Type I von Neumann algebras from gravitational path integrals: Ryu–Takayanagi as entropy without holography

Based on: arXiv:2310.02189 with **Xi Dong**, **Donald Marolf** and **Zhencheng Wang**

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QISS — The Quantum Information Structure of Spacetime
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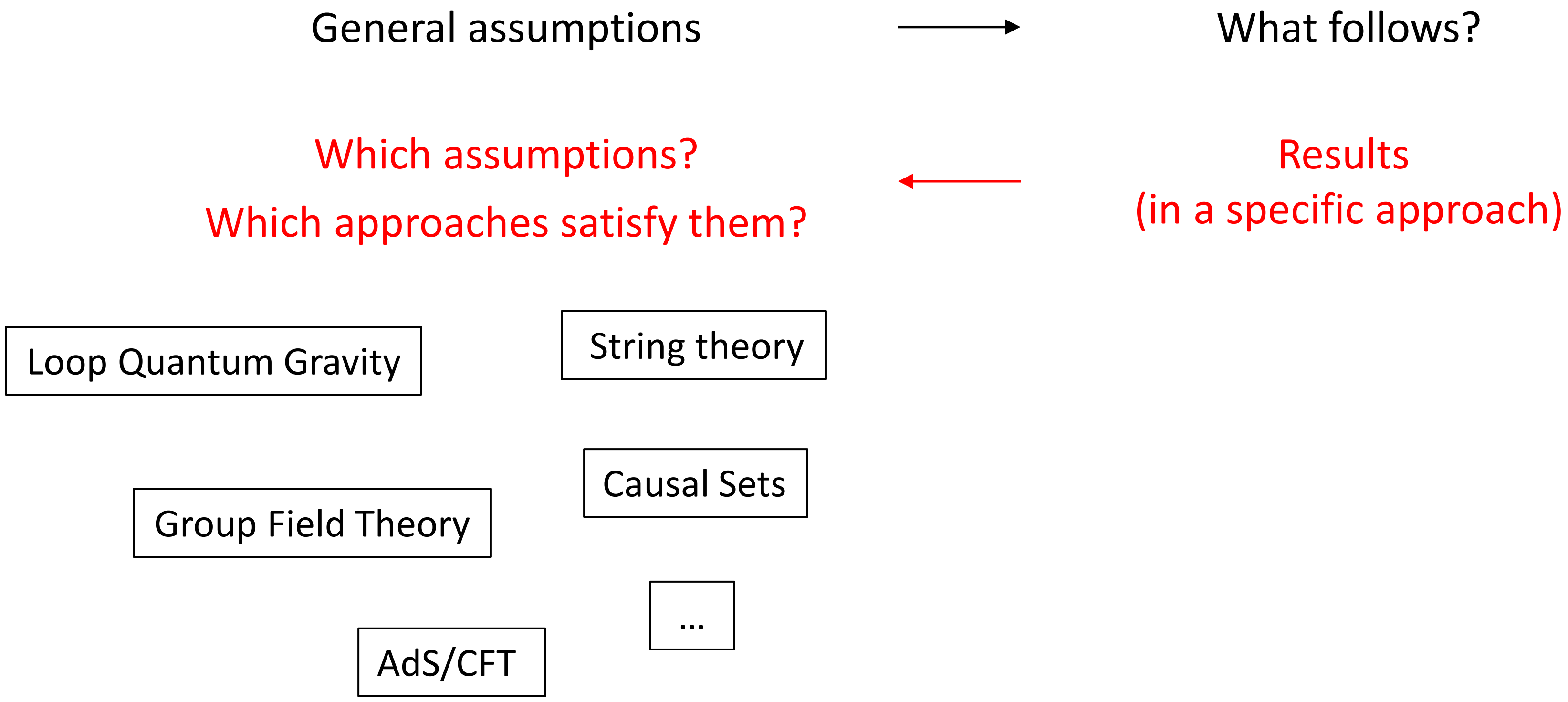
Quantum Gravity Across Different Approaches

General assumptions → What follows?

Quantum Gravity Across Different Approaches



Quantum Gravity Across Different Approaches

General assumptions → What follows?

Which assumptions?
Which approaches satisfy them? ← Results (in a specific approach)

Loop Quantum Gravity

String theory

Causal Sets

Group Field Theory

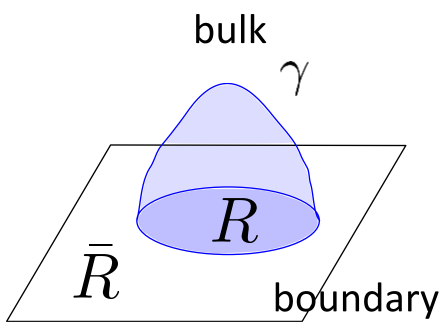
AdS/CFT

...

Entropy in quantum gravity:

Ryu–Takayanagi formula

$$S(R) = \frac{A(\gamma)}{4G}$$



Proposed for holographic entanglement entropy in AdS/CFT

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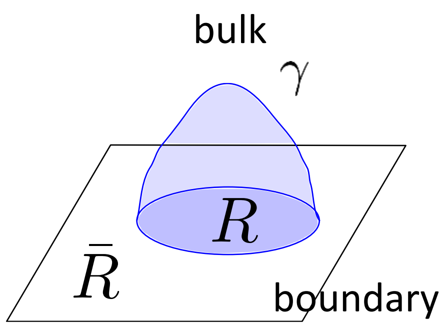

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Can we understand it without holography?

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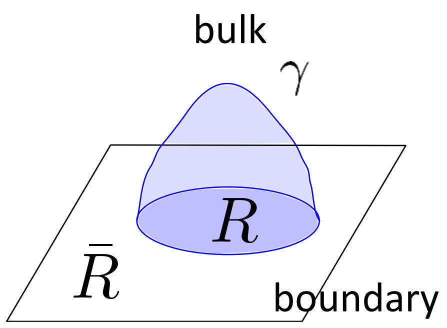


Proposed for holographic entanglement entropy in AdS/CFT

Can we understand it without holography?

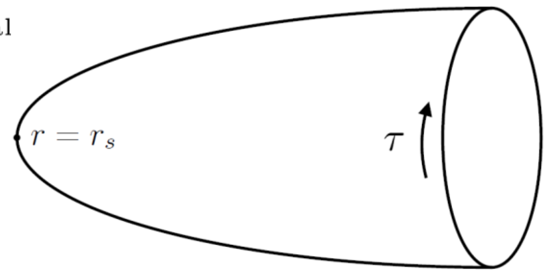
YES!

RT from the Gravitational Path Integral

Ryu–Takayanagi was originally proposed to compute holographic entanglement entropy in AdS spacetime, but the present understanding of the formula is much more general!

