Effective Field Theory for Extreme Mass Ratios

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QCD Meets Gravity 2023, CERN
11 December 2023

Based on 2308.14832, 23xx.xxxxx
(with C. Cheung, J. Wilson-Gerow, J. Parra-Martinez, I. Rothstein)

Part 1

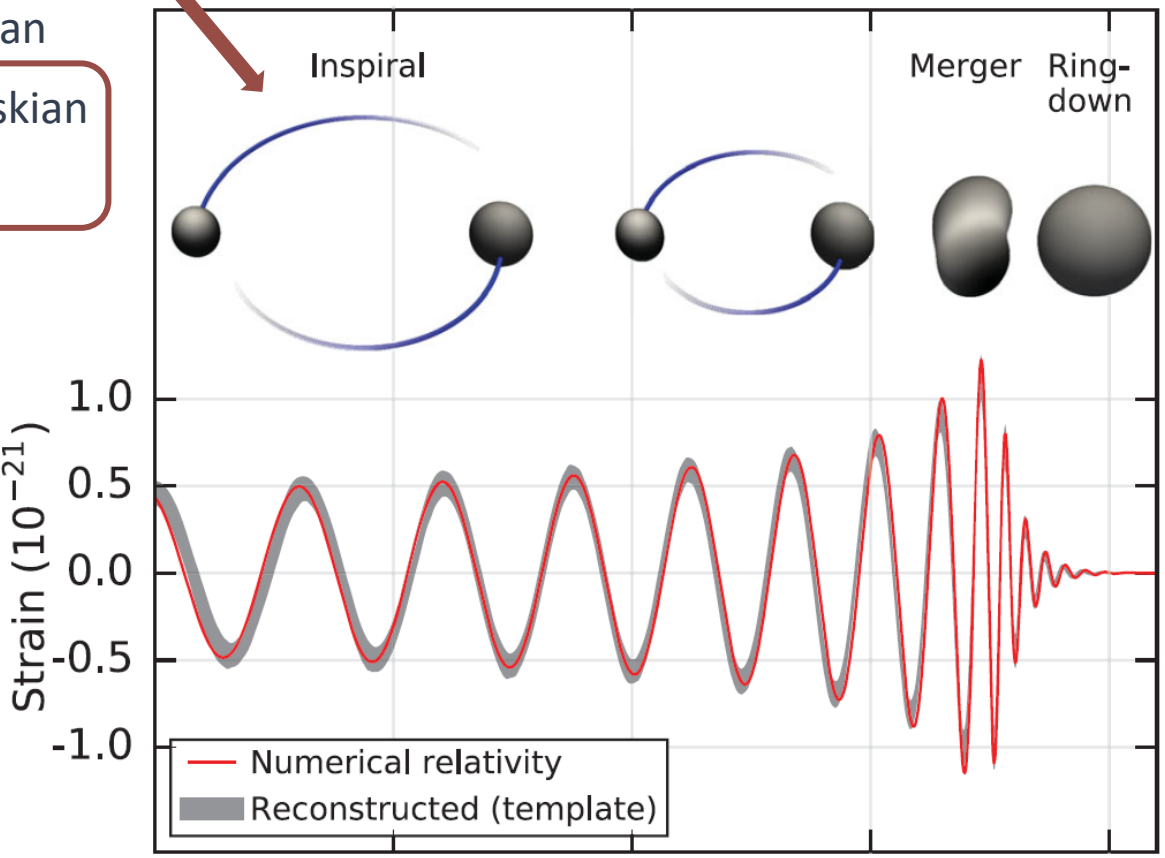
Motivation: gravitational wave physics

Perturbation theory:

Post-Newtonian

Post-Minkowskian

Self-force



LIGO Scientific Collaboration and Virgo Collaboration (2016)

