Generating primordial fluctuations from modified teleparallel gravity

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[Gonzalez-Espinoza, Otalora, PLB **809**, 2020]
[Gonzalez-Espinoza, Herrera, Otalora, Saavedra, EPJC **81**, 2021]

Content

Slow-Roll Inflation

The action for the inflaton field ϕ, minimally coupled to Einstein gravity is

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right] \tag{1}$$

where R is curvature scalar derived from the metric of the space-time $g_{\mu\nu}$ and $V(\phi)$ is the inflationary scalar potential.

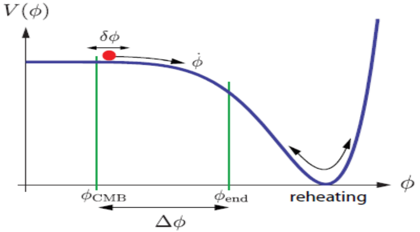

Figure: 1. Scalar potential. This classical dynamics solve the problems of the Big Bang standard model, e.g. the horizon problem, the flatness problem, etc [Baumann arXiv:0907.542].

Quantum Fluctuations During Inflation

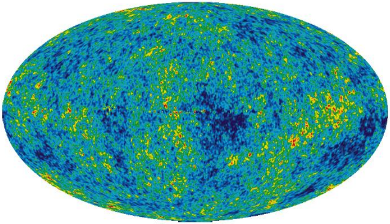

Figure: 2 . Inflation also explains why the CMB has small inhomogeneities.

The power spectrum of the curvature fluctuation

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}, \quad \mathcal{P}_s \equiv \frac{k^3}{2\pi^2} P_{\mathcal{R}} \tag{2}$$

And the spectral index

$$n_s - 1 \equiv \frac{d \ln \mathcal{P}_s}{d \ln k} \tag{3}$$

The power spectrum for tensor perturbations

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_h \quad \mathcal{P}_t \equiv \frac{k^3}{\pi^2} P_h \tag{4}$$

And the spectral index

$$n_t \equiv \frac{d \ln \mathcal{P}_t}{d \ln k} \tag{5}$$

The tensor-to-scalar ratio is

$$r \equiv \frac{\mathcal{P}_t}{\mathcal{P}_s} \tag{6}$$

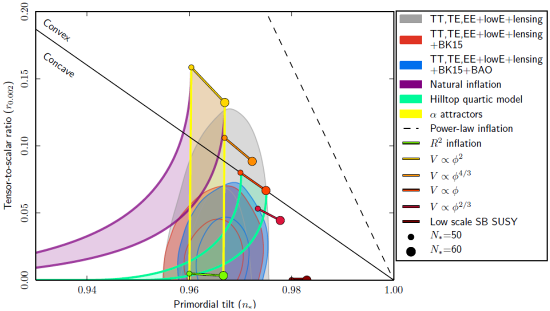

Figure: 3. $n_s - r$ plane and latest Planck data [Akrami *et al.* Planck, 2020].

Teleparallel Gravity

The dynamical variable is $e_A(x^\mu)$. This is related to metric by

$$g_{\mu\nu} = e^A_{\ \mu}\, e^B_{\ \nu}\, \eta_{AB} \tag{7}$$

where $e^A_{\ \mu}$ are the tetrad components in a coordinate base. The action of TG is given by [Aldrovandi, Pereira, Springer 2013)]

$$S = -\frac{M_{pl}^2}{2} \int d^4x\, e\, T \tag{8}$$

being T the torsion scalar, $e = \det\left(e^A_{\ \mu}\right) = \sqrt{-g}$. The torsion scalar is defined as

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\mu\rho} - T_{\rho\mu}{}^{\rho} T^{\nu\mu}{}_{\nu}, \quad \text{where} \tag{9}$$

$$T^{\rho}{}_{\mu\nu} \equiv e_A{}^{\rho} \left[\partial_\mu e^A_{\ \nu} - \partial_\nu e^A_{\ \mu} + \omega^A{}_{B\mu}\, e^B_{\ \nu} - \omega^A{}_{B\nu}\, e^B_{\ \mu}\right] \tag{10}$$

are the components of torsion tensor. The spin connection of TG is

$$\omega^A{}_{B\mu} = \Lambda^A{}_D(x)\partial_\mu \Lambda_B{}^D(x) \quad \rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} = e_A{}^{\rho}\left(\partial_\nu e^A_{\ \mu} + \omega^A{}_{B\nu} e^B_{\ \mu}\right) \tag{11}$$

which is the so-called Weitzenböck connection. Then, it can be shown that

$$T = -R - e^{-1}\partial_\mu (e T^{\nu\mu}{}_{\nu}) \tag{12}$$

This equation shows that TG and GR are equivalent theories.

