Effective Field Theory for Extreme Mass Ratios

Nabha Shah
California Institute of Technology

QCD Meets Gravity 2023, CERN
11 December 2023

Based on 2308.14832, 23xx.xxxxx
(with C. Cheung, J. Wilson-Gerow, J. Parra-Martinez, I. Rothstein)

Part 1
Motivation: gravitational wave physics

Perturbation theory:

- Post-Newtonian
- Post-Minkowskian
- Self-force

LIGO Scientific Collaboration and Virgo Collaboration (2016)
New results have been obtained for conservative and radiative dynamics, non-spinning and spinning blackholes, finite size effects, etc.
Post-Minkowskian (PM) perturbation theory

\[ V = c_1 \left( \frac{G}{r} \right) + c_2 \left( \frac{G}{r} \right)^2 + \cdots \]

Generically: \( c_n(p^2, p \cdot r) \)

Isotropic: \( c_n(p^2) \)

An expansion in loops with all orders in velocity at each loop order
Expansion in mass ratio or the self-force (SF) expansion

\[ \lambda = \frac{m_1}{m_2} = \frac{m_L}{m_H} \]

\[ \mathcal{M} = \lambda^0 \mathcal{M}_{0SF} + \lambda^1 \mathcal{M}_{1SF} + \lambda^2 \mathcal{M}_{2SF} + \cdots \]

Interplays with the post-Minkowskian expansion


\[ \mathcal{M}_{1PM} \sim G^2 m_L^2 m_H^2 \]

\[ \mathcal{M}_{2PM} \sim G^2 m_L^2 m_H^2 (m_H^2 + m_L^2) \]

\[ \mathcal{M}_{3PM} \sim G^3 m_L^2 m_H^2 (m_H^2 + m_L^2) \]

\[ \mathcal{M}_{4PM} \sim G^4 m_L^2 m_H^2 (m_H^3 + m_L^3) \]

\[ \mathcal{M}_{5PM} \sim G^5 m_L^2 m_H^2 (m_H^4 + m_L^4) \]
Expansion in mass ratio or the self-force (SF) expansion

\[ \lambda = \frac{m_1}{m_2} = \frac{m_L}{m_H} \]

\[ \mathcal{M} = \lambda^0 \mathcal{M}_{0SF} + \lambda^1 \mathcal{M}_{1SF} + \lambda^2 \mathcal{M}_{2SF} + \cdots \]

Are there ways to maximize data extraction?
Quantum Tree Graphs and the Schwarzschild Solution

M. J. Duff*

Physics Department, Imperial College, London SW7, England
(Received 7 July 1972)

\[ g_{\mu\nu} = \eta_{\mu\nu} + G + G^2 + G^3 + G^3 + \cdots \]

\[ g^{00} = -1 - \frac{2GM}{r} - \frac{2(GM)^2}{r^2} + O(G^3) \]

\[ g^{ij} = \left(1 - \frac{2GM}{r} + \frac{3(GM)^2}{r^2}\right) \eta^{ij} - \frac{(GM)^2}{r^2} x^i x^j + O(G^3) \]

The metric encodes all order PM data and can be interpreted as a sum of infinite flat-space diagrams

Expand known solutions
Probe dynamics from the geodesic equation

The **geodesic equation** encodes perturbative scattering information for a test particle in a background at all orders in the PM expansion.

![Geodesic Equation Diagram]

Convenient coordinates can be chosen

\[
\bar{g}_{\mu\nu}(x) = (1 + f)^4 \eta_{\mu\nu} + \left[ \left( \frac{1 - f}{1 + f} \right)^2 - (1 + f)^4 \right] u_{H\mu} u_{H\nu} \quad \text{where} \quad f = \frac{GM}{2r}
\]

\[
\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{2GM}{r} (\eta_{\mu\nu} - 2u_{H\mu} u_{H\nu}) + \frac{1}{2} \left( \frac{GM}{r} \right)^2 \left( 3\eta_{\mu\nu} + u_{H\mu} u_{H\nu} \right) + \cdots
\]
Perturbations away from a probe in Schwarzschild

1. At 0SF:

perturbative corrections away from a non-spinning black hole binary system

Examples:
 tidal distortions
 higher derivative corrections to general relativity interactions with a charged body


