Relativistic electrodynamics in a medium

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Electromagnetism

- Unification of electricity and magnetism by Maxwell:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \tag{1a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1d}$$

- The Lorentz force law,

$$F = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \tag{2}$$

- The classical behaviour of **E** and **B** fields *in a vacuum* is governed by (1) and (2).

EM in Special Relativity

- Maxwell's equations are Lorentz-invariant.
- Introducing the field strength tensor, $F_{\mu\nu}$, eqns. (1) are equivalent to

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu \tag{3a}$$

$$\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0 \tag{3b}$$

- $F_{\mu\nu}$ is defined in terms of the four–potential, A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4}$$

- The electric and magnetic fields are obtained via,

$$E^\mu = F^{\mu\nu} u_\nu \tag{5a}$$

$$B^\mu = -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho} u_\nu F_{\lambda\rho} \tag{5b}$$

- The field strength tensor has the equivalent form,

$$F^{\mu\nu} = E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\lambda\rho} B_\lambda u_\rho \tag{6}$$

EM in a medium

- In a material medium, one must distinguish between bound and free charges and currents.
- The *inhomogeneous* Maxwell's equations are modified:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{7a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{7b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7c}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{7d}$$

- We introduce the fields,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{8a}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{8b}$$

EM in Special Relativity in a medium

- Are Maxwell's equations in a medium Lorentz-invariant?
- The constitutive relations (8) are certainly not.
- Define new four-vectors,

$$D^\alpha \equiv \epsilon E^\alpha \tag{9a}$$

$$H^\alpha \equiv \frac{1}{\mu} B^\alpha \tag{9b}$$

- Introduce a 'dielectric field strength tensor',

$$G^{\mu\nu} \equiv D^\mu u^\nu - D^\nu u^\mu + \epsilon^{\mu\nu\lambda\rho} H_\lambda u_\rho. \tag{10}$$

EM in Special Relativity in a medium

- The inhomogeneous Maxwell equations in a medium in special relativity are now,

$$\partial_\mu G^{\mu\nu} = \frac{1}{c} J^{\nu}_{\mathrm{free}} \tag{11}$$

- These are sourced by the *free* 4-current densities. Note the non-trivial relation between $F^{\mu\nu}$ and $G^{\mu\nu}$.
- Eqn. (11) can be derived from the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} G^{\mu\nu} - \frac{1}{c} J^{\mu}_{\mathrm{free}} A_\mu \tag{12}$$

- The EM stress tensor in a medium is

$$T^{\mu\nu} = -\eta_{\alpha\beta} F^{\mu\alpha} G^{\nu\beta} + \frac{1}{4} F_{\alpha\beta} G^{\alpha\beta} \tag{13}$$

Applications

- Magnetised neutron stars; *General-relativistic* EM in a medium
- Cerenkov radiation from charged particles
- Magnetic monopole interactions
- Dark matter in compact stars
- Milli-charged dark matter

References

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