Generalized scalar-torsion $f(T,\phi)$ gravity

We start with the action [Hohmann,Järv, Ualikhanova PRD 97, 2018]

$$S = \int d^4x\, e\, [f(T,\phi) + P(\phi)X] \tag{13}$$

where f is an arbitrary function of ϕ and the torsion scalar T, and also $X = -\partial_\mu\phi\partial^\mu\phi/2$. Varying the action with respect to the tetrad field $e^A_{\ \mu}$

$$f_{,T}G_{\mu\nu} + S_{\mu\nu}{}^{\rho}\partial_\rho f_{,T} + \frac{1}{4}g_{\mu\nu}\left(f - Tf_{,T}\right) + \frac{P}{4}\left(g_{\mu\nu}X + \partial_\mu\phi\partial_\nu\phi\right) = 0 \tag{14}$$

These equations are not symmetric. Under an infinitesimal Lorentz transformation the variation of the action is

$$\delta S = \int d^4x e\partial_\rho f_{,T} S_{\mu\nu}{}^{\rho}\xi^{\mu\nu} \text{ where } \xi^{\mu\nu} = e_A{}^{\mu} e_B{}^{\nu}\xi^{AB},\ \ \xi^{AB} = -\xi^{BA} \tag{15}$$

The condition $\delta S = 0$ for arbitrary ξ^{AB} leads us to

$$\partial_\rho f_{,T} S_{[\mu\nu]}{}^{\rho} = 0 \tag{16}$$

For TG one has $\partial_\rho f_{,T} = 0$, and then local Lorentz invariance is restored. For MTG one has $\partial_\rho f_{,T} \neq 0$ and then it corresponds to a set of six equations.

Cosmological Background

We impose the standard homogeneous and isotropic background geometry

$$e^A_{\ \mu} = \mathrm{diag}(1, a, a, a) \ \ \rightarrow \ \ ds^2 = -dt^2 + a^2\,\delta_{ij}dx^i dx^j \tag{17}$$

which gives the flat FRW metric This is a proper tetrad and then we can choose $\omega^A_{\ B\mu} = 0$ [Kršš́ák, Saridakis, CQG. **33**, 2016]. We obtain the background equations

$$f(T,\phi) - P(\phi)X - 2Tf_{,T} = 0 \tag{18}$$

$$f(T,\phi) + P(\phi)X - 2Tf_{,T} - 4\dot{H}f_{,T} - 4H\dot{f}_{,T} = 0 \tag{19}$$

$$P(\phi)\ddot{\phi} + 3P(\phi)H\dot{\phi} + P_{,\phi}X - f_{,\phi} = 0 \tag{20}$$

where $H \equiv \dot{a}/a$ and $T = 6H^2$. We introduce the slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad \delta_{PX} = -\frac{P(\phi)X}{2H^2 f_{,T}}, \quad \delta_{f_{,T}} = \frac{\dot{f}_{,T}}{f_{,T}H}, \quad \delta_P = \frac{\dot{P}}{HP}, \quad \delta_\phi = \frac{\ddot{\phi}}{H\dot{\phi}} \tag{21}$$

Then, the background equations (18) and (19) can be written as

$$\epsilon = \delta_{PX} + \delta_{f_{,T}} \quad \text{where} \quad \delta_{f_{,T}} = \delta_{f\dot{H}} + \delta_{fX} \tag{22}$$

where we have defined

$$\delta_{f\dot{H}} = \frac{f_{,TT}\dot{T}}{Hf_{,T}}, \quad \delta_{fX} = \frac{f_{,T\phi}\dot{\phi}}{Hf_{,T}} \tag{23}$$

