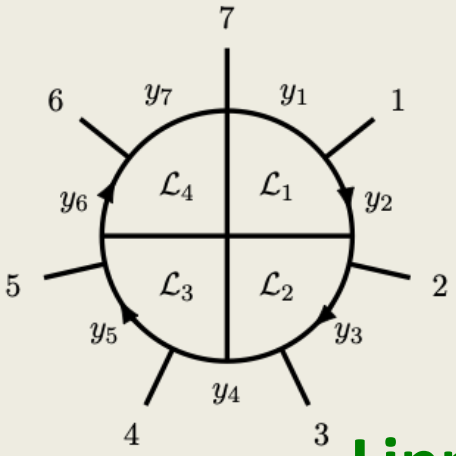
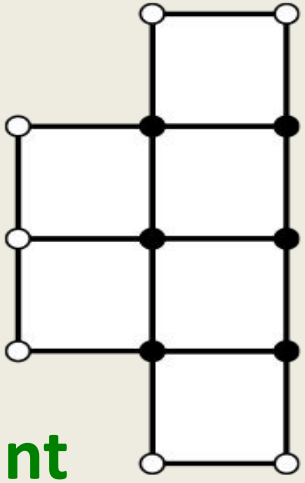
Landau Singularities of Ziggurat Graphs



Anastasia Volovich

Brown and Harvard



Lippstreu, Spradlin, Yelleshpur Srikant

2211.16425 and 2305.17069

QCD Meets Gravity, CERN, December 2023

Goal and Motivation

The goal is to investigate what can be said about the singularity structure of massless planar 7-point Feynman integrals in arbitrary QFTs using the Landau equations.

A primary motivation for this work is to provide a better foundation for the amplitude bootstrap program in N=4 Yang-Mills.

Outline

1. N=4 Yang-Mills symbol alphabet (review)

2. Symbol alphabet from Landau equations

Planar N=4 Yang-Mills Amplitudes

- **Planar N=4 Yang-Mills scattering amplitudes have been computed to very high loop order.**

- **They have many remarkable properties, that have sparked interest from mathematicians working on combinatorics, algebraic geometry and number theory.**

- **At the same time, several methods that have been developed for N=4 Yang-Mills are directly applicable to, and have greatly aided, computations relevant to actual experiments.**

- Current/future programs
- Past programs
- Apply to a program
- Organize a program

Mailing List

Solving N=4 super-Yang-Mills theory via Scattering Amplitudes: January 8, 2024 – March 8, 2024

