Choi-Defined Resource Theories

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Quantum does it better!

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- Unifying theme in quantum information: **quantum is a resource**.
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This idea is made mathematically rigorous with **resource theories**.
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Only a subset of operations can be performed (free operations), dictated by the physical setting:

- the identity channel is free;
- swapping two resource systems is free;
- the composition of free operations is free;
- the tensor product of free operations is free;
- discarding a system is free.
States as resources

**Free states:** states that can be prepared at no cost
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Free states: states that can be prepared at no cost

Main question

Can $\rho$ be converted into $\sigma$ with free operations?

Indeed, we can reach a larger set of states from it.

What if we have just a set of free states? How can we reconstruct the free operations too?

CRNG operations [Chitambar & Gour]

Operations sending free states to free states even when they're applied only to half of a bipartite free state.
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The Choi matrix of $\mathcal{N}_{B\rightarrow C} \circ \mathcal{M}_{A\rightarrow B}$ is given by the link product of their Choi matrices $N_{CB}$ and $M_{BA}$:

$$N_{CB} \ast M_{BA} = \text{tr}_B \left[ (N_{CB} \otimes 1_A) \left( 1_C \otimes M_{BA}^{T_B} \right) \right].$$
The **CD operations** are all and only the quantum channels such that their renormalized Choi matrix is a free state.

**Theorem**

The answer is positive iff

1. $d_A \Phi_{AA'}$ is free;
2. If $\rho_A$ and $\mu_{BA}$ are free states, and $\mu_{BA}$ is the renormalized Choi matrix of a quantum channel, then $d_A \mu_{BA}^{\ast} \rho_A$ is a free state.

In this case CD operations coincide with CRNG operations.
CD operations [Zanoni & CMS]

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Can we always construct a resource theory in this way?

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Theorem

The answer is positive iff

1. \( \frac{1}{d_A} \Phi_{AA'} \) is free;
2. If \( \rho_A \) and \( \mu_{BA} \) are free states, and \( \mu_{BA} \) is the renormalized Choi matrix of a quantum channel, then \( d_A \mu_{BA} * \rho_A \) is a free state.

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Conclusions

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thus providing a concrete construction for CRNG operations.
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- Start from a set of free states...
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- We determined necessary and sufficient conditions for this construction.
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- thus providing a concrete construction for CRNG operations.

Many resource theories have this property: NPT and SEP entanglement, magic, imaginarity...