Scalar Perturbations

We start from ADM decomposition [Wu, Geng,PRD **86**, 2012]

$$e^0{}_\mu = (N, \mathbf{0}) \;\; e^a{}_\mu = (N^a, h^a{}_i) \;\; e_0{}^\mu = \left(1/N, -N^i/N\right) \;\; e_a{}^\mu = \left(0, h_a{}^i\right) \tag{24}$$

where $N^i = h_a{}^i N^a$, with $h^a{}_j h_a{}^i = \delta^i_j$. In the uniform field gauge, $\delta\phi = 0$, we take

$$N = 1 + \alpha, \quad N^a = a^{-1} e^{-\mathcal{R}} \delta^a{}_i \partial^i \psi, \quad h^a{}_i = a e^{\mathcal{R}} \delta^a{}_j \delta^j{}_i \tag{25}$$

which gives the corresponding perturbed metric

$$ds^2 = -\left[(1+\alpha)^2 - a^{-2} e^{-2\mathcal{R}} (\partial\psi)^2\right] dt^2 + 2\partial_i \psi dt dx^i + a^2 e^{2\mathcal{R}} \delta_{ij} dx^i dx^j \tag{26}$$

The additional degrees of freedom can be incorporated through

$$\Lambda^A{}_B = (e^\chi)^A{}_B = \delta^A{}_B + \chi^A{}_B + \frac{1}{2}\chi^A{}_C \chi^C{}_B + \mathcal{O}(\chi^3) \tag{27}$$

$$e'^A_\mu = (e^\chi)^A{}_B \, e^B{}_\mu = e^A{}_\mu + \chi^A{}_B e^B{}_\mu + \frac{1}{2}\chi^A{}_C \chi^C{}_B e^B{}_\mu + \mathcal{O}(\chi^3) \tag{28}$$

$\chi_{AB} = -\chi_{BA}$ is parameterized as $\chi^0{}_B = (0, \chi_b)$, $\chi^a{}_B = (\chi^a, B^a{}_b)$ and then

$$\chi^i = h_a{}^i \chi^a = \partial_i \beta + \chi_i^{(T)} \text{ and } B_{ij} = h^a{}_i h^b{}_j B_{ab} = -B_{ji} = -\epsilon_{jik} B^k \tag{29}$$

where β is a scalar mode [Wu,PLB **762**,2016]

Second order action

Expanding the action up to second order and integrating out α, $\partial^2\psi$ and $\partial^2\beta$

$$S^{(2)} = \int dtd^3xa^3Q_s\left[\dot{\mathcal{R}}^2 - \frac{c_s^2}{a^2}(\partial\mathcal{R})^2 - m^2\mathcal{R}^2\right] \quad \text{where} \tag{30}$$

$$Q_s = \frac{PX}{H^2}, \quad c_s^2 = 1, \quad \eta_{\mathcal{R}} = \frac{m^2}{3H^2} = \delta_{f,T}\left[1 + \left(1 + \frac{\delta_{fX}}{\delta_{PX}}\right)\frac{\delta_{f,T}}{\delta_{f\dot{H}}}\right] \tag{31}$$

This represents the effects of local Lorentz violation in MTG.
For the non-minimally coupled scalar field model

$$\delta_{fX} = \frac{f_{,T\phi}\dot{\phi}}{Hf_{,T}} \neq 0 \text{ and } \delta_{f\dot{H}} = \frac{f_{,TT}\dot{T}}{Hf_{,T}} = 0 \quad \rightarrow \quad \eta_{\mathcal{R}} = \infty \tag{32}$$

However, this can be solved in the presence of a non-linear coupling. Also, in the absence of coupling between T and ϕ, one has

$$\delta_{fX} = \frac{f_{,T\phi}\dot{\phi}}{Hf_{,T}} = 0 \quad \rightarrow \quad \eta_{\mathcal{R}} = 2\delta_{f,T} = 2\delta_{f\dot{H}} \sim \mathcal{O}(\epsilon) \tag{33}$$