Post-Minkowskian (PM) perturbation theory

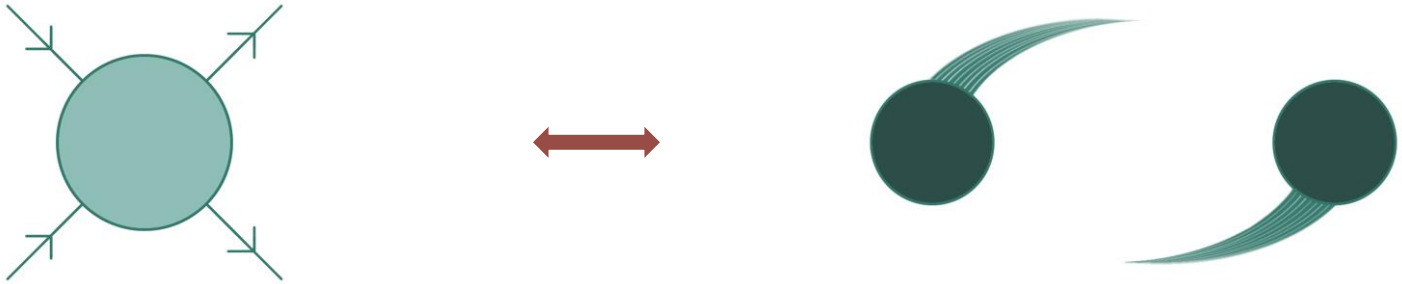
$$V = c_1 \left(\frac{G}{r}\right) + c_2 \left(\frac{G}{r}\right)^2 + \cdots$$

Generically: $c_n(\mathbf{p}^2, \mathbf{p}\cdot\mathbf{r})$

Isotropic: $c_n(\mathbf{p}^2)$

Relativistic scattering

Classical bound state dynamics



New results have been obtained for conservative and radiative dynamics, non-spinning and spinning blackholes, finite size effects, etc.

Post-Minkowskian (PM) perturbation theory

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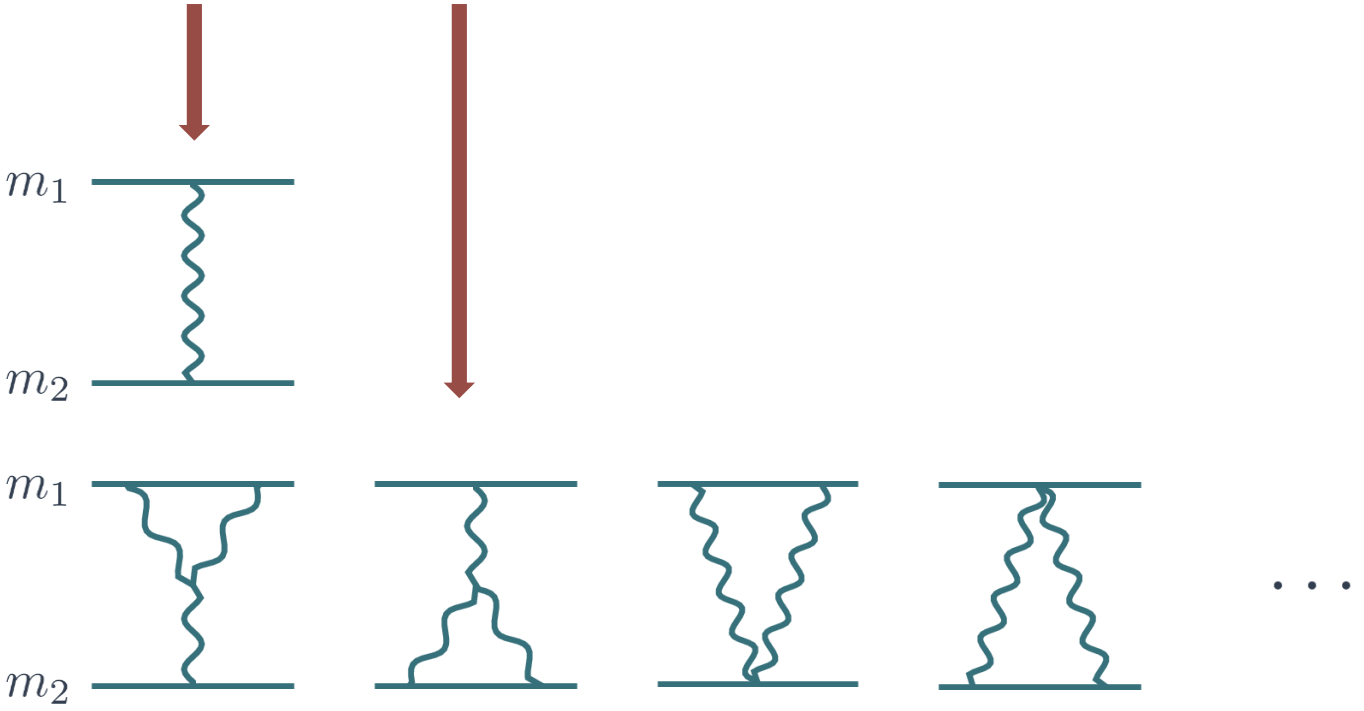