Gibbons-Hawking (1977)

$$Z(\beta) = \text{Path integral on the Euclidean black hole} \sim e^{I_{\text{classical}}}$$

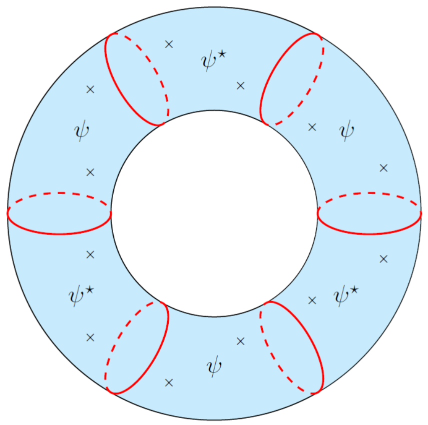


$$S = (1 - \beta\partial\beta)\log Z(\beta) = \frac{A_{\text{horizon}}}{4G}$$

non-trivial part of variation near $r = r_s$

Lewkowycz-Maldacena (2013)

- Path integral prescription for the construction of the state
- Replica trick:
  - consider n copies of the system and compute $\mathrm{Tr}[\rho^n]$
  - analytically continue in n and compute the entropy as



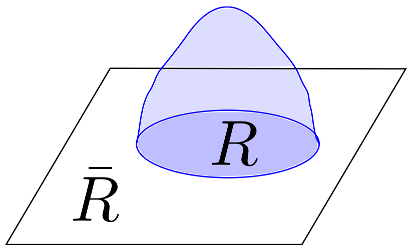
$$S = (1 - n\partial n)\log \mathrm{Tr}[\rho^n]\Big|_{n=1} = \frac{A(\gamma)}{4G}$$

non-trivial part of variation near γ

Where to start?

General assumptions ⟶ **Ryu–Takayanagi formula** without holography?

?



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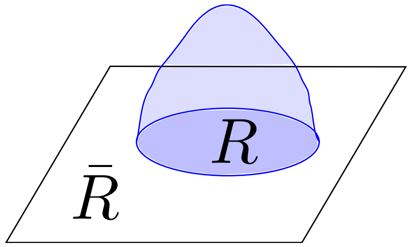
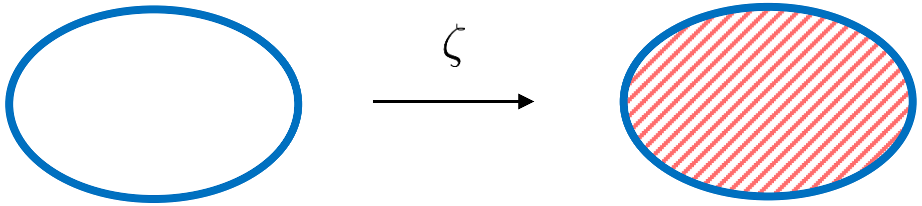


- A UV-complete theory of quantum gravity should contain a map



boundary conditions

M

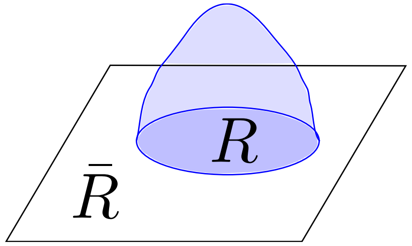
number

$\zeta(M)$

“transition amplitude”

Where to start?

General assumptions ? → **Ryu–Takayanagi formula** without holography?



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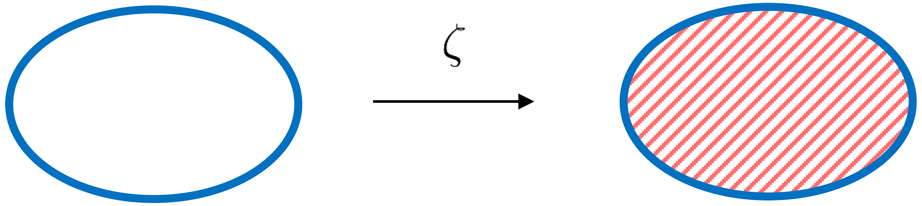


boundary conditions M → number $\zeta(M)$ "transition amplitude"

- We might call this map a (Euclidean) gravitational path integral
- It might look like

$$\zeta(M) = \int_{\mathrm{bc}:M} \mathcal{D}g e^{-S[g]}$$

not a requirement!

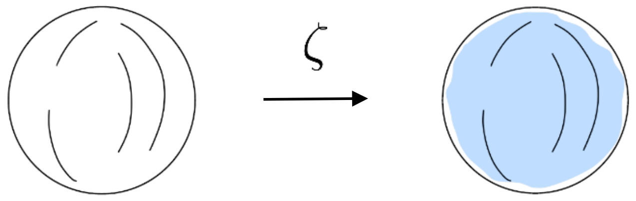
→ assumptions for the gravitational path integral

Outline

1. Axioms
2. Hilbert Space
3. Operator Algebras
4. Type I von Neumann Factors
5. Entropy (with state-counting interpretation)

Axioms

(Euclidean) Gravitational Path Integral



$$M \supset g_M, \phi_M$$

boundary conditions
"source-manifold"

$$\zeta(M) = \int_{\mathrm{bc}:M} \mathcal{D}g\mathcal{D}\phi e^{-S[g,\phi]}$$

Axioms

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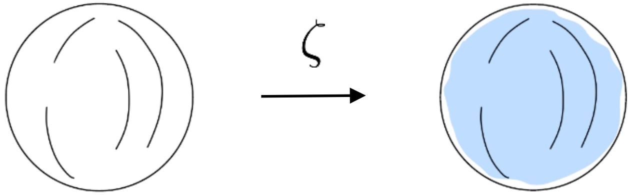

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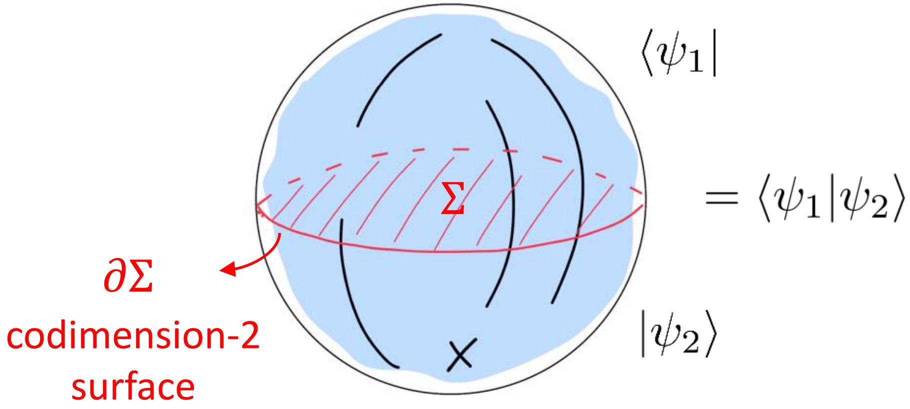

Axioms

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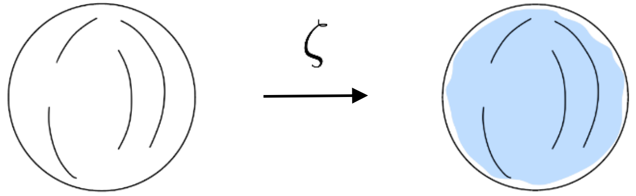

$$M \supset g_M, \phi_M \qquad \zeta(M) = \int_{\text{bc}:M} \mathcal{D}g\mathcal{D}\phi e^{-S[g,\phi]}$$

boundary conditions
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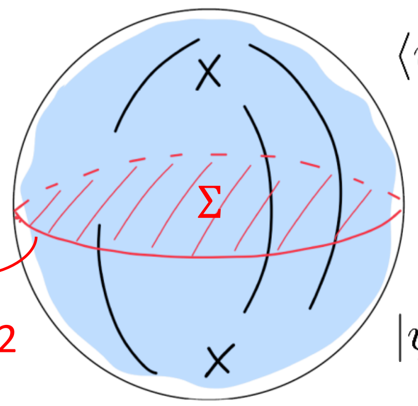