This is the explicit mass term arising in $f(T)$ gravity plus scalar field. Thus, for TG, $f \sim T$, one has

$$\delta_{f\dot{H}} = \frac{f_{,TT}\dot{T}}{Hf_{,T}} = 0 \quad \rightarrow \quad \eta_{\mathcal{R}} = 0 \tag{34}$$

which is consistent with the local Lorentz invariance of TG

Mukhanov-Sasaki equation

We introduce the Mukhanov variable $v \equiv z\mathcal{R}$, and $z^2 = 2a^2 Q_s$ and $d\tau = dt/a$

$$S^{(2)} = \frac{1}{2}\int d\tau d^3x \left[(v')^2 - c_s^2(\partial v)^2 - M^2 v^2\right], \quad M^2 = a^2 m^2 - \frac{z''}{z} \tag{35}$$

where $m^2 = 3H^2\eta_{\mathcal{R}}$. Varying the action and using the Fourier expansion

$$v_k'' + \left[k^2 - \frac{1}{\tau^2}\left(\tilde{\nu}^2 - \frac{1}{4}\right)\right] v_k = 0, \quad \tilde{\nu} = \nu - \eta_{\mathcal{R}} = \frac{3}{2} + \epsilon + \frac{1}{2}\eta - \eta_{\mathcal{R}} \tag{36}$$

For $\tilde{\nu}$ constant and real, the exact solution to (36) is

$$v_k(\tau) = \sqrt{-\tau}\left[C_1 H_{\tilde{\nu}}^{(1)}(-k\tau) + C_2 H_{\tilde{\nu}}^{(2)}(-k\tau)\right] \tag{37}$$

where $H_{\tilde{\nu}}^{(1)}$ and $H_{\tilde{\nu}}^{(2)}$ are the Hankel's functions [Riotto, Notes 2003]. By imposing the Bunch-Davies vacuum $v_k(\tau) = e^{-ik\tau}/\sqrt{2k}$ at $k \gg aH$ $(-k\tau \ll 1)$

$$v_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}\left(\tilde{\nu}+\frac{1}{2}\right)}(-\tau)^{\frac{1}{2}} H_{\tilde{\nu}}^{(1)}(-k\tau) \tag{38}$$

On super-horizon scales $k \ll aH$ $(-k\tau \to 0)$ one finds

$$v_k(\tau) = e^{i\frac{\pi}{2}\left(\tilde{\nu}-\frac{1}{2}\right)} 2^{\tilde{\nu}-\frac{3}{2}} \frac{\Gamma(\tilde{\nu})}{\Gamma\left(\frac{3}{2}\right)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\tilde{\nu}} \tag{39}$$

Primordial Scalar Power Spectrum

We obtain on superhorizon scales [Gonzalez-Espinoza, Otalora, PLB **809**, 2020]

$$|\mathcal{R}_k| \simeq \frac{H}{2\sqrt{k^3 Q_s}} \left(\frac{k}{aH}\right)^{-\epsilon-\eta/2+\eta_{\mathcal{R}}} \simeq \frac{H_k}{2\sqrt{k^3 Q_{sk}}} \left[1 + \eta_{\mathcal{R}} \ln\left(\frac{k}{aH}\right)\right] \quad (40)$$

where $$\eta = \frac{\dot{Q}_s}{HQ_s} = \delta_P + 2\delta_\phi + 2\epsilon \quad (41)$$

and H_k and Q_{sk} are the values of H and Q_s at $k = aH$.
The scalar power spectrum is given by

$$\mathcal{P}_s(k) \equiv \frac{k^3}{2\pi^2} |\mathcal{R}_k(\tau)|^2 \simeq \frac{H_k^2}{8\pi^2 Q_{sk}} \left[1 + 2\eta_{\mathcal{R}} \ln\left(\frac{k}{aH}\right)\right] \quad (42)$$

Then the consequence of local Lorentz violation is a slight logarithmic time-dependence of $\mathcal{R}$ and $\mathcal{P}_s$ at superhorizon scales.
Finally, the scale-dependence of the scalar power spectrum is

$$n_s - 1 \equiv \left.\frac{d\ln\mathcal{P}_s(k)}{d\ln k}\right|_{k=aH} = -2\epsilon - \eta + 2\eta_{\mathcal{R}} \quad (43)$$

This carries out the effects of local Lorentz violation on the scalar power spectrum through the term $2\eta_{\mathcal{R}}$, at first-order in slow-roll approximation.

