wave equation for U :

$$U_{tt} - U_{rr} + \frac{\Omega_t}{\Omega} U_t - \frac{\Omega_r}{\Omega} U_r = \frac{1}{2} f(\phi) \frac{\phi'^2}{\phi^2} \left(\Omega_t^2 - \Omega_r^2 \right)$$

- U sourced by the scalar field
- has amplitude dampening / friction
- no dispersion

- Pick the free functions conveniently : $\Psi(u) = -\frac{u}{2}, \quad \Xi(v) = \frac{v}{2}$ → $\Omega = r$

→ $$U_{tt} - U_{rr} - \frac{U_r}{r} = -\frac{1}{2} f(\phi) \frac{\phi'^2}{\phi^2}$$

inhomogeneous Einstein-Rosen wave eq.,
ER wave sourced by the scalar field

Last metric function given by the same eqs. as for Brans-Dicke :

$$K_t = 2rU_tU_r + \frac{r\phi_r}{\phi} U_t,$$

$$K_r = r \left(U_t^2 + U_r^2 \right) - \frac{\phi_r}{\phi} + \frac{r\phi_r}{\phi} U_r + (\omega + 2) \frac{r\phi_r^2}{2\phi^2}.$$

Summary

- Cylindrical symmetry interesting conceptually and has possible astrophysical applications : strings, galactic dark matter filaments, AGN / quasar jets.

- Backreaction of weak cylindrical waves through the Isaacson procedure gives the radiating Levi-Civita (Rao) spacetime, sourced by the incoherent superposition of electromagnetic or gravitational waves with random phases and polarisations (null dust).

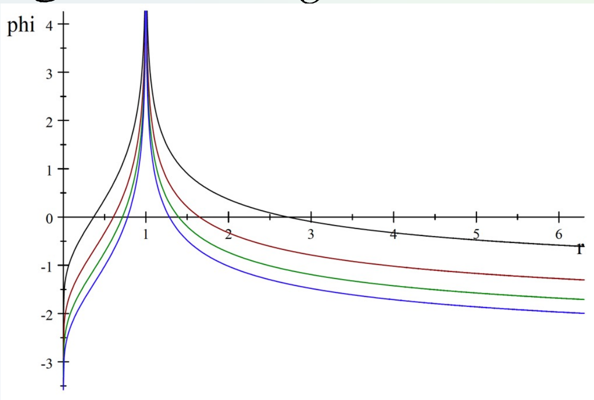
- Einstein-Rosen waves are exact (not weak) GWs, worth studying.

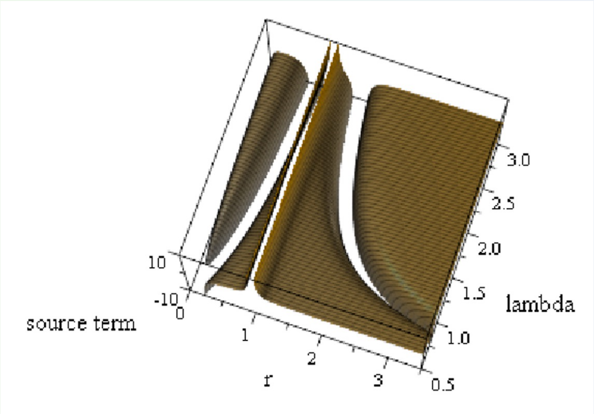
- Their ground state: the Levi-Civita metric conveniently described with the Komar mass density of the axis, which represents a strong singularity. Einsteinian and Newtonian gravity diverges heavily with increasing Komar mass density, the Newtonian part containing eventually only acceleration-type contribution (homogeneous gravitational field).

- Inhomogeneous Einstein-Rosen wave equation identified for generalised Brans-Dicke theories. The ER wave (tensorial contribution) is sourced by the scalar.

Prospects

- Solve the inhomogeneous ER wave equation for meaningful choices of $\omega(\phi)$
- First try: $f = 1 - \phi$ leads to $\phi = \ln \left| \ln \left(\mu r^{\lambda} \right)^{-1} \right|^{-1}$, singular not only in the origin, but also at some finite cylinder radius →





Then the source term in the inhomogeneous ER eq. also has singularities, but has compact support

- Find a particular solution of the inhomogeneous ER eq. + generic ER solution from GR → the general solution
- Possibility to solve numerically the system of all equations
- Incorporate a potential into the discussion, in order to be able to match to a pulsating cylinder (string) as source of the ER GWs

(Leandros' suggestion, thanks !)

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