$\langle\psi|$

$= \langle\psi|\psi\rangle \geq 0$

$|\psi\rangle$

$\partial\Sigma$
codimension-2
surface

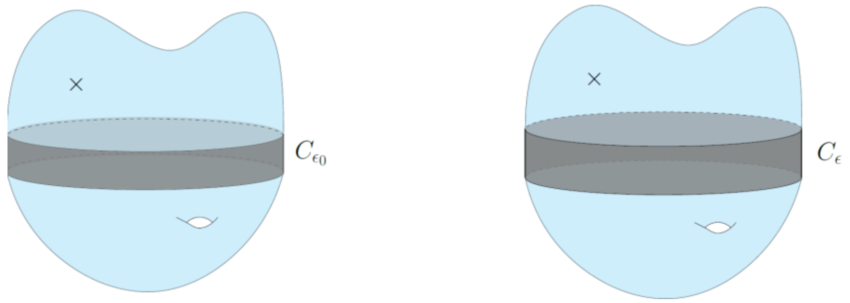
Axioms

1. **Finiteness:** The path integral gives a well-defined map ζ from boundary conditions defined by smooth manifolds to the complex numbers $\mathbb{C}$

2. **Reality:** ζ is a real function of (possibly complex) boundary conditions, i.e. $[\zeta(M)]^* = \zeta(M^*)$

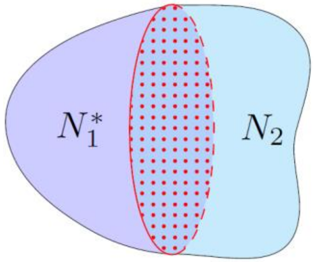
3. **Reflection Positivity**: ζ is reflection-positive

4. **Continuity:** if the boundary manifold contains a cylinder of size ε, ζ is continuous under changes of ε

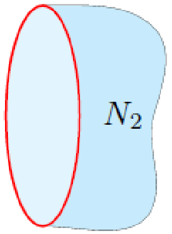


5. **Factorization:** $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\zeta(M_2)$

Hilbert Space

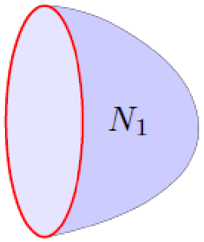


$M_{N_1^* N_2}$



$= |N_2\rangle$

B^{d-2}



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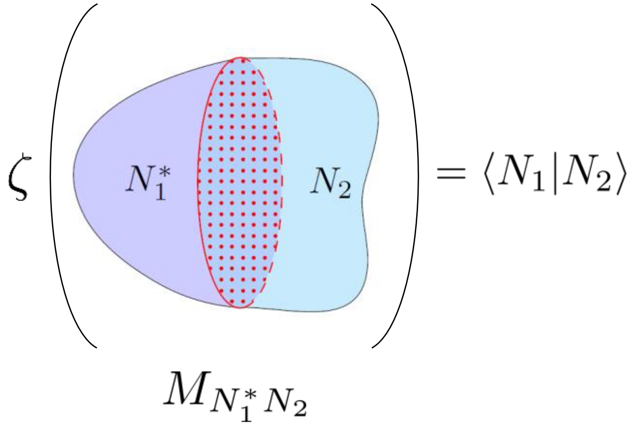
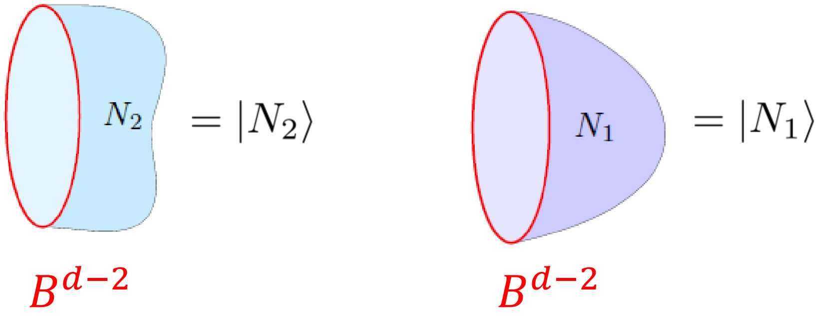
Hilbert Space





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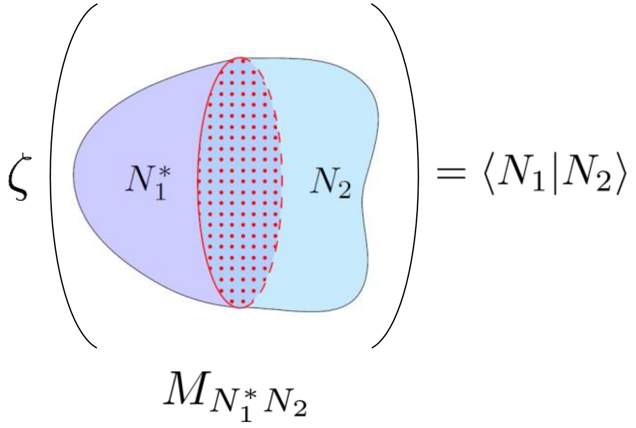
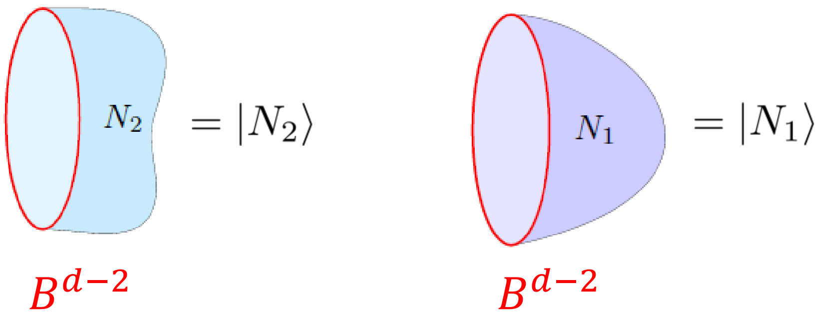




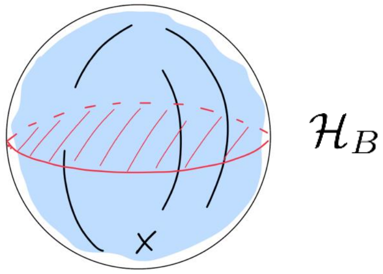
source-manifolds with boundary B

- Define the pre-Hilbert space $H_B = \{|N\rangle : N \in \underline{Y}_B^{d-1}\}$
- Consider the quotient of H_B by its null space and complete the result: Hilbert space $\mathcal{H}_B$
- $\mathcal{H}_B$ is the B-sector of the full quantum gravity Hilbert space:

$$\mathcal{H}_{QG} = \bigoplus_B \mathcal{H}_B$$

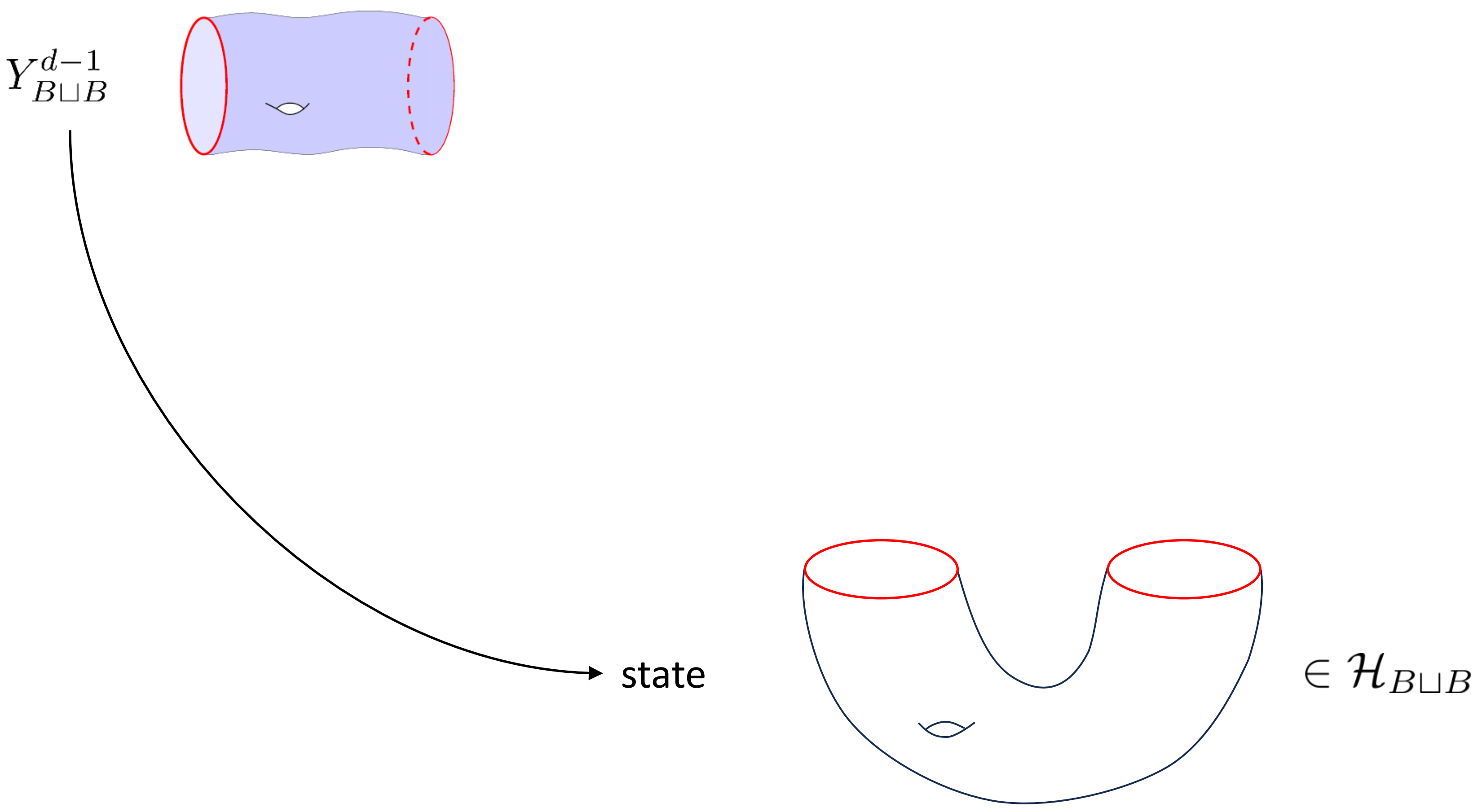
Outline

1. Axioms ✓
2. Hilbert Space ✓



3. Operator Algebras
4. Type I von Neumann Factors
5. Entropy (with state-counting interpretation)

Algebra



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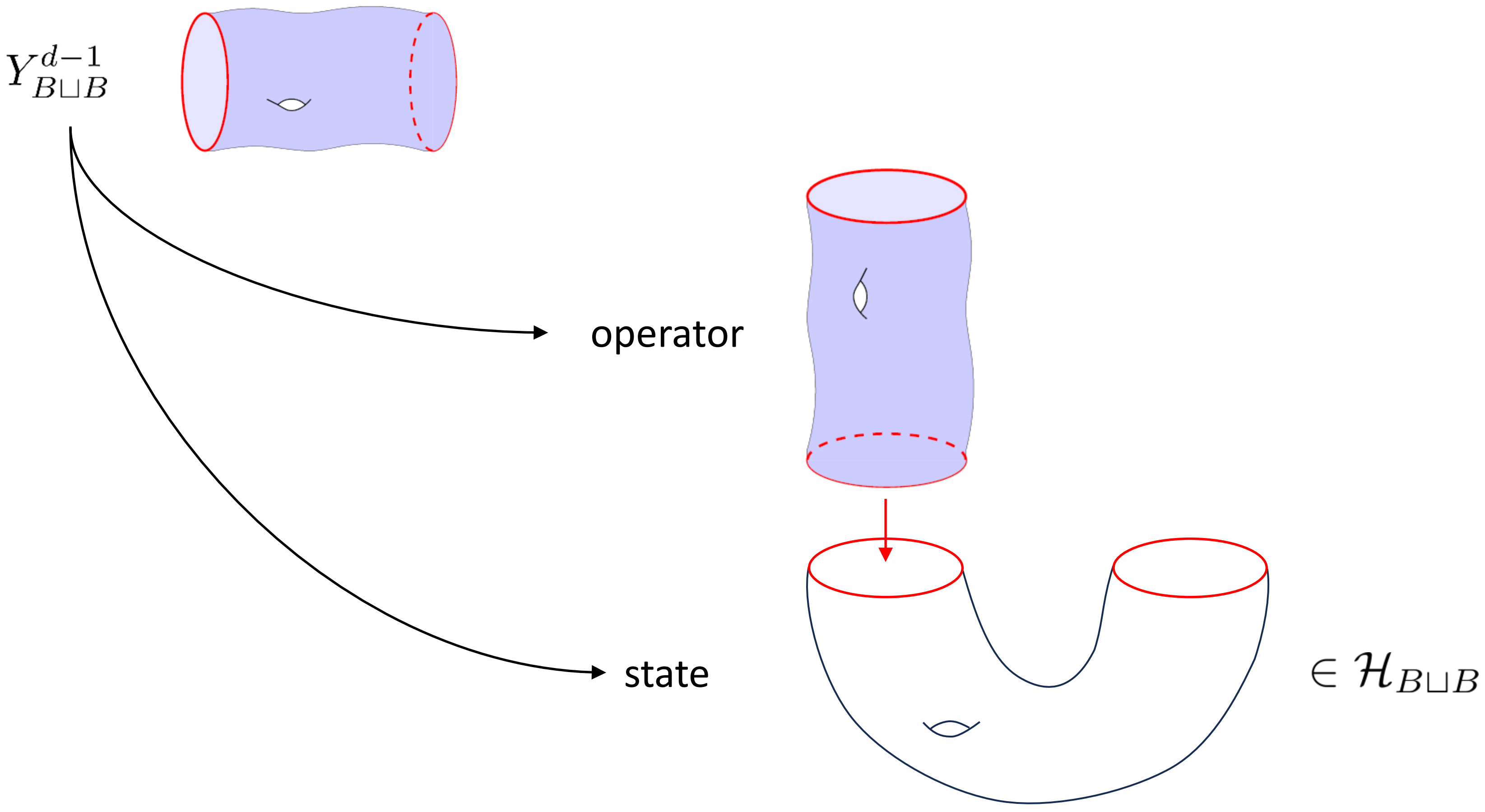