Primordial Tensor Power Spectrum

Then, using the tetrad formalism we find the second-order action for the tensor modes, $h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij}$, in the way

$$S_T = \sum_\lambda \int dt d^3x a^3 Q_T \left[\dot{h}_\lambda^2 - \frac{c_T^2}{a^2} \left(\partial h_\lambda\right)^2 \right] \text{ where } Q_T = -\frac{1}{2} f_{,T}, \quad c_T^2 = 1 \quad (44)$$

The non-ghost condition is satisfied only for $f_{,T} < 0$. There are no additional propagating modes in the quadratic action. Thus, the power spectrum for tensor perturbations becomes

$$\mathcal{P}_T = \frac{H_k^2}{2\pi^2 Q_{Tk}} \quad (45)$$

with H_k and Q_{Tk} the values of H and Q_T at $k = aH$.
Tensor-to-scalar ratio, evaluated at the horizon crossing, is given by

$$r = \frac{\mathcal{P}_T}{\mathcal{P}_s} \simeq 16\delta_{PX} = 16\left(\epsilon - \delta_{f_{,T}}\right) \quad (46)$$

Also, we can obtain the consistency relation

$$r = 8\left(-n_T - 3\delta_{f_{,T}}\right) \text{ where } n_T \equiv \left.\frac{d\ln \mathcal{P}_T}{d\ln k}\right|_{k=aH} = -2\epsilon - \delta_{f_{,T}} \quad (47)$$

This is agreement with the standard inflation limit where $r = -8n_T$.

Reconstructing inflation

In the slow-roll regime, ϵ, δ_{PX}, $\delta_\phi \ll 1$, and we have (Eqs. (18) and (20))

$$\begin{aligned} f(T,\phi) &\simeq 2Tf_{,T}, && (48)\\ 3P(\phi)H\dot{\phi} &\simeq f_{,\phi}. && (49)\end{aligned}$$

We consider [Gonzalez-Espinoza, Herrera, Otalora, Saavedra, EPJC **81**, 2021]

$$f(T,\phi) = -\frac{M_{pl}^2}{2}T - G(T)F(\phi) - V(\phi), \quad (50)$$

where $G(T) \sim T^{1+s}$, s a constant. Thus, the background equations (48), (49) take the form

$$\begin{aligned} \frac{M_{pl}^2}{2}T + (2s+1)FT^{s+1} &\simeq V, && (51)\\ \frac{\dot{\phi}}{M_{pl}H} &\simeq -\left[\frac{2GF_{,\phi}}{M_{pl}T} + \frac{2V_{,\phi}}{M_{pl}T}\right]. && (52)\end{aligned}$$

Below, we introduce the number of $e-$folds N

$$N = \log(a_f/a) = \int_t^{t_f} H dt = \int_\phi^{\phi_f} \frac{H}{\dot{\phi}} d\phi. \quad (53)$$

Assuming $T = T(\phi)$, $\phi = \phi(N)$, and performing

$$V_{,\phi} = \frac{V_{,N}}{\phi_{,N}}, \quad F_{,\phi} = \frac{F_{,N}}{\phi_{,N}}, \tag{54}$$

and

$$V_{,\phi\phi} = \frac{V_{,NN}}{\phi_{,N}^2} - \frac{V_{,N}\phi_{,NN}}{\phi_{,N}^3}, \quad F_{,\phi\phi} = \frac{F_{,NN}}{\phi_{,N}^2} - \frac{F_{,N}\phi_{,NN}}{\phi_{,N}^3}, \tag{55}$$

from Eq. (52) we find

$$\phi_{,N} = \sqrt{\frac{2}{T}\left(F_{,N}G + V_{,N}\right)}\ , \tag{56}$$

and therefore

$$\begin{aligned} n_s(N) - 1 \quad &= \quad T_{,N}\left[\frac{1}{T} - \frac{F_{,N}G_{,T}}{F_{,N}G + V_{,N}}\right] + \frac{V_{,NN}}{F_{,N}G + V_{,N}} + \\ &\frac{F_{,NN}G}{F_{,N}G + V_{,N}} - \frac{4\left(F_{,N}G + V_{,N}\right)}{T\left(M_{pl}^2 + 2FG_{,T}\right)} - \\ &\frac{2F_{,N}\left(G - 2TG_{,T}\right) + V_{,N}}{T\left(M_{pl}^2 + 2F\left(2TG_{,TT} + G_{,T}\right)\right)} - \frac{2F_{,N}^2G_{,T}^2}{FG_{,TT}\left(F_{,N}G + V_{,N}\right)}, \end{aligned} \tag{57}$$