2. Beyond 0SF:

Effective field theory in a mass ratio expansion
Schematic procedure at 0SF

Geodesic equation, Hamiltonian

\[ g^{\mu\nu} P_\mu P_\nu + m^2 - \lambda \mathcal{O}(g, P) = 0, \quad P_\mu = (H, p_r, p_\theta, p_\phi) \]

\[ H(p, r, J) \rightarrow H^{\text{iso}}(p, r) \]

Amplitude

\[ \mathcal{M}(p, r) = \frac{1}{2\sqrt{(p^2 + m^2)}}(\bar{p}(r)^2 - p^2) + \text{iterations} \]

\[ H^{\text{iso}}(\bar{p}(r), r) = E = \sqrt{p^2 + m^2} \]

The extraction procedure is purely algebraic thanks to the impetus formula

Bern, Cheung, Roiban, Shen, Solon, Zeng (2018, 2019); Kälin, Porto (2020); Bjerrum-Bohr, Cristofoli, Damgaard (2020)
An EFT formalism for extreme mass ratios

1. Starting point: probe motion in curved spacetime, $\bar{x}_L$ and $\bar{g}_{\mu\nu}(x)$

2. Accounting for heavy particle motion with $x_H$

3. A systematic expansion in $\lambda = \frac{m_L}{m_H}$: 
   
   $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$

   $x_i^\mu = \bar{x}_i^\mu + \delta x_i^\mu$

4. Integrate out the heavy particle fluctuation, $\delta x_H$

5. Use resulting effective action: 
   background field + corrective operators
   to calculate quantities of interest (e.g. on-shell radial action)
Electromagnetism
Background configuration: test particle in an electromagnetic field

Coulomb potential: \( \vec{A}_\mu(x) = \frac{z_H m_H u_H \mu}{4\pi r} \) where \( r = \sqrt{(u_H x)^2 - x^2} \)

and \( \ddot{x}_L^\mu - z_L \vec{F}^{\mu\nu}(\vec{x}_L) \dot{x}_L^\nu = 0 \) is the light particle equation of motion

Assume \( z_i = \frac{q_i}{m_i} \) of equal size so that forces scale with mass

For the heavy particle: \( \ddot{x}_H^\mu - z_H \vec{F}^{\mu\nu}(\vec{x}_H) \dot{x}_H^\nu = 0 \)

vanish in dimensional regularization

Background motion of the heavy particle is simply \( \vec{x}_H^\mu(\tau) = u_H^\mu \tau \)
Expanding the action in orders of the mass ratio

\[ S_{EM} = m_H \int d\tau \left[ -\frac{1}{2} \dot{x}_H^2 - z_H \dot{x}_H^\mu A_\mu(x_H) - \lambda \left( \frac{1}{2} \dot{x}_L^2 + z_L \dot{x}_L^\mu A_\mu(x_L) \right) \right] \]

\[ + \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \quad \text{light particle action is suppressed by } \lambda \]

\[ A_\mu = \bar{A}_\mu + \delta A_\mu \]

\[ x_i^\mu = \bar{x}_i^\mu + \delta x_i^\mu \]

terms linear in \( \delta x_H \) and \( \delta g_{\mu\nu} \) vanish

dimensional regularization sets terms with \( \bar{A}_\mu(x_H) \) to zero

\[ S_{EM} = \bar{S}_{EM} + \delta S_{EM} \]

\[ \delta S_{EM} = m_H \int d\tau \left[ -\frac{1}{2} \delta \dot{x}_H^2 - z_H \delta x_H^\mu \dot{x}_H^\nu \delta F_{\mu\nu}(\bar{x}_H) - \lambda z_L \dot{x}_L^\mu \delta A_\mu(\bar{x}_L) \right] \]

\[ + \int d^4x \left[ -\frac{1}{4} \delta F_{\mu\nu} \delta F^{\mu\nu} \right] + \cdots \]
Effective action and recoil operator at 1SF

\[ \delta S_{\text{EM}} = m_H \int d\tau \left[ -\frac{1}{2} \delta \dot{x}_H^2 - z_H \delta x_H^\mu \dot{x}_H^\nu \delta F_{\mu\nu}(\vec{x}_H) - \lambda z_L \dot{x}_L^\mu \delta A_\mu(\vec{x}_L) \right] \]

\[ + \int d^4x \left[ -\frac{1}{4} \delta F_{\mu\nu} \delta F^{\mu\nu} \right] + \cdots \]

