On the SU(2) gauge symmetry in Loop Quantum Cosmology

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Three points program

- Introduction to nondiagonal Bianchi models [Montani, MB ‘23]
  - Minisuperspace and Ashtekar variables
  - Flux quantization procedure

- Abelianization of the Gauss constraint [Montani, MB ‘23]
  - Gauge freedom and canonical transformation
  - Revised Gauss Constraint and Quantum-level implications

- Yang-Mills approach for the cosmological sector [MB ‘24] [MB ‘24]
Nondiagonal Bianchi models
Minisuperspace

Globally hyperbolic spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$

Homogeneous space $\Sigma$ prescription $q_{ij}(t,x) = \eta_{ij}(t)\omega^I_i(x)\omega^J_j(x)$

Nondiagonal metric decomposition $\eta_{IJ} = \Gamma_{AB} R^A_I R^B_J$

Maurer-Cartan equation $d\omega^I + \frac{1}{2} f^I_{JK} \omega^J \omega^K = 0$

Lie algebra generators $[\xi_I, \xi_J] = f^K_{IJ} \xi_K$

Metric configuration variables $\{a_1, a_2, a_3, \theta, \psi, \phi\}$
Nondiagonal Bianchi models

Ashtekar variables

Lagrangian

\[ L_{ADM} = N |\text{det}(\omega^i_J)| \sqrt{\text{det}(\Gamma_{AB})} \left[ \bar{R} + \frac{1}{4N^2} \left( \Gamma^A_{BC} \Gamma^B_{CD} \bar{\Gamma}_{AB} \bar{\Gamma}_{CD} + 2 \Gamma^A_{BC} \Gamma^B_{CD} (R \Lambda)^B_A (R \Lambda)^C_B + 2 (R \Lambda)^B_A (R \Lambda)^C_B \right) + 2N^A N^B (f^I_{AJ} f^J_{BI} + \eta^{IJ} \eta_{KL} f^K_{AI} f^L_{BJ}) + 4N^K \eta^{IJ} \eta_{KL} f^L_{KI} - \Gamma^I_{IJ} \Gamma^K_{KL} \right] \]

Ashtekar connection

\[ A^a_i = \left[ \frac{1}{2} \bar{a}^{abc} a_c \Lambda^j_b R^K_k f^j_{ki} - \frac{1}{4} \bar{a}^{abc} a_c \eta_{ij} \Lambda^K_b \Lambda^L_c e^j_{lk} + \frac{\gamma}{2N} a_{(a)} R^a_A (\eta^{IJ} \eta_{JI} + N^A \eta^{LK} \eta_{IJ} f^J_{AK} + N^A f^L_{AI}) \right] \omega^i_i \]

Electric field

\[ E^i_a = |\text{det}(\omega^i_J)| \text{sgn}(a_{(a)}) |a_b a_c | \Lambda^a_i \xi^i_i \]
Nondiagonal Bianchi I model

Ashtekar variables

Lagrangian

\[ L_{ADM} = N|\det(\omega_i^I)|\sqrt{\det(\Gamma_{AB})} \left[ K + \frac{1}{4N^2} (\Gamma^{AC}\Gamma^{BD}\Gamma_{AB} \Gamma_{CD} + 2\Gamma^{AB}\Gamma_{CD}(R\Lambda)^B_A (R\Lambda)^C_B + 2(R\Lambda)^B_C (R\Lambda)^C_B) + 2N^A N^B (f_{AIJ} f_{BIJ}^J + \eta^I_{LI} \eta_{KL} f_{AI}^K f_{BI}^L) + 4N^K \eta^I_{LI} \eta^J_{LJ} f_{KI}^L - \Gamma^{IJ} \Gamma_{IJ} \Gamma^{KL} \Gamma_{KL} \right] \]

Ashtekar connection

\[ A_i^a = \left[ \frac{1}{2} \varepsilon^{abc} \frac{a_c}{a_b} \Lambda^j_b R^c_l f_{ij}^l - \frac{1}{4} \varepsilon^{abc} \frac{1}{a_b a_c} \eta_{ij} \Lambda^K_b \Lambda^L_c f_{LK}^l + \frac{\gamma}{2N} a^{(a)} R^a_L \left( \eta^L_{JI} \eta_{JI} + N^A \eta^L_{LJ} \eta_{JI} f_{AK}^J + N^A f_{AI}^L \right) \right] \omega_i^I \]

Electric field

\[ E^i_a = |\det(\omega_i^I)| \text{sgn}(a^{(a)}) |a_b a_c| \Lambda^l_a \varepsilon^i_l \]

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Nondiagonal Bianchi I model

Flux quantization

Quantization in the flux polarization as in [Ashtekar, Wilson-Ewing ‘09]

The fluxes computed on the faces of the fiducial cell

Basis states of the Hilbert space: $|p_1, p_2, p_3, \theta, \psi, \phi\rangle$

Credits: Beatrice Gorga

Geometric operators depend on diagonal fluxes only!
Abelianization of the Gauss constraint