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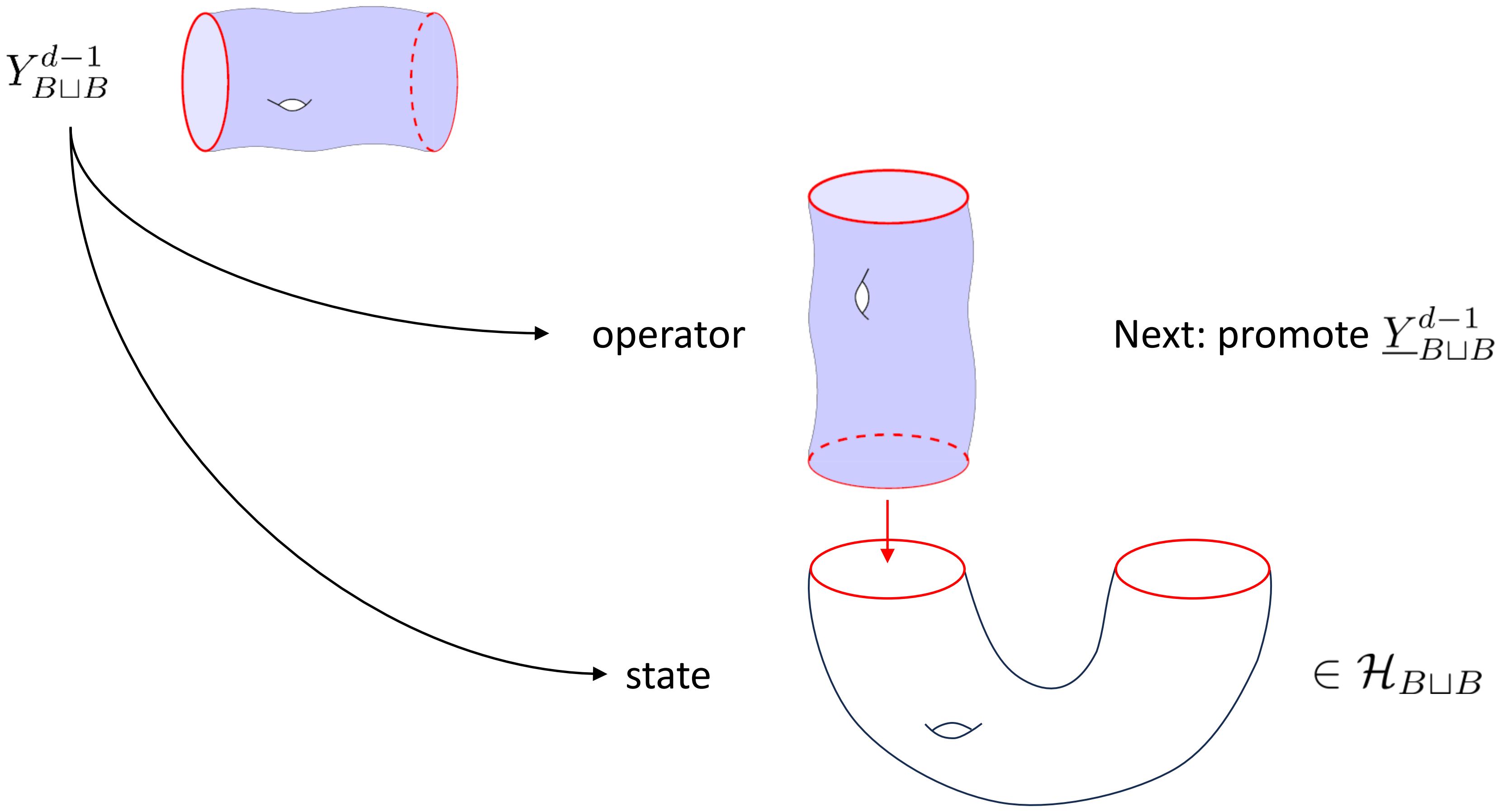
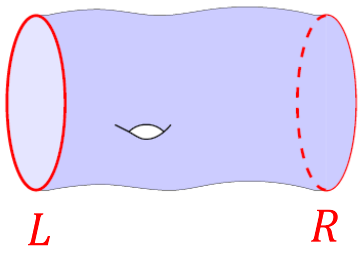
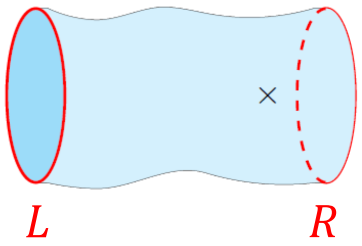


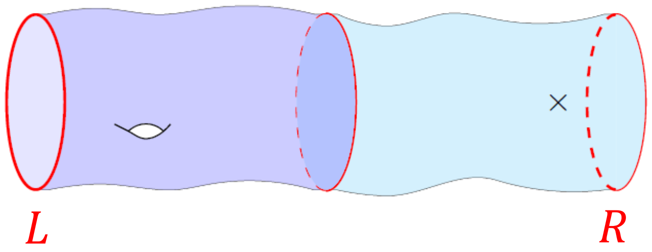
Next: promote $\underline{Y}^{d-1}_{B \sqcup B}$ to an algebra

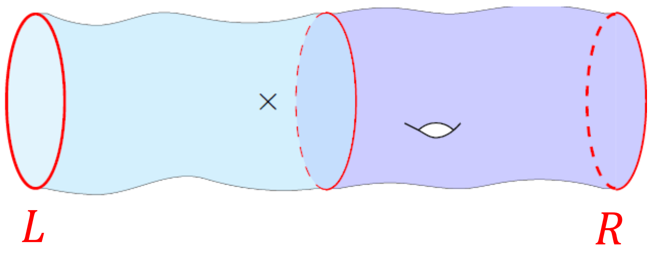
Algebra

- On the set $\underline{Y}^{d-1}_{B\sqcup B}$ we define a *left product* and a *right product*:

$a =$ 

$b =$ 

<u>left product:</u> $(\cdot_L)$ $a \cdot_L b =$ 

<u>right product:</u> $(\cdot_R)$ $a \cdot_R b =$ 

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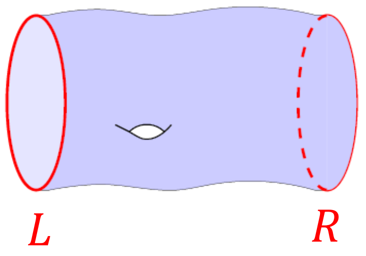
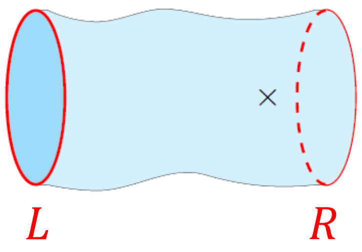
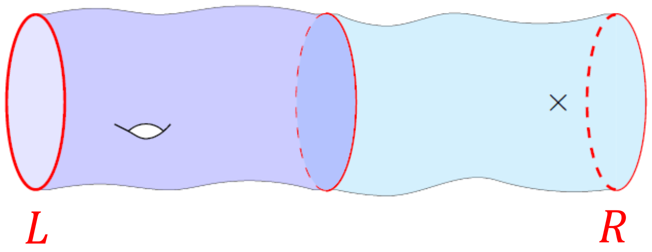
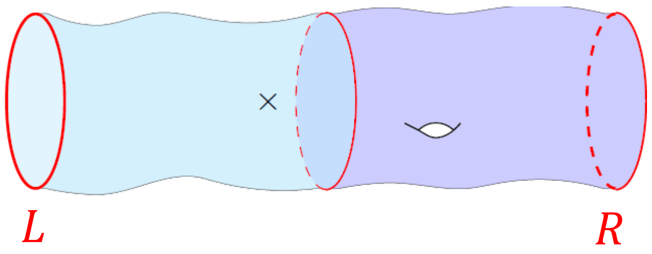
$a =$ 