An expansion in loops with all orders in velocity at each loop order

Expansion in mass ratio or the self-force (SF) expansion

$$\lambda = \frac{m_1}{m_2} = \frac{m_L}{m_H}$$

$$\mathcal{M} = \lambda^0 \mathcal{M}_{\rm 0SF} + \lambda^1 \mathcal{M}_{\rm 1SF} + \lambda^2 \mathcal{M}_{\rm 2SF} + \cdots$$

Interplays with the post-Minkowskian expansion

Vines, Steinhoff, Buonanno (2019), Siemonsen, Vines (2019); Bini, Damour, Geralico (2019, 2020), Damour (2020), Antonelli, Kavanagh, Khalil, Steinhoff, Vines (2020)

$$\mathcal{M}_{\rm 1PM} \sim G m_L^2 m_H^2 \quad (1)$$

$$\mathcal{M}_{\rm 2PM} \sim G^2 m_L^2 m_H^2 (m_H + m_L)$$

0SF

$$\mathcal{M}_{\rm 3PM} \sim G^3 m_L^2 m_H^2 (m_H^2 + m_H m_L + m_L^2)$$

$$\mathcal{M}_{\rm 4PM} \sim G^4 m_L^2 m_H^2 (m_H^3 + m_H^2 m_L + m_H m_L^2 + m_L^3)$$

1SF

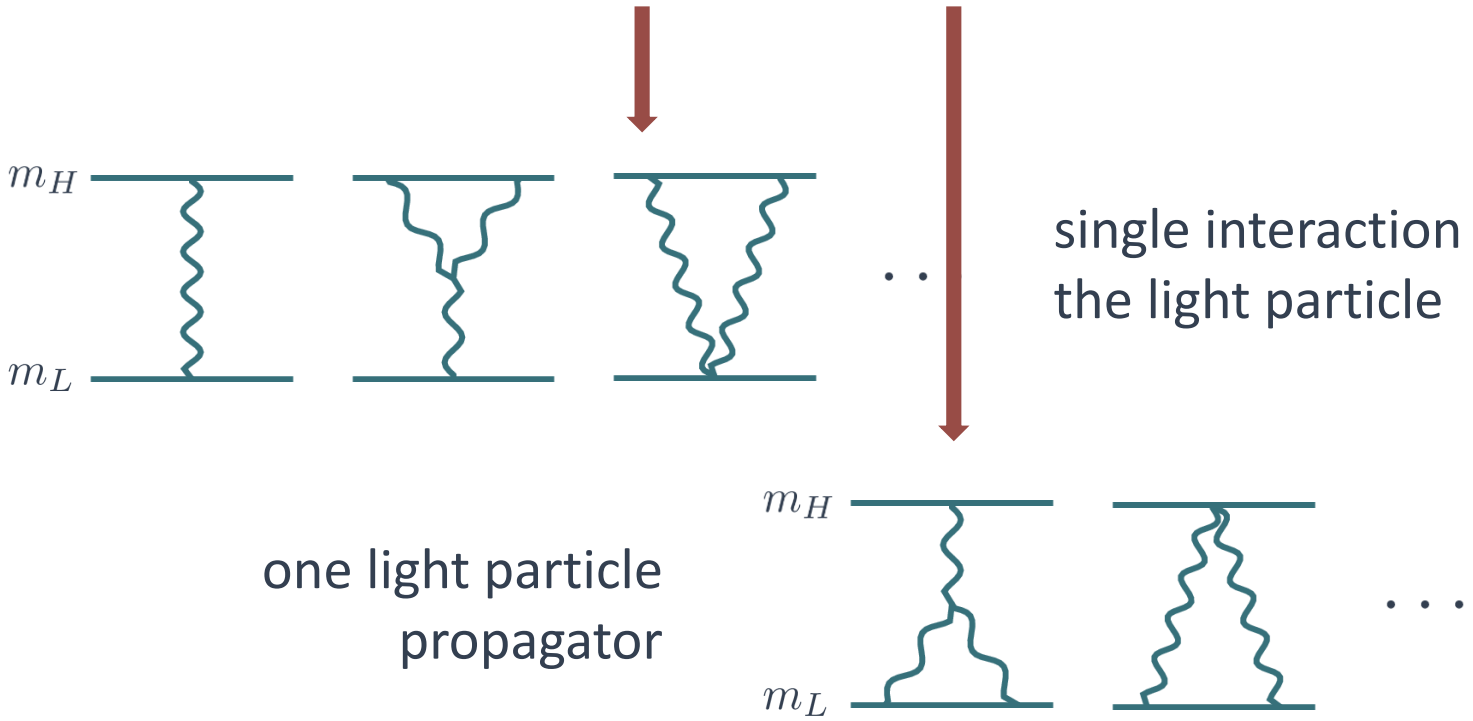
$$\mathcal{M}_{\rm 5PM} \sim G^5 m_L^2 m_H^2 (m_H^4 + m_H^3 m_L + m_H^2 m_L^2 + m_H m_L^3 + m_L^4)$$

