Choi-Defined Resource Theories

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This idea is made mathematically rigorous with resource theories.

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  - the tensor product of free operations is free;
  - discarding a system is free.

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CRNG operations [Chitambar & Gour]

Operations sending free states to free states even when they're applied only to half of a bipartite free state.

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Link product [Chiribella et al.]

The Choi matrix of $\mathcal{N}_{B\to C} \circ \mathcal{M}_{A\to B}$ is given by the link product of their Choi matrices N_{CB} and M_{BA}:

$$N_{CB} * M_{BA} = \mathrm{tr}_B \left[(N_{CB} \otimes \mathbb{1}_A) \left(\mathbb{1}_C \otimes M_{BA}^{T_B} \right) \right].$$

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The answer is positive iff

1. $\frac{1}{d_A}\Phi_{AA'}$ *is free;*
2. *If* ρ_A *and* μ_{BA} *are free states, and* μ_{BA} *is the renormalized Choi matrix of a quantum channel, then* $d_A\mu_{BA} * \rho_A$ *is a free state.*

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In this case CD operations coincide with CRNG operations.

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Many resource theories have this property: NPT and SEP entanglement, magic, imaginarity...

References

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