Tolman-Ehrenfest’s criterion of thermal equilibrium extended to conformally static spacetimes

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1. Intro + motivation
2. How to extend T-E to conformally static spacetimes
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Tolman-Ehrenfest criterion

Assume:

- static spacetime

\[ ds^2 = g_{00}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j \quad (i, j, k = 1, 2, 3) \]

- test fluid at rest in the static frame, supported by pressure (4-velocity \( u^\mu \) is parallel to the timelike Killing vector)

- thermal equilibrium \( \leftrightarrow \) heat flux density \( q^a = 0 \) in the dissipative fluid model

\[ T_{ab} = \rho u_a u_b + Ph_{ab} + \pi_{ab} + q_a u_b + q_b u_a \]

(here \( h_{ab} = g_{ab} + u_a u_b \) = 3-metric, \( u^a \) = 4-velocity)
then the fluid is in thermal equilibrium iff its temperature $\mathcal{T}$ satisfies

$$\mathcal{T} \sqrt{-g_{00}} = T_0 \quad \text{constant} \quad \text{(T-E criterion)}$$

Since heat = energy = mass,

heat sinks in a gravitational field and the temperature of the fluid cannot be uniform.

Fluid is hotter where gravity is stronger, $\vec{\nabla} \mathcal{T} \neq 0$

(Tolman 1928, 1930; Tolman & Ehrenfest 1930)
Connection with materials science: thermal transport in a material due to a temperature gradient mimicked by Luttinger (1964) as a counterbalancing weak gravitational field restoring thermal equilibrium in the presence of $\nabla T$.

T-E criterion applied in neutron stars (Laskos-Patsos + 2022; Kim & Lee 2022; Guo + 2022)

T-E recently revisited by Santiago & Visser 2018-19: attempts to generalize T-E to stationary spacetimes are doomed (temperature is non-unique).

T-E gives insight into thermal physics + gravity.

No progress on T-E since 1930: here we generalize T-E to conformally static static spacetimes with metric

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad g_{ab \text{ static}}, \quad \Omega > 0$$
Example 1: static conformal factor $\Omega = \Omega(x^i)$

This is almost trivial but still useful: the conformally related metric $\tilde{g}_{ab}$ is also static and the T-E criterion applies,

$$\tilde{T} \sqrt{-\tilde{g}_{00}} = \text{const.} = \tilde{T} \Omega \sqrt{-g_{00}}$$

$$\rightarrow \tilde{T} = \frac{T}{\Omega}$$

Is this rule more general (assuming that thermal equilibrium is maintained?) YES
Example 2: Cosmic Microwave Background in FLRW universe

All FLRW universes are conformally flat. Consider spatially flat FLRW

\[ ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \]
\[ = a^2(\eta) \left( -d\eta^2 + dx^2 + dy^2 + dz^2 \right) \]

with conformal factor \( \Omega(\eta) = a(\eta) \) scale factor. After radiation domination, the CMB becomes a test fluid decoupled from matter and evolves in local thermal equilibrium (reaction rates > Hubble rate \( H \)) with temperature scaling as \( T \sim 1/a \) to maintain the Planck distribution:

\[ u(\nu, T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} \]

where \( \lambda_{physical} = a\lambda_{comoving} \) and \( \nu \sim 1/a \). So, \( T \sim 1/\Omega \) again.
Conformally static metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$, $g_{ab}$ static; use coordinates $(t, x^i)$ adapted to the time symmetry $(g_{\mu \nu}, t = 0, g_{0i} = 0)$. The 4-velocity of the fluid is always normalized,

$$\tilde{g}_{ab} \tilde{u}^a \tilde{u}^c = -1 \quad \rightarrow \quad \tilde{u}^c = \frac{u^c}{\Omega}, \quad \tilde{u}_c = \Omega u_c$$

Eckart’s law of heat conduction for dissipative fluids reads

$$\tilde{q}_a = -\tilde{K} \tilde{h}_{ab} \left( \tilde{\nabla}^b \tilde{T} + \tilde{T} \tilde{a}^b \right)$$

