Tolman-Ehrenfest's criterion of thermal equilibrium extended to conformally static spacetimes

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Outline

1. Intro + motivation
2. How to extend T-E to conformally static spacetimes
3. Conclusions

Tolman-Ehrenfest criterion

Assume:

- static spacetime

$$ds^2 = g_{00}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j \quad (i, j, k = 1, 2, 3)$$

- test fluid at rest in the static frame, supported by pressure (4-velocity u^μ is parallel to the timelike Killing vector)
- thermal equilibrium $\leftrightarrow$ heat flux density $q^a = 0$ in the dissipative fluid model

$$T_{ab} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a$$

(here $h_{ab} \equiv g_{ab} + u_a u_b =$ 3-metric, $u^a =$ 4-velocity) then the fluid is in thermal equilibrium iff its temperature $\mathcal{T}$ satisfies

$$\boxed{\mathcal{T}\sqrt{-g_{00}} = T_0 \quad \text{constant} \qquad \text{(T-E criterion)}}$$

Since heat $=$ energy $=$ mass,

heat sinks in a gravitational field and the temperature of the fluid cannot be uniform.

Fluid is hotter where gravity is stronger, $\vec{\nabla}\mathcal{T} \neq 0$

(Tolman 1928, 1930; Tolman & Ehrenfest 1930)

Connection with materials science: thermal transport in a material due to a temperature gradient mimicked by Luttinger (1964) as a counterbalancing weak gravitational field restoring thermal equilibrium in the presence of $\vec{\nabla}\mathcal{T}$.

T-E criterion applied in neutron stars (Laskos-Patsos + 2022; Kim & Lee 2022; Guo + 2022)

T-E recently revisited by Santiago & Visser 2018-19: attempts to generalize T-E to stationary spacetimes are doomed (temperature is non-unique).

T-E gives insight into thermal physics + gravity.

No progress on T-E since 1930: here we generalize T-E to conformally static spacetimes with metric

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \,, \quad g_{ab} \text{ static}, \quad \Omega > 0$$

Two examples showing the way

Example 1: static conformal factor $\Omega = \Omega(x^i)$

This is almost trivial but still useful: the conformally related metric $\tilde{g}_{ab}$ is also static and the T-E criterion applies,

$$\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}} = \text{const.} = \tilde{\mathcal{T}}\Omega\sqrt{-g_{00}}$$

$$\rightarrow \boxed{\tilde{T} = \frac{\mathcal{T}}{\Omega}}$$

Is this rule more general (assuming that thermal equilibrium is maintained?) YES

Example 2: Cosmic Microwave Background in FLRW universe

All FLRW universes are conformally flat. Consider spatially flat FLRW

$$
\begin{aligned}
ds^2 &= -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \\
&= a^2(\eta)\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)
\end{aligned}
$$

with conformal factor $\Omega(\eta) = a(\eta)$ scale factor. After radiation domination, the CMB becomes a test fluid decoupled from matter and evolves in local thermal equilibrium (reaction rates $>$ Hubble rate H) with temperature scaling as $\mathcal{T} \sim 1/a$ to maintain the Planck distribution:

$$
u(\nu, \mathcal{T}) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{K\mathcal{T}}} - 1}
$$

where $\lambda_{physical} = a\lambda_{comoving}$ and $\nu \sim 1/a$. So, $\boxed{\mathcal{T} \sim 1/\Omega}$ again.

TEST FLUIDS IN CONFORMALLY STATIC SPACETIMES

Conformally static metric $\tilde{g}_{ab} = \Omega^2 \, g_{ab}$, g_{ab} static; use coordinates (t, x^i) adapted to the time symmetry ($g_{\mu\nu,t} = 0$, $g_{0i} = 0$). The 4-velocity of the fluid is always normalized,

$$\tilde{g}_{ab}\tilde{u}^a\tilde{u}^c = -1 \quad \rightarrow \quad \tilde{u}^c = \frac{u^c}{\Omega}\,, \; \tilde{u}_c = \Omega u_c$$

Eckart's law of heat conduction for dissipative fluids reads

$$\tilde{q}_a = -\tilde{\mathcal{K}}\tilde{h}_{ab}\left(\tilde{\nabla}^b\tilde{\mathcal{T}} + \tilde{\mathcal{T}}\tilde{a}^b\right)$$

We derive the T-E criterion from Eckart's law: thermal equilibrium $\equiv \tilde{q}_a = 0$ ($\mathcal{T}$ *can* depend on time) (Misner, Thorne, Wheeler 1973)

Use Buchdahl's identity $a^c = \nabla^c \ln \sqrt{-g_{00}}$ for the 4-acceleration of a particle in a static spacetime. In general, $\tilde{a}^c \neq \tilde{\nabla}^c \ln \sqrt{-\tilde{g}_{00}}$ (when the 4-force has a component parallel to the 4-acceleration, or for particle of variable mass, rockets, solar sails, timelike geodesics mapped to the conformal frame in scalar-tensor or dilaton gravity), but it is always

$$\tilde{h}_{ab}\tilde{a}^b = \tilde{h}_{ab}\tilde{\mathcal{K}}\tilde{\nabla}^b \ln \sqrt{-\tilde{g}_{00}} \,,$$

Proof: one computes

$$\tilde{a}_a = a_a + h_{ab}\,\frac{\nabla^b \Omega}{\Omega}$$

which leads to $\tilde{g}_{ab}\tilde{a}^a\tilde{u}^b = 0$. Then,

$$\begin{aligned}
\tilde{q}_a &= -\tilde{\mathcal{K}}\tilde{h}_{ab}\left(\tilde{\nabla}^b\tilde{\mathcal{T}} + \tilde{\mathcal{T}}\tilde{a}^b\right) \\
&= -\tilde{\mathcal{K}}\tilde{h}_{ab}\tilde{\mathcal{T}}\left(\tilde{\nabla}^b \ln\tilde{\mathcal{T}} + \tilde{\nabla}^b \ln\sqrt{-\tilde{g}_{00}}\right) \\
&= -\tilde{\mathcal{K}}\tilde{h}_{ab}\tilde{\mathcal{T}}\tilde{\nabla}^b \ln\left(\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}}\right)
\end{aligned}$$

and thermal equilibrium $\tilde{q}_a = 0$ implies that $\tilde{\nabla}^b \ln\left(\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}}\right)$ is parallel to $\tilde{u}^b$. Then $\tilde{\mathcal{K}}\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}}$ must depend only on time,

$$\tilde{\mathcal{T}}\sqrt{-\tilde{g}_{00}} = f(t)$$

where $f(t)$ is an integration function $\rightarrow$

$$\tilde{\mathcal{T}} = \frac{f(t)}{\Omega\sqrt{-g_{00}}} = \frac{f(t)\mathcal{T}}{\Omega\left(\mathcal{T}\sqrt{-g_{00}}\right)} = \text{const.}\,\frac{f(t)\mathcal{T}}{\Omega}$$

The product const.$\times f(t)$ is fixed by the fact that, if $\Omega = 1$, the conformal transformation must reduce to the identity with $\tilde{\mathcal{T}} = \mathcal{T}$ and const.$\times f(t) = 1 \;\; \rightarrow$

$$\boxed{\tilde{\mathcal{T}} = \frac{\mathcal{T}}{\Omega}}$$

The prescription $\tilde{\mathcal{T}} = \frac{\mathcal{T}}{\Omega}$ agrees with:

- The Sultana-Dyer black hole, conformal to Schwarzschild with $\Omega = a$ scale factor of the FLRW universe in which the black hole is embedded, has temperature

$$\tilde{\mathcal{T}} = \frac{1}{8\pi m a} + ... = \frac{\mathcal{T}}{\Omega} + ...$$

(Saida + 2007; Majhi 2014; Bhattacharya & Majhi 2016)

- According to an heuristic argument by Dicke, after the conformal transformation the two metrics are equivalent provided that the scaling of units is taken into account. Then length $\sim \Omega$, time $\sim \Omega$, mass $\sim$ energy $\sim \tilde{\mathcal{K}}\tilde{\mathcal{T}} \sim \Omega^{-1}$ so $\tilde{\mathcal{T}} \sim \Omega^{-1}$.

CONCLUSIONS

- First generalization of the T-E criterion since 1930, but *thermal equilibrium must be maintained* (not guaranteed at all by differential geometry)
- $\tilde{\mathcal{T}} = \frac{\mathcal{T}}{\Omega}$ makes sense compared with the temperature scaling of the CMB $\mathcal{T} \sim 1/a$; Dicke's qualitative argument $K\mathcal{T} \sim$ mass $\sim \Omega^{-1}$; the temperature of the conformally-Schwarzschild black hole of Sultana-Dyer embedded in FLRW.
- Further generalization of the T-E criterion seems unlikely: if a non-static geometry shakes a fluid, equilibrium thermodynamics and the concept of temperature itself are likely to fail.

THANK YOU