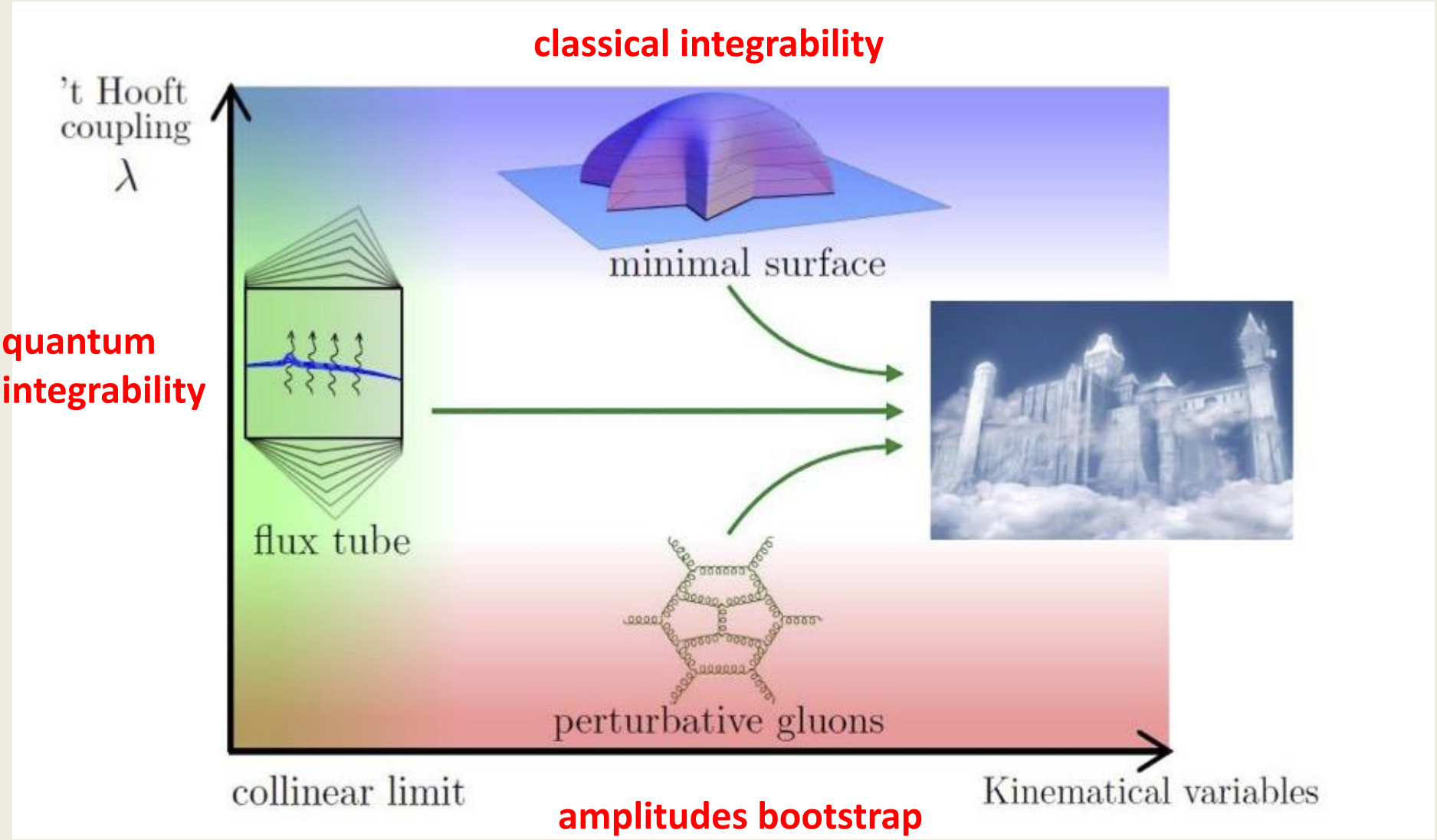
January 26, 2023 by Claudia Landgrover

Organizing by:

- Benjamin Basso (Ecole Normale Superieure),
- Lance Dixon (SLAC/Stanford U.)
- Jaroslav Trnka (UC Davis)
- Anastasia Volovich (Brown)

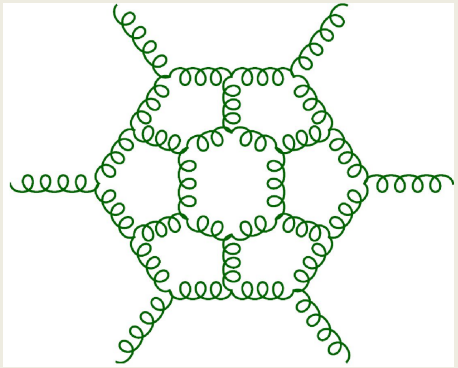
This program will bring together physicists and mathematicians with expertise in different facets of particle scattering in planar N=4 super-Yang-Mills theory, in order to try to solve the theory for generic values of the coupling and kinematical variables.

Solving N=4 Yang-Mills



Status: N=4 Yang-Mills amplitudes

n-point L-loop MHV/NMHV



- n<6 all loops — Bern, Dixon, Smirnov 0505205
- n=6 MHV/NHMV through 7/8-loops — Dixon, Liu 2308.08199
- n=7 through 4-loops — Caron-Huot, Dixon, Drummond, Dulat, Foster, Gurdogan, von Hippel, McLeod, Papathanasiou, review: 2005.06735
- All n MHV through 2-loops — Caron-Huot 1105.5606
- n=8 MHV through 3-loop — Li, Zhang 2110.00350
- n=8, 9 NMHV through 2-loops — He, Li, Zhang 1911.01290, 2009.11471

6 and 7-points: Amplitudes Bootstrap

Caron-Huot, Dixon, Drummond, Dulat, Gurdogan, von Hippel, McLeod, Papathanasiou, 2005.06735

Write down the answer as linear combo of functions and then determine the coefficients by solving a system of linear constraints.