We derive the T-E criterion from Eckart’s law: thermal equilibrium $\equiv \tilde{q}_a = 0$ $(\tilde{T}$ can depend on time) (Misner, Thorne, Wheeler 1973)
Use Buchdahl’s identity $a^c = \nabla^c \ln \sqrt{-g_{00}}$ for the 4-acceleration of a particle in a static spacetime. In general, $\ddot{a}^c \neq \ddot{\nabla}^c \ln \sqrt{-\tilde{g}_{00}}$ (when the 4-force has a component parallel to the 4-acceleration, or for particle of variable mass, rockets, solar sails, timelike geodesics mapped to the conformal frame in scalar-tensor or dilaton gravity), but it is always

$$\tilde{h}_{ab} \ddot{a}^b = \tilde{h}_{ab} \tilde{\kappa} \tilde{\nabla}^b \ln \sqrt{-\tilde{g}_{00}},$$

Proof: one computes

$$\ddot{a}_a = a_a + h_{ab} \frac{\nabla^b \Omega}{\Omega}$$
which leads to $\tilde{g}_{ab}\tilde{u}^a\tilde{u}^b = 0$. Then,

$$
\tilde{q}_a = -\tilde{K}\tilde{h}_{ab}\left(\tilde{\nabla}^b\tilde{T} + \tilde{T}\tilde{a}^b\right)
= -\tilde{K}\tilde{h}_{ab}\tilde{T}\left(\tilde{\nabla}^b\ln\tilde{T} + \tilde{\nabla}^b\ln\sqrt{-\tilde{g}_{00}}\right)
= -\tilde{K}\tilde{h}_{ab}\tilde{T}\tilde{\nabla}^b\ln\left(\tilde{T}\sqrt{-\tilde{g}_{00}}\right)
$$

and thermal equilibrium $\tilde{q}_a = 0$ implies that $\tilde{\nabla}^b\ln\left(\tilde{T}\sqrt{-\tilde{g}_{00}}\right)$ is parallel to $\tilde{u}^b$. Then $\tilde{K}\tilde{T}\sqrt{-\tilde{g}_{00}}$ must depend only on time,

$$
\tilde{T}\sqrt{-\tilde{g}_{00}} = f(t)
$$

where $f(t)$ is an integration function →
\[ \tilde{T} = \frac{f(t)}{\Omega \sqrt{-g_{00}}} = \frac{f(t) T}{\Omega (T \sqrt{-g_{00}})} = \text{const.} \frac{f(t) T}{\Omega} \]

The product const. \( \times f(t) \) is fixed by the fact that, if \( \Omega = 1 \), the conformal transformation must reduce to the identity with \( \tilde{T} = T \) and const. \( \times f(t) = 1 \) →

\[ \tilde{T} = \frac{T}{\Omega} \]
The prescription $\tilde{T} = \frac{T}{\Omega}$ agrees with:

- The Sultana-Dyer black hole, conformal to Schwarzschild with $\Omega = a$ scale factor of the FLRW universe in which the black hole is embedded, has temperature

$$\tilde{T} = \frac{1}{8\pi ma} + ... = \frac{T}{\Omega} + ...$$

(Saida + 2007; Majhi 2014; Bhattacharya & Majhi 2016)

- According to an heuristic argument by Dicke, after the conformal transformation the two metrics are equivalent provided that the scaling of units is taken into account. Then length $\sim \Omega$, time $\sim \Omega$, mass $\sim$ energy $\sim \tilde{K} \tilde{T} \sim \Omega^{-1}$ so $\tilde{T} \sim \Omega^{-1}$. 
Conclusions

First generalization of the T-E criterion since 1930, but *thermal equilibrium must be maintained* (not guaranteed at all by differential geometry)

\[ \tilde{T} = \frac{T}{\Omega} \] makes sense compared with the temperature scaling of the CMB \( T \sim 1/a \); Dicke’s qualitative argument \( KT \sim \text{mass} \sim \Omega^{-1} \); the temperature of the conformally-Schwarzschild black hole of Sultana-Dyer embedded in FLRW.

Further generalization of the T-E criterion seems unlikely: if a non-static geometry shakes a fluid, equilibrium thermodynamics and the concept of temperature itself are likely to fail.
THANK YOU