$b =$ 

<u>left product:</u> $(\cdot_L)$ $\quad a \cdot_L b =$ 

<u>right product:</u> $(\cdot_R)$ $\quad a \cdot_R b =$ 

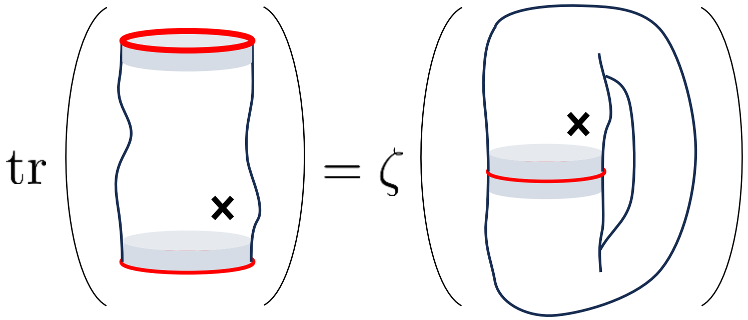
- For convenience $\quad ab := a \cdot_L b = b \cdot_R a$

- The set $\underline{Y}^{d-1}_{B\sqcup B}$ equipped with the left (right) product defines a left (right) surface algebra A_L (A_R)

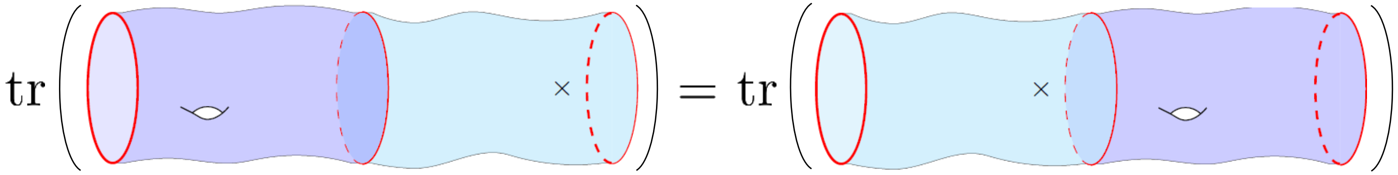
Trace

- The path integral defines a trace operation:

$$\mathrm{tr} : A_{L/R} \to \mathbb{C}$$



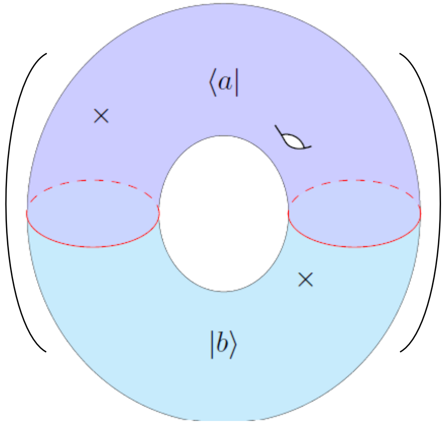
- It satisfies the cyclic property:



Trace

- The trace on A_L and A_R corresponds to the inner product on $\mathcal{H}_{B\sqcup B}$:

$$\langle a|b\rangle = \mathrm{tr}(a^\star b) = \zeta\left(\right)$$



- It is positive-definite: $\mathrm{tr}(a^\star a) = \zeta\left(M(a^\star a)\right) = \langle a|a\rangle \geq 0$

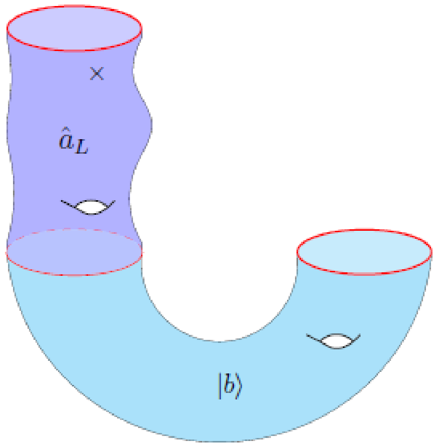
Axiom 3

- We can prove the trace inequality $\mathrm{tr}(aa^\star bb^\star) \leq \mathrm{tr}(a^\star a)\mathrm{tr}(b^\star b)$

Operator algebras

- We define a representation of the left surface algebra on the Hilbert space: given $a \in A_L$ there is an associated operator $\hat{a}_L \in \hat{A}_L$ such that

$$\hat{a}_L \left|b\right\rangle = \left|a \cdot_L b\right\rangle = \left|ab\right\rangle$$



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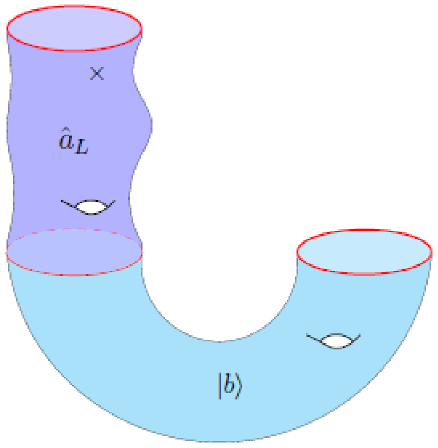
- These operators are **bounded**:

$$|\hat{a}_L|b\rangle|^2 = \langle ab|ab\rangle = \mathrm{tr}(a^\star abb^\star) \leq \mathrm{tr}(a^\star a)\mathrm{tr}(bb^\star) = \mathrm{tr}(a^\star a)\langle b|b\rangle$$

trace inequality



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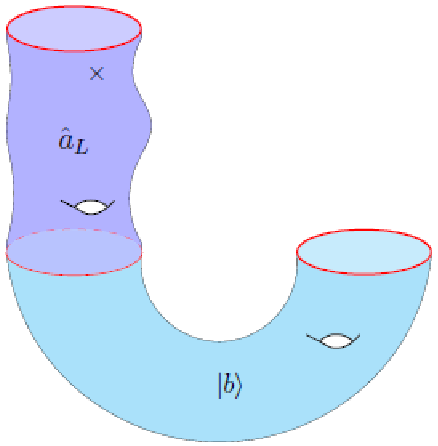

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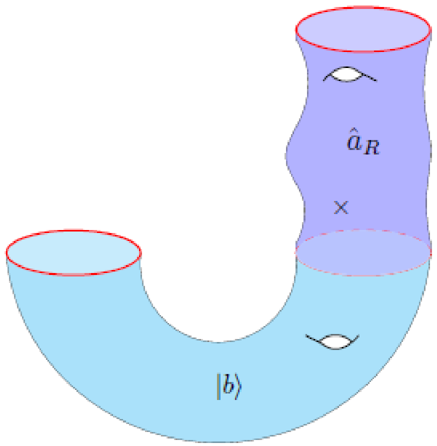
trace inequality

- We can similarly define a representation $\hat{A}_R$ of A_R :



$$\hat{a}_R \, |b\rangle = |a \cdot_R b\rangle = |b \cdot_L a\rangle = |ba\rangle$$

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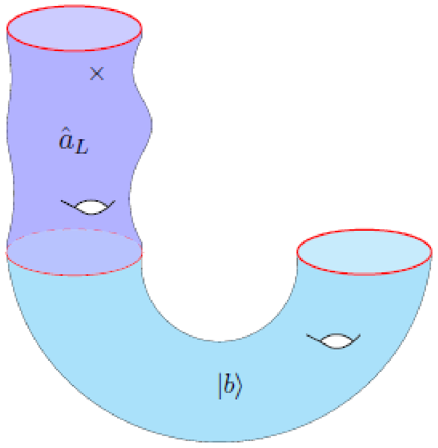

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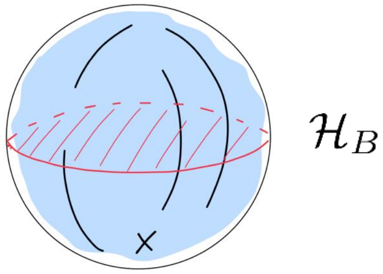
trace inequality

- The operator algebras $\hat{A}_{L/R}$ get a trace from the trace on $A_{L/R}$:

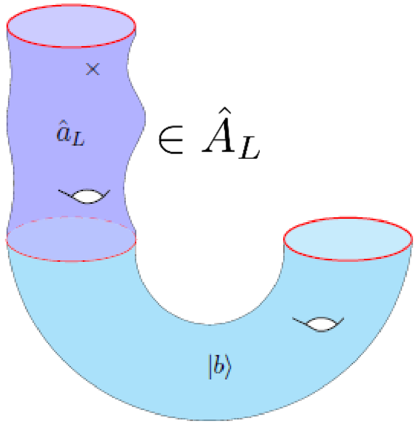
$$\mathrm{tr}(\hat{a}) := \mathrm{tr}(a)$$

Outline

1. Axioms ✓
2. Hilbert Space ✓



3. Operator Algebras ✓



4. Type I von Neumann Factors ←
5. Entropy (with state-counting interpretation)

Type I von Neumann algebras

- We constructed $\hat{A}_L, \hat{A}_R =$ commuting algebras of bounded operators on $\mathcal{H}_{B\sqcup B}$
- We can complete $\hat{A}_L, \hat{A}_R$ to von Neumann algebras $\mathcal{A}_L, \mathcal{A}_R$ by taking the closure in the weak (or strong) operator topology (or taking the double commutant of $\hat{A}_{L/R}$)

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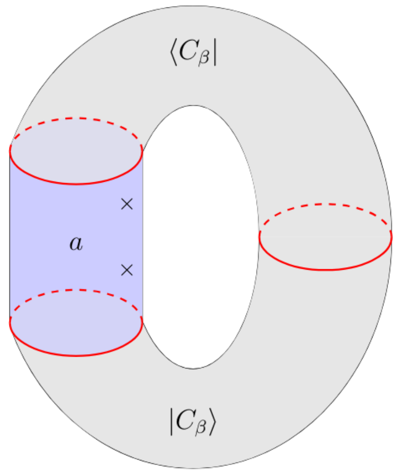
- We show that the trace defined on $\hat{A}_{L/R}$ can be extended to (all positive elements of) the von Neumann algebra:

$$\mathrm{tr}(a) = \lim_{\beta \downarrow 0} \langle C_\beta | a | C_\beta \rangle$$



- We can study the structure of the von Neumann algebras via properties of the trace!

Type I von Neumann algebras

- We can prove that the trace is

1) Faithful $\mathrm{tr}(a) = 0 \text{ iff } a = 0$
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Some known results on von Neumann algebras:

- Every von Neumann algebra is a direct sum or integral of factors (algebras with trivial center)
- These factors can be type I, II or III
- There is *no faithful, normal and semifinite trace on type III* ⇒ we cannot have type III
- on type II, for any faithful, normal and semifinite trace there are *nonzero projections with arbitrarily small trace* ⇒ we cannot have type II

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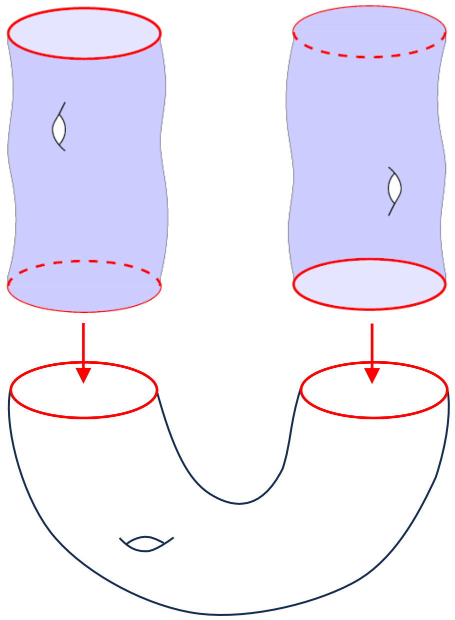
- Therefore, $\mathcal{A}_{L/R}$ is a direct sum/integral of type I factors!
- The spectrum of $z \in \mathcal{Z}_L$ (center of $\mathcal{A}_L$) is discrete

$$\mathcal{A}_L = \bigoplus_{\mu} \mathcal{A}_{L,\mu}$$

Type I von Neumann algebras

- $\mathcal{A}_L, \mathcal{A}_R$ are each other commutants on $\mathcal{H}_{B\sqcup B}$, and so they have the same center $\mathcal{Z}$

$$\mathcal{A}_L = \bigoplus_\mu \mathcal{A}_{L,\mu}$$



$$\mathcal{A}_R = \bigoplus_\mu \mathcal{A}_{R,\mu}$$

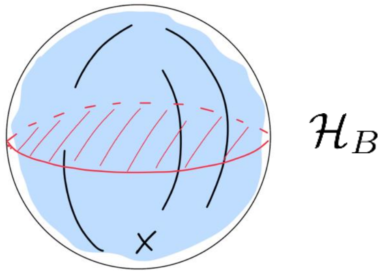
- $\mathcal{H}_{B\sqcup B}$ can be decomposed into eigenspaces of $\mathcal{Z}$

$$\mathcal{H}_{B\sqcup B} = \bigoplus_\mu \mathcal{H}^\mu_{B\sqcup B}$$

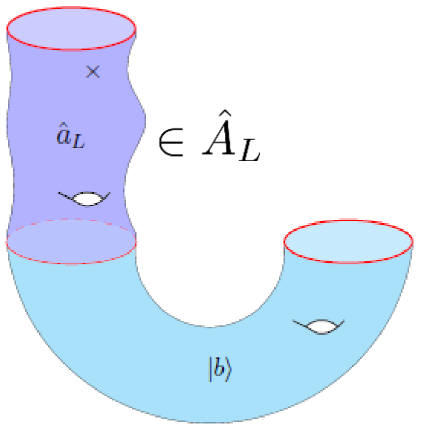
with $\mathcal{H}^\mu_{B\sqcup B} = \mathcal{H}^\mu_{B\sqcup B,L} \otimes \mathcal{H}^\mu_{B\sqcup B,R}$

Outline

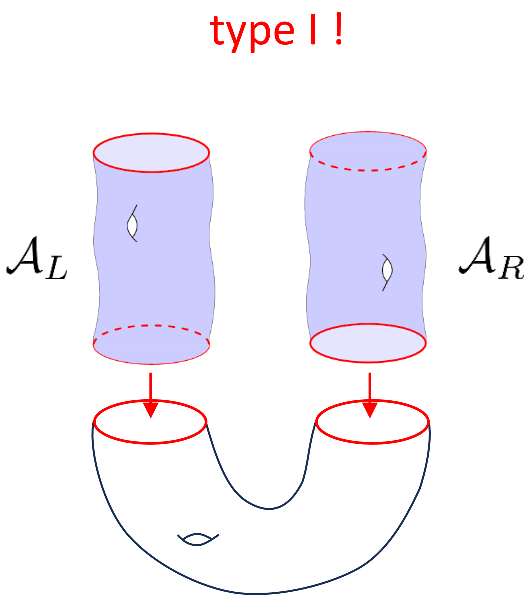
1. Axioms ✓
2. Hilbert Space ✓



3. Operator Algebras ✓



4. Type I von Neumann Factors ✓



5. Entropy (with state-counting interpretation) ⬅

Trace Normalization

- Faithful, normal, semifinite traces on type I algebras are unique up to an overall normalization constant. Therefore, on a given μ-sector

$$\mathrm{tr}(a) = n_\mu \mathrm{Tr}_\mu(a)$$

positive integer!

- We define the extended Hilbert space factors:

$$\widetilde{\mathcal{H}}^\mu_{B\sqcup B,L/R} := \mathcal{H}^\mu_{B\sqcup B,L/R} \otimes \mathcal{H}_{n_\mu}$$

"hidden sector"

where $\mathrm{tr} = \widetilde{\mathrm{Tr}}_\mu$!

- The full extended Hilbert space:

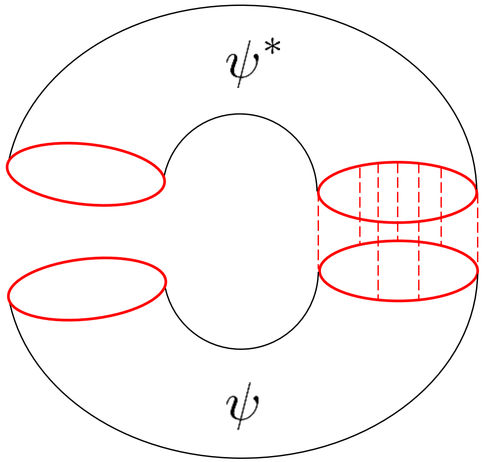
$$\widetilde{\mathcal{H}}_{B\sqcup B} := \bigoplus_{\mu\in\mathcal{I}} \left(\widetilde{\mathcal{H}}^\mu_{B\sqcup B,L} \otimes \widetilde{\mathcal{H}}^\mu_{B\sqcup B,R}\right)$$

⇒ The hidden sectors allow to interpret the path integral trace as a Hilbert space trace

Entropy

✓ The trace tr defines an entropy on the left/right B

- Given a state $|\psi\rangle \in \mathcal{H}_{B\sqcup B}$ we can define a reduced density operator $\rho_\psi \in \mathcal{A}_L$



- The von Neumann entropy is $S^L_{vN}(\psi) = \mathrm{tr}(-\rho_\psi \ln \rho_\psi)$

Entropy

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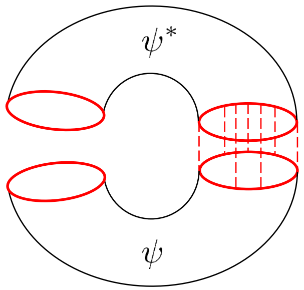

$$S_{vN}^{L}(\psi) = \mathrm{tr}(-\rho_\psi \ln \rho_\psi)$$

✓ Thanks to the relation $\mathrm{tr} = \tilde{\mathrm{Tr}}_\mu$ this entropy has a state-counting interpretation as left entropy on the extended Hilbert space $\widetilde{\mathcal{H}}_{B \sqcup B}$

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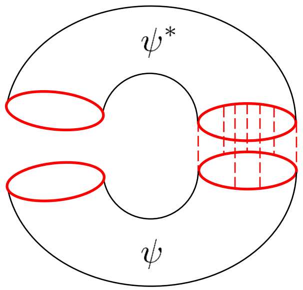

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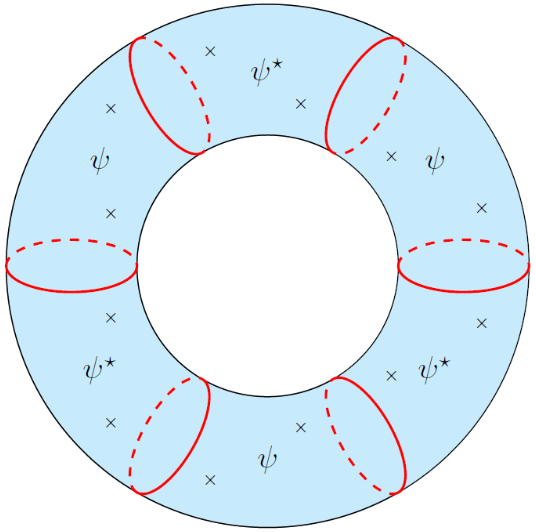
✓ We can compute this entropy via the replica trick:

$$\mathrm{tr}(\rho_\psi^n) = \zeta\left(M([\psi\psi^\star]^n)\right)$$

$$S^L_{vN}(\psi) = (1 - n\partial n)\log \mathrm{tr}(\rho_\psi^n)\Big|_{n=1}$$



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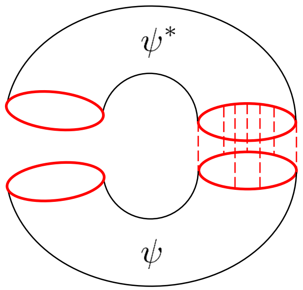

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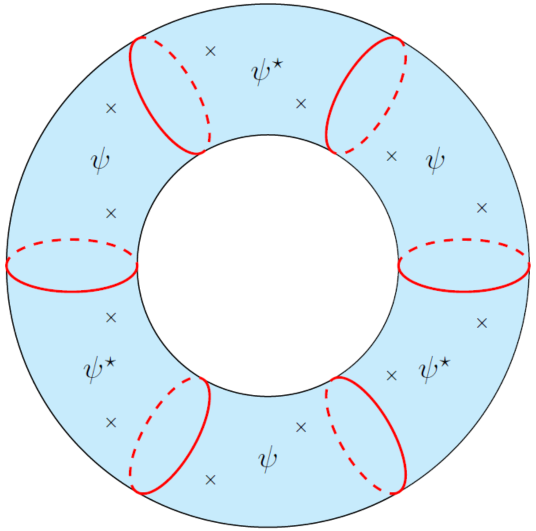
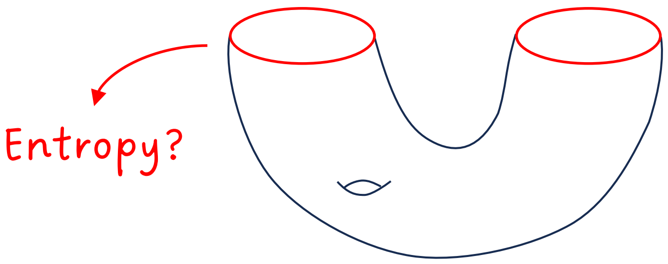
$$= \frac{A(\gamma)}{4G} \quad \textbf{RT}$$



✓ If the theory admits a semiclassical limit described by Einstein-Hilbert or JT gravity, we can argue (by following Lewkowycz-Maldacena) that in such a limit the entropy is given by the Ryu-Takayanagi entropy

Conclusions



$$\mathcal{H}_{B\sqcup B} \overset{?}{\neq} \mathcal{H}_B \otimes \mathcal{H}_B$$

- A gravitational path integral satisfying a simple and familiar set of axioms defines type I von Neumann algebras of observables associated with codimension-2 boundaries.

- The path integral also defines a trace and entropy on these algebras.

- The Hilbert space on which the algebras act decomposes as

$$\mathcal{H}_{B\sqcup B} = \bigoplus_{\mu} \mathcal{H}^{\mu}_{B\sqcup B,L} \otimes \mathcal{H}^{\mu}_{B\sqcup B,R}$$

- The path integral trace is equivalent to a standard trace on an extended Hilbert space: $\mathrm{tr} = \tilde{\mathrm{Tr}}_{\mu}$.

- This provides a state-counting interpretation of the entropy, even when the gravitational theory is not known to have a holographic dual.

- In the semiclassical limit, the entropy is given by the Ryu-Takayanagi formula.

Thanks for the attention!