Gauge freedom

\[ G_\alpha \big|_{\varrho \, M_t} = 0 \]

\[ \varrho \, M_t = \{ a_1, a_2, a_3, \theta, \psi, \phi, \pi_1, \pi_2, \pi_3, \pi_\theta, \pi_\psi, \pi_\phi \} \]

M. Bojowald’s suggestion in [Bojowald ‘00, ‘13]

\[
\begin{align*}
A_i^{(a)}(t, x) &= \phi_i^{(a)}(t)\omega_i^I(x) \\
E_a^I(t, x) &= |\text{det}(\omega(x))|p_a^I(t)\xi^I_I(x)
\end{align*}
\]

\[
G_\alpha = \epsilon_{abc}\Phi^b_I p_c^I
\]

Recover the gauge freedom adding a rotation

\[ \varrho \, M_t = \{ a_1, a_2, a_3, \theta, \psi, \phi, \alpha, \beta, \gamma, \pi_1, \pi_2, \pi_3, \pi_\theta, \pi_\psi, \pi_\phi, \pi_\alpha, \pi_\beta, \pi_\gamma \} \]

Three abelian constraints

\[
\begin{align*}
\pi_\alpha &= 0 \\
\pi_\beta &= 0 \\
\pi_\gamma &= 0
\end{align*}
\]

Mismatch in the number of degrees of freedom!
Abelianization of the Gauss constraint
Canonical transformation

Lie condition $\phi^a_i \pi^i_n = 0$ provides, perturbative in configurational variables, a linear dependence between Gauss constraint and gauge momenta

**Ansatz**

Gauss constraint is linear in the gauge momenta $G_a = L_{ag} \pi_g$

System of 9 independent equations

$$\epsilon_{abc} = L_{ag} (O^t)^c_d \frac{\partial O^d}{\partial q_g}$$

$$L_{ag} = \begin{pmatrix} -\csc \beta \cos \gamma & \sin \gamma & \cot \beta \cos \gamma \\ \csc \beta \sin \gamma & \cos \gamma & -\cot \beta \sin \gamma \\ 0 & 0 & 1 \end{pmatrix}$$

Admits a unique solution!
Abelianization of the Gauss constraint
The Abelian constraints

The Gauss constraint is recast into three abelian constraints, namely the gauge momenta

\[ G_\alpha = \begin{pmatrix} -\csc \beta \cos \gamma \pi_\alpha & \sin \gamma \pi_\beta & \cot \beta \cos \gamma \pi_\gamma \\ \csc \beta \sin \gamma \pi_\alpha & \cos \gamma \pi_\beta & -\cot \beta \sin \gamma \pi_\gamma \\ 0 & 0 & \pi_\gamma \end{pmatrix} \]

This feature holds at the quantum level \( \hat{G}_\alpha |\Psi\rangle = 0 \iff \hat{\pi}_g |\Psi\rangle = 0 \)

The wavefunction factorizes \( |\Psi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3, \alpha, \beta, \gamma) = \varphi(\alpha, \beta, \gamma) \Phi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3) \)

\[ \hat{\pi}_g |\Psi\rangle = 0 \Rightarrow \varphi = \text{const} \]

The Hilbert space previously defined is the gauge-invariant one

From a SU(2) symmetry, three U(1) appear!
Cosmological sector of Loop Quantum Gravity
A Yang-Mills approach

\[ P^{Spin}(\Sigma) \]
\[ \downarrow \]
\[ P^{SO}(\Sigma) \]
\[ \downarrow \]
\[ \Sigma \]

**Yang-Mills variables**
- Connection \( \omega \) is a 1-form on \( P^{Spin}(\Sigma) \) with value in the Lie algebra of SU(2)
- Dreibein \( e \) is a section in \( P^{SO}(\Sigma) \)

**Ashtekar variables**
- Connection \( A \) is the local field \( A = e^* \omega \)
- Electric field \( E \) is built from the dreibein \( E = \sqrt{q} d^3x \otimes e \)

The request of homogeneity for \( \omega \) yields to a homogeneous geometry for \( \Sigma \)
Configurational space $\mathcal{A} = \{A \mid A = e^*\omega, \omega \text{ homogeneous}\}$

The set of constraints are the same of LQG

Spin-network states as cylindric functions on $\mathcal{A}$

Some properties analogous to the usual cosmological states naturally emerge:

- the spin networks are homogeneous, namely the curves of the graph are integral curves of linear combinations of $\xi_i$
- the invariant states bring pointwise holonomy
Conclusions

- The diagonal quantization in LQC is quite general within the minisuperspace approach.

- The Abelianization of the quantum theory is a feature of the minisuperspace. The three U(1) symmetries arise from decomposing the Gauss constraint in three abelian ones.

- We can identify a cosmological sector with the same constraints as LQG and perform a quantization that yields spin-network states exhibiting properties akin to those in LQC.
Thank you for your attention

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