Integrate out \( \delta x_H \) using \( \delta \ddot{x}_H^\mu - z_H \delta F^{\mu\nu}(\vec{x}_H) \dot{x}_H^\nu = 0 \)

\[ \delta S_{\text{EM}}^{\text{eff}} = R_{\text{EM}} + \int d^4x \left( -\frac{1}{4} \delta F_{\mu\nu} \delta F^{\mu\nu} - \delta A_\mu \bar{J}_L^\mu \right) \]

\[ R_{\text{EM}} = -\frac{1}{2} z_H^2 m_H \int d\tau \dot{x}_H^\alpha \delta F_{\alpha\mu}(\vec{x}_H) \frac{1}{\partial^2 \tau} \dot{x}_H^\beta \delta F_{\beta \mu}(\vec{x}_H) \]

The non-local in time recoil operator accounts for the wobble of the heavy particle.
Effective action and recoil operator at 2SF

For light particle fluctuation:

\[ m_L \int d\tau \left[ -\frac{1}{2} \delta \dot{x}_L^2 - z_L \delta x_L^\mu \dot{x}_L^\nu \delta F_{\mu\nu}(\bar{x}_L) \right. \]

\[ -\frac{1}{2} z_L \delta x_L^\mu \delta \dot{x}_L^\nu \bar{F}_{\mu\nu}(\bar{x}_L) - \frac{1}{2} z_L \delta x_L^\rho \delta x_L^\nu \dot{x}_L^\mu \partial_\rho \bar{F}_{\mu\nu}(\bar{x}_L) \]

The light particle experiences **recoil** and deviation from its trajectory but continues to propagate in the background.

Heavy particle recoil:

\[ \mathcal{R}^{(2SF)}_{EM} = \frac{z_H^2 m_H}{2} \int d\tau \left[ \delta E^\alpha(\bar{x}_H) \frac{1}{\partial_\tau^2} \delta F_{\alpha\mu}(\bar{x}_H) \frac{1}{\partial_\tau} \delta E^\mu(\bar{x}_H) \right. \]

\[ + \delta E^\alpha(\bar{x}_H) \frac{1}{\partial_\tau^2} \partial_\mu \delta E_\alpha(\bar{x}_H) \frac{1}{\partial_\tau^2} \delta E^\mu(\bar{x}_H) \]

(where \( \delta E^\mu = \dot{x}_H^\nu \delta F^{\mu\nu} \))
Types of interactions

1SF:

\[ m_L \text{ background trajectory source} \]

\[ \text{Heavy particle recoil operator} \]

2SF:

\[ \text{Deviation and propagation} \]

\[ \text{Heavy particle recoil operator} \]
Gravity
Effective action at 1SF

\[ S_{GR} = \sum_{i=H,L} m_i \int d\tau \left[ -\frac{1}{2} \dot{x}_i^\mu \dot{x}_i^\nu g_{\mu\nu}(x_i) \right] + \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G} R \right] \]

Self-energy contributions are dropped and the action at 1SF is

\[ \delta S_{GR}^{\text{eff}} = \mathcal{R}_{GR} - \int d^4x \sqrt{-g} \left[ \frac{1}{32\pi G} \left( \frac{1}{2} \tilde{\nabla}_\rho \delta g_{\mu\nu} \tilde{\nabla}^\rho \delta g^{\mu\nu} + \cdots + \frac{1}{2} \delta g_{\mu\nu} \tilde{T}^{\mu\nu} \right) \right] \]

\[ \mathcal{R}_{GR} = -\frac{1}{2} m_H \int d\tau \dot{x}_H^\alpha \dot{x}_H^\beta \delta \Gamma_{\alpha\beta}(\bar{x}_H) \frac{1}{\partial^2_{\tau}} \dot{x}_H^\gamma \dot{x}_H^\delta \delta \Gamma_{\mu\gamma\delta}(\bar{x}_H) \]

The action describes interactions of a probe particle with fluctuations of the background, heavy particle recoil encoded in a nonlocal operator

Expanding known curved spacetime solutions provides simplification
Types of interactions

1SF:
- Geodesic source
- Recoil operator

2SF:
- Geodesic deviation
- Recoil operator
- Self-interactions
The interplay between PM and SF expansions provides an avenue for furthering calculations for the gravitational two-body problem.

At OSF, the Schwarzschild metric and geodesic motion carry all order perturbative information.

EFT setup systematically describes perturbative corrections beyond test-particle dynamics needed at higher SF orders while leveraging the resummation provided by the background configurations.
Thank You!