0SF 1SF 2SF 3SF 4SF

Expansion in mass ratio or the self-force (SF) expansion

$$\lambda = \frac{m_1}{m_2} = \frac{m_L}{m_H}$$

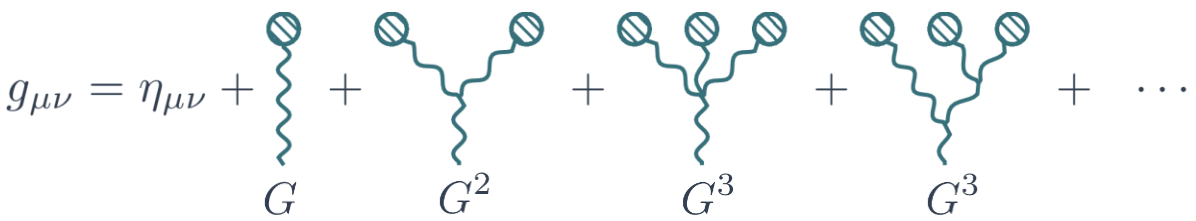
$$\mathcal{M} = \lambda^0 \mathcal{M}_{0\mathrm{SF}} + \lambda^1 \mathcal{M}_{1\mathrm{SF}} + \lambda^2 \mathcal{M}_{2\mathrm{SF}} + \cdots$$



Are there ways to maximize data extraction?

Quantum Tree Graphs and the Schwarzschild Solution

M. J. Duff*
Physics Department, Imperial College, London SW7, England
(Received 7 July 1972)



$$g^{00} = -1 - \frac{2GM}{r} - \frac{2(GM)^2}{r^2} + O(G^3)$$

$$g^{ij} = \left(1 - \frac{2GM}{r} + \frac{3(GM)^2}{r^2}\right)\eta^{ij} - \frac{(GM)^2}{r^2}\frac{x^i x^j}{r^2} + O(G^3)$$

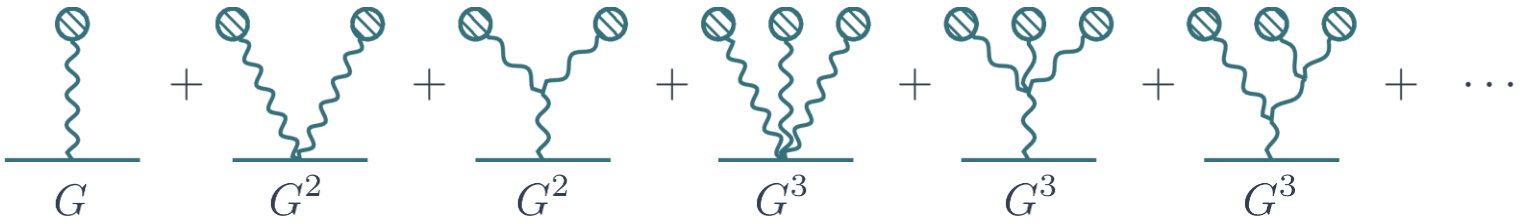
The metric encodes all order PM data

and can be interpreted as a sum of infinite flat-space diagrams

Expand known solutions

Probe dynamics from the geodesic equation

The geodesic equation encodes perturbative scattering information for a test particle in a background at all orders in the PM expansion



Convenient coordinates can be chosen

$$\bar{g}_{\mu\nu}(x) = (1+f)^4\ \eta_{\mu\nu} + \left[\left(\frac{1-f}{1+f}\right)^2 - (1+f)^4\right] u_{H\mu}u_{H\nu} \quad \text{where} \quad f = \frac{GM}{2r}$$

$$\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{2GM}{r}(\eta_{\mu\nu} - 2u_{H\mu}u_{H\nu}) + \frac{1}{2}\left(\frac{GM}{r}\right)^2 (3\eta_{\mu\nu} + u_{H\mu}u_{H\nu}) + \cdots$$

Perturbations away from a probe in Schwarzschild

1. At 0SF:

perturbative corrections away from a non-spinning black hole binary system

Examples:
tidal distortions
higher derivative corrections to general relativity
interactions with a charged body

Cheung, Shah, Solon (arXiv:2010.08568)

2. Beyond 0SF:

Effective field theory in a mass ratio expansion

Schematic procedure at 0SF

Geodesic equation, Hamiltonian

$$g^{\mu\nu}P_\mu P_\nu + m^2 - \lambda\mathcal{O}(g,P) = 0\,, \quad P_\mu = (H, p_r, p_\theta, p_\phi)$$

$$H(p,r,J) \quad \rightarrow \quad H^{\rm iso}(p,r)$$

Amplitude

$$\mathcal{M}(p,r) = \frac{1}{2\sqrt{(p^2+m^2)}}(\overline{p}(r)^2 - p^2) + \text{iterations}$$

$$H^{\rm iso}(\overline{p}(r), r) = E = \sqrt{p^2+m^2}$$

The extraction procedure is purely algebraic thanks to the *impetus formula*

Bern, Cheung, Roiban, Shen, Solon, Zeng (2018, 2019);
Kälin, Porto (2020); Bjerrum-Bohr, Cristofoli, Damgaard (2020)

An EFT formalism for extreme mass ratios

1. Starting point: probe motion in curved spacetime, $\bar{x}_L$ and $\bar{g}_{\mu\nu}(x)$

2. Accounting for heavy particle motion with x_H

3. A systematic expansion in $\lambda = \dfrac{m_L}{m_H}$: $\quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, $\quad x_i^\mu = \bar{x}_i^\mu + \delta x_i^\mu$

4. Integrate out the heavy particle fluctuation, δx_H

5. Use resulting effective action:
   background field + *corrective operators*
   to calculate quantities of interest (e.g. on-shell radial action)

Electromagnetism

Background configuration: test particle in an electromagnetic field

Coulomb potential: $\bar{A}_\mu(x) = \dfrac{z_H m_H u_{H\mu}}{4\pi r}$ where $r = \sqrt{(u_H x)^2 - x^2}$

and $\ddot{\bar{x}}^\mu_L - z_L \bar{F}^{\mu\nu}(\bar{x}_L)\dot{\bar{x}}_{L\nu} = 0$ is the light particle equation of motion

Assume $z_i = \dfrac{q_i}{m_i}$ of equal size so that forces scale with mass

For the heavy particle: $\ddot{\bar{x}}^\mu_H - z_H \bar{F}^{\mu\nu}(\bar{x}_H)\dot{\bar{x}}_{H\nu} = 0$

$\left.\dfrac{1}{r}\right|_0$ Divergence!

vanish in dimensional regularization

Background motion of the heavy particle is simply $\bar{x}^\mu_H(\tau) = u^\mu_H \tau$

Expanding the action in orders of the mass ratio

$$S_{\rm EM} = m_H \int d\tau \left[-\tfrac{1}{2}\dot{x}_H^2 - z_H \dot{x}_H^\mu A_\mu(x_H) - \lambda \left(\tfrac{1}{2}\dot{x}_L^2 + z_L \dot{x}_L^\mu A_\mu(x_L) \right) \right]$$
$$+ \int d^4x \left[-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

light particle action is suppressed by λ

$$A_\mu = \bar{A}_\mu + \delta A_\mu$$

$$x_i^\mu = \bar{x}_i^\mu + \delta x_i^\mu$$

terms linear in δx_H and $\delta g_{\mu\nu}$ vanish

dimensional regularization sets terms with $\bar{A}_\mu(x_H)$ to zero

$$S_{\rm EM} = \bar{S}_{\rm EM} + \delta S_{\rm EM}$$

$$\delta S_{\rm EM} = m_H \int d\tau \left[-\tfrac{1}{2}\delta\dot{x}_H^2 - z_H \delta x_H^\mu \dot{\bar{x}}_H^\nu \delta F_{\mu\nu}(\bar{x}_H) - \lambda z_L \dot{\bar{x}}_L^\mu \delta A_\mu(\bar{x}_L) \right]$$
$$+ \int d^4x \left[-\tfrac{1}{4} \delta F_{\mu\nu} \delta F^{\mu\nu} \right] + \cdots$$

Effective action and recoil operator at 1SF

$$\delta S_{\rm EM} = m_H \int d\tau \left[-\tfrac{1}{2}\delta\dot{x}_H^2 - z_H \delta x_H^\mu \dot{\bar{x}}_H^\nu \delta F_{\mu\nu}(\bar{x}_H) - \lambda z_L \dot{\bar{x}}_L^\mu \delta A_\mu(\bar{x}_L) \right]$$
$$+ \int d^4x \left[-\tfrac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu} \right] + \cdots$$

Integrate out δx_H using $\delta\ddot{x}_H^\mu - z_H \delta F^{\mu\nu}(\bar{x}_H)\dot{\bar{x}}_{H\nu} = 0$

$$\delta S_{\rm EM}^{\rm eff} = \mathcal{R}_{\rm EM} + \int d^4x \left(-\tfrac{1}{4}\delta F_{\mu\nu}\delta F^{\mu\nu} - \delta A_\mu \bar{J}_L^\mu \right)$$

$$\mathcal{R}_{\rm EM} = -\frac{1}{2} z_H^2 m_H \int d\tau\, \dot{\bar{x}}_H^\alpha \delta F_{\alpha\mu}(\bar{x}_H) \frac{1}{\partial_\tau^2} \dot{\bar{x}}_H^\beta \delta F_\beta{}^\mu(\bar{x}_H)$$

The non-local in time recoil operator accounts for the wobble of the heavy particle

Effective action and recoil operator at 2SF

For light particle fluctuation:

$$m_L \int d\tau \Big[-\tfrac{1}{2}\delta\dot{x}_L^2 - z_L \delta x_L^\mu \dot{\bar{x}}_L^\nu \delta F_{\mu\nu}(\bar{x}_L) - \tfrac{1}{2} z_L \delta x_L^\mu \delta\dot{x}_L^\nu \bar{F}_{\mu\nu}(\bar{x}_L) - \tfrac{1}{2} z_L \delta x_L^\rho \delta x_L^\nu \dot{\bar{x}}_L^\mu \partial_\rho \bar{F}_{\mu\nu}(\bar{x}_L) \Big]$$

The light particle experiences recoil and deviation from its trajectory but continues to propagate in the background

Heavy particle recoil:

$$\mathcal{R}_{\rm EM}^{(2{\rm SF})} = \frac{z_H^2 m_H}{2} \int d\tau \Big[\delta E^\alpha(\bar{x}_H) \frac{1}{\overleftarrow{\partial_\tau^2}} \delta F_{\alpha\mu}(\bar{x}_H) \frac{1}{\overrightarrow{\partial_\tau}} \delta E^\mu(\bar{x}_H) + \delta E^\alpha(\bar{x}_H) \frac{1}{\overleftarrow{\partial_\tau^2}} \partial_\mu \delta E_\alpha(\bar{x}_H) \frac{1}{\overrightarrow{\partial_\tau^2}} \delta E^\mu(\bar{x}_H) \Big]$$

(where $\delta E^\mu = \dot{\bar{x}}_{H\nu} \delta F^{\mu\nu}$)

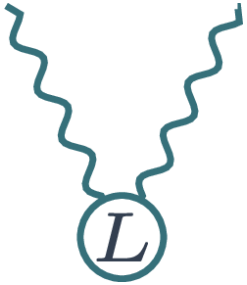
Types of interactions

1SF:

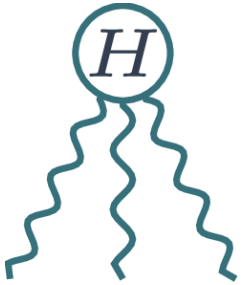
m_L background trajectory source

Heavy particle recoil operator

2SF:



Deviation and propagation



Heavy particle recoil operator

Gravity

Effective action at 1SF

$$S_{\rm GR} = \sum_{i=H,L} m_i \int d\tau \left[-\tfrac{1}{2}\dot{x}_i^\mu \dot{x}_i^\nu g_{\mu\nu}(x_i)\right] + \int d^4x\sqrt{-g}\left[-\tfrac{1}{16\pi G}R\right]$$

Self-energy contributions are dropped and the action at 1SF is

$$\delta S^{\rm eff}_{\rm GR} = \mathcal{R}_{\rm GR} - \int d^4x\sqrt{-\bar{g}}\left[\tfrac{1}{32\pi G}(\tfrac{1}{2}\bar{\nabla}_\rho \delta g_{\mu\nu}\bar{\nabla}^\rho \delta g^{\mu\nu} + \cdots + \tfrac{1}{2}\delta g_{\mu\nu}\bar{T}^{\mu\nu}\right]$$

$$\mathcal{R}_{\rm GR} = -\frac{1}{2}m_H \int d\tau\, \dot{\bar{x}}_H^\alpha \dot{\bar{x}}_H^\beta \delta\Gamma^\mu{}_{\alpha\beta}(\bar{x}_H)\frac{1}{\partial_\tau^2}\dot{\bar{x}}_H^\gamma \dot{\bar{x}}_H^\delta \delta\Gamma_{\mu\gamma\delta}(\bar{x}_H)$$

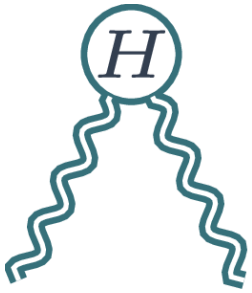
The action describes interactions of a probe particle with fluctuations of the background, heavy particle recoil encoded in a nonlocal operator

Expanding known curved spacetime solutions provides simplification

Types of interactions

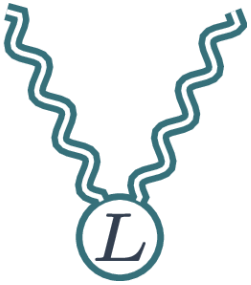
1SF:

Geodesic source



Recoil operator

2SF:



Geodesic deviation

Recoil operator

Self-interactions

The interplay between PM and SF expansions provides an avenue for furthering calculations for the gravitational two-body problem

At 0SF, the Schwarzschild metric and geodesic motion carry all order perturbative information

EFT setup systematically describes perturbative corrections beyond test-particle dynamics needed at higher SF orders while leveraging the resummation provided by the background configurations

Thank You!