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathscr{H}_6$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Symbol Alphabet

- **One input into the 6 and 7-points amplitudes bootstrap is the symbol alphabet**

- **6-points: 15 letters <a a+1 b c>**
- **7-points: 49 letters <a a+1 b c>**

<1(23)(45)(67)>

<1(27)(34)(56)> +cyclic

$$\mathbf{Z_i^A} = (\mathbf{Z_i^1}, \mathbf{Z_i^2}, \mathbf{Z_i^3}, \mathbf{Z_i^4}) \in \mathbf{P^3}$$

$$\langle ijkl \rangle \equiv \langle Z_i Z_j Z_k Z_l \rangle = \det(Z_i Z_j Z_k Z_l)$$

Goncharov Spradlin Vergu AV

Golden Goncharov Spradlin Vergu AV

Caron-Huot

8, 9, ..n-points: $\bar{Q}$ equation

$$\bar{Q}_a^A = \sum_{i=1}^{n} \chi_i^A \frac{\partial}{\partial Z_i^a}$$

$$\bar{Q}_a^A R_{n,k} = \frac{\Gamma_{\text{cusp}}}{4} \, \text{Res}_{\epsilon=0} \int_{\tau=0}^{\tau=\infty} \left(\text{d}^{2|3} \mathcal{Z}_{n+1} \right)_a^A [R_{n+1,k+1} - R_{n,k} R_{n+1,1}^{\text{tree}}] + \text{cyclic}$$

relates

$$R_{n,k}^{(L)} \text{ to } R_{n+1,k+1}^{(L-1)}$$

Caron-Huot 1105.5606

He, Li, Zhang 1911.01290
2009.11471

Li, Zhang 2110.00350

8-point symbol alphabet

180+24 RATIONAL LETTERS Li, Zhang 2110.00350

- $\binom{8}{4} - 2 = 68$: all $\langle abcd\rangle$ except $\langle 1357\rangle$ and $\langle 2468\rangle$;
- 1 cyclic class of $\langle 12(345)\cap(678)\rangle$;
- 7 cyclic classes of $\langle 1(ij)(kl)(mn)\rangle$ with $2 \le i < j < k < l < m < n \le 8$;
  5 cyclic classes of $\langle 1(28)(kl)(mn)\rangle$ with $2 < k < l < m < n < 8$;
- 5 cyclic classes of $\langle \bar{2}\cap\bar{4}\cap(568)\cap\bar{8}\rangle, \langle \bar{2}\cap\bar{4}\cap\bar{6}\cap(681)\rangle, \langle(127)\cap(235)\cap\bar{5}\cap\bar{7}\rangle, \langle(127)\cap\bar{3}\cap(356)\cap\bar{7}\rangle, \langle\bar{2}\cap(278)\cap(346)\cap\bar{6}\rangle$.

$$\bar{a} = (a-1, a, a+1)$$
$$\langle a(bc)(de)(fg)\rangle := \langle abde\rangle\langle acfg\rangle - \langle acde\rangle\langle abfg\rangle,$$
$$\langle ab(cde)\cap(fgh)\rangle := \langle abde\rangle\langle cfgh\rangle + \langle abec\rangle\langle dfgh\rangle + \langle abcd\rangle\langle efgh\rangle,$$
$$\langle(a_1b_1c_1)\cap(a_2b_2c_2)\cap(a_3b_3c_3)\cap(a_4b_4c_4)\rangle := \langle(a_1b_1c_1)\cap(a_2b_2c_2), (a_3b_3c_3)\cap(a_4b_4c_4)\rangle$$

2 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle$$

\+ 1 cyclic

9-point symbol alphabet

He, Li, Zhang 2009.11471

59 x 9 RATIONAL LETTERS

- 13 cyclic classes of $\langle 12kl\rangle$ for $3 \leq k < l \leq 8$ but $(k,l) \neq (6,7),(7,8)$;
- 7 cyclic classes of $\langle 12(ijk)\cap(lmn)\rangle$ for $3 \leq i < j < k < l < m < n \leq 9$;
- 8 cyclic classes of $\langle \bar{2}\cap(245)\cap\bar{6}\cap(691)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{6}\cap(892)\rangle$, $\langle \bar{2}\cap(346)\cap\bar{2}\cap(782)\rangle$, $\langle \bar{2}\cap(245)\cap\bar{7}\cap(791)\rangle$, $\langle \bar{2}\cap(245)\cap(568)\cap\bar{8}\rangle$, $\langle \bar{2}\cap(245)\cap(569)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(245)\cap(679)\cap\bar{9}\rangle$, $\langle \bar{2}\cap(256)\cap(679)\cap\bar{9}\rangle$;
- 10 cyclic classes of $\langle 1(i\,i{+}1)(j\,j{+}1)(k\,k{+}1)\rangle$ for $2 \leq i,\ i+1 < j,\ j+1 < k \leq 8$;
- 6 cyclic classes $\langle 1(2i)(j\,j{+}1)(k9)\rangle$ for $3 \leq i < j,\ j+1 < k \leq 8$, but $(i,k) \neq (3,8),(4,7)$;
- 14 cyclic classes of $\langle 1(29)(ij)(k\,k{+}1)\rangle$ for $3 < i < j \leq 8,\ 3 \leq k \leq i-2$ or $j+1 \leq k \leq 7$;
- 1 cyclic class of $\langle 1,(56)\cap\bar{3},(78)\cap\bar{3},9\rangle$.

11 x 9 ALGEBRAIC LETTERS (SQUARE ROOTS)

$$\Delta_{1357} = (\langle 1256\rangle\langle 3478\rangle - \langle 1278\rangle\langle 3456\rangle - \langle 1234\rangle\langle 5678\rangle)^2 - 4\langle 1234\rangle\langle 3456\rangle\langle 5678\rangle\langle 1278\rangle \quad + \quad \text{8 cyclic}$$

Symbol Letters from Mathematics

- **Cluster Algebras** Golden, Goncharov, Spradlin, Vergu, AV
- **Tropical Geometry** Drummond, Foster, Gurdogan, Kalousios; Henke, Papathanasiou
- **Dual Polytopes** Arkani-Hamed, Lam, Spradlin
- **Plabic Graphs** Mago, Schreiber, Spradlin, Srikant, AV; He, Li
- **Tensor Diagrams** Ren, Spradlin, AV
- **Scattering Diagrams** Herderschee
- **Schubert Problem** Yang
- **A-determinant** Dlapa, Helmer, Papathanasiou, Tellander

Symbol Letters from Physics

Can we have a general derivation of these letters?

Symbol letters encode information about the location of singularities of an amplitude.

Landau formulated the sufficient conditions for a Feynman integral to have singularities decades ago…

Outline

1. N=4 Yang-Mills symbol alphabet (review)

2. Symbol alphabet from Landau equations

Landau Singularities

Landau 1959, ELOP

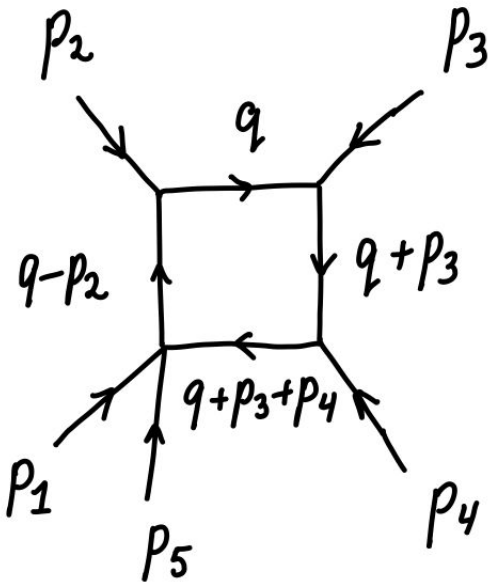
Landau Equations for a given Feynman integral = set of kinematic constraints that are necessary for the appearance of a pole or a branch point in the integrated function

$$\sum_{i \in \text{loop}} \alpha_i q_i^\mu = 0 \quad \forall \text{ loops},$$

$$\alpha_i (q_i^2 - m_i^2) = 0 \quad \forall i.$$

Landau Singularities occur for external momenta such that the Landau equations have solutions.

Example



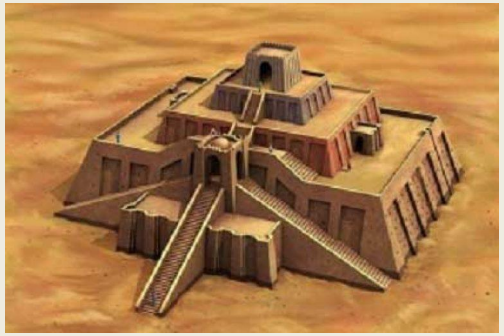
$$\int \frac{d^4 q}{q^2 (q-p_2)^2 (q+p_3)^2 (q+p_3+p_4)^2}$$

$$\begin{cases} (q-p_2)^2 = q^2 = (q+p_3)^2 = (q+p_3+p_4)^2 \\ \alpha_1 (q-p_2) + \alpha_2 q + \alpha_3 (q+p_3) + \alpha_4 (q+p_3+p_4) = 0 \end{cases}$$

$$(p_2+p_3)^2 (p_3+p_4)^2 = 0$$

OR

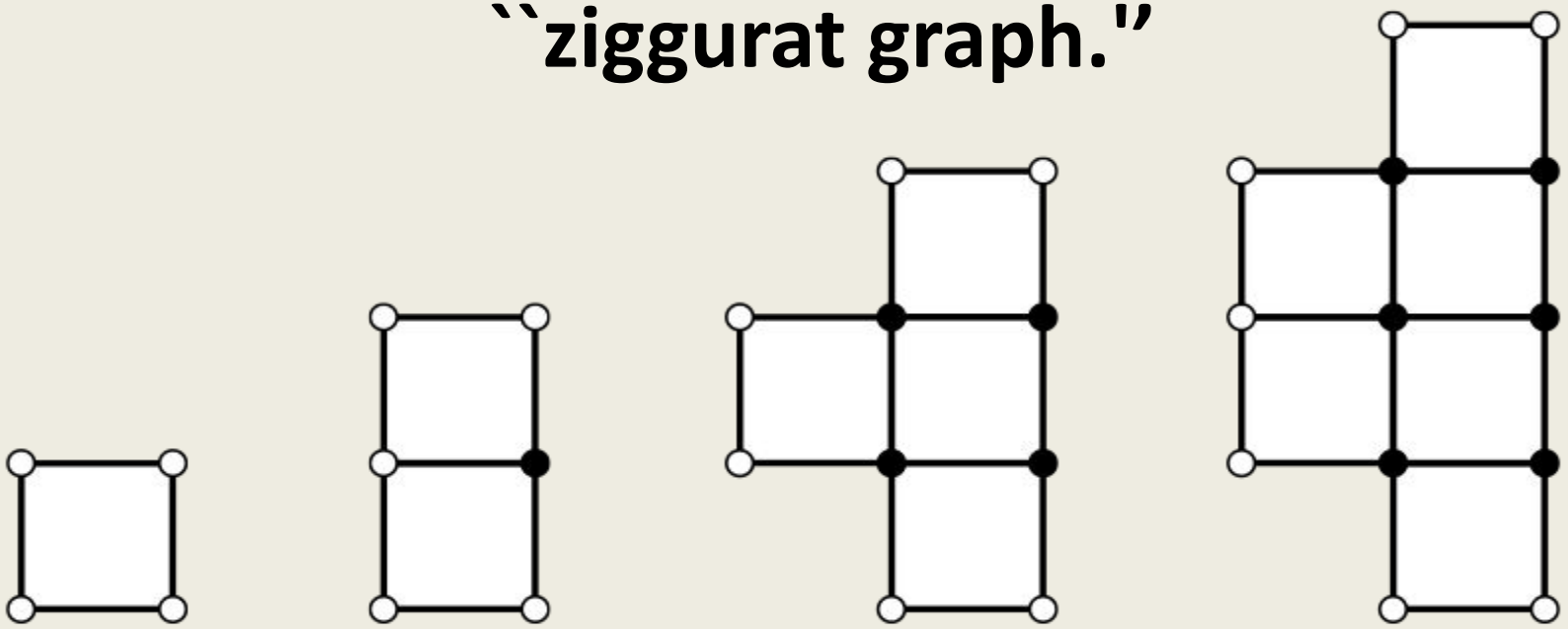
$$\langle 1234 \rangle \langle 2345 \rangle = 0$$



Ziggurat Graphs

Prlina, Spradlin, Stanojevic 1805.11617

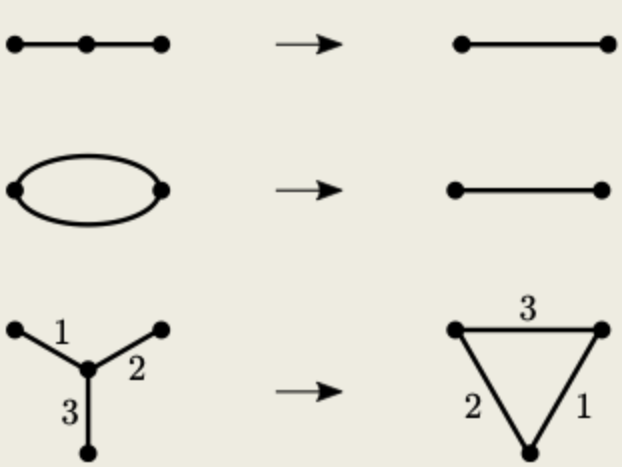
Landau singularities of any n-particle amplitude in any massless, planar theory are a subset of those of a special type of ``ziggurat graph.''



Landau singularities from Ziggurats

- **Landau equations are invariant under circuit moves**

Prlina, Spradlin, Stanojevic



- **Any graph can be reduced to ziggurat (or a relaxation) by these moves.**

Gitler

Six-point Ziggurat

Prlina, Spradlin, Stanojevic