and

$$r(N) \quad = \quad \frac{16\left(F_{,N}G + V_{,N}\right)}{T\left(M_{pl}^2 + 2FG_{,T}\right)}. \tag{58}$$

High-energy limit

In order to obtain $T = T(\phi)$ we use the approximation

$$\frac{M_{pl}^2}{2} T + (2s+1)FT^{s+1} \simeq (2s+1)FT^{s+1} = V, \quad \rightarrow \quad T \simeq \left[\frac{V}{(2s+1)F}\right]^{1/(s+1)}. \quad (59)$$

Applying this limit to $n_s(N)$ (57) and $r(N)$ (58), and combining the resulting expressions with $r_{,N}$ we obtain

$$F(N) = F_0 \exp\left[\int_{N_0}^{N} \frac{s\left(-3r \pm \sqrt{A}\right)}{32}\, dN'\right], \quad (60)$$

$$V(N) = V_0 \exp\left[\int_{N_0}^{N} \frac{(7s+4)r \mp s\sqrt{A}}{32(1+2s)}\, dN'\right], \quad (61)$$

where

$$A = \frac{r\left[64(2s+1)(1-n_s) - (15s+8)r\right]}{s} + \frac{64(2s+1)r_{,N}}{s} > 0, \quad (62)$$

being that $V_0 > 0$ and $F_0 > 0$ are two integration constants defined as $F(N = N_0) = F_0$ and $V(N = N_0) = V_0$, in which N_0 is such that $N > N_0 > 0$.

Examples

Following [Chiba, PTEP, 073E02, 2015] and [Herrera, PRD, **98**, 023542 (2018)]:

$$\text{Example 1:} \quad n_s(N) = 1 - \frac{2}{N}, \quad \text{and} \quad r(N) = \frac{q}{N}, \tag{63}$$

In this case we obtain:

$$F(\phi) = F_0(\phi - \phi_0)^{\sigma_1}, \quad V(\phi) = V_0(\phi - \phi_0)^{\sigma_2}, \tag{64}$$

where $N(\phi = \phi_0) = N_0$, and F_0, V_0, σ_1 and σ_2 are constants, being that σ_1 and σ_1 depend on s and q.

$$\text{Example 2:} \quad n_s(N) = 1 - \frac{2}{N}, \quad \text{and} \quad r(N) = \frac{q}{N(N+\gamma)}, \tag{65}$$

which leads us to

$$F(\phi) \simeq F_0 \left[1 - \frac{\gamma p_1 e^{-\eta_1(\phi-\phi_0)}}{N_0}\right], \quad V(\phi) \simeq V_0 \left[1 - \frac{\gamma p_2 e^{-\eta_1(\phi-\phi_0)}}{N_0}\right], \tag{66}$$

where p_1, p_2, η_1 and η_2 are constants, and we have assumed $\gamma/N \ll 1$.

Conclusions

- The breaking of local Lorentz symmetry in MTG leads to the existence of additional degrees of freedom that can induce new imprints on the inflationary observables.
- The additional tensor modes are completely cancelled out from the second order action for tensor perturbations, remaining only the usual transverse massless graviton modes, propagating at the speed of light.
- In the second order action for the curvature perturbation, it is observed the emergence of an explicit mass term, which represents the effects of local Lorentz violation. This leads to a slight logarithmic time-dependence of $\mathcal{R}$ and $\mathcal{P}_s$ at superhorizon scales.
- The new contributions to the tensor-to-scalar ratio r coming from MTG can either lower its value bringing it to values more compatible with observations, or raise it too much and then leave it outside the allowed contour regions from the latest Planck data.
- Using very well-known attractors for n_s and r, one can reconstruct the non-minimal coupling and the scalar potential, thus providing results in agreement with observations.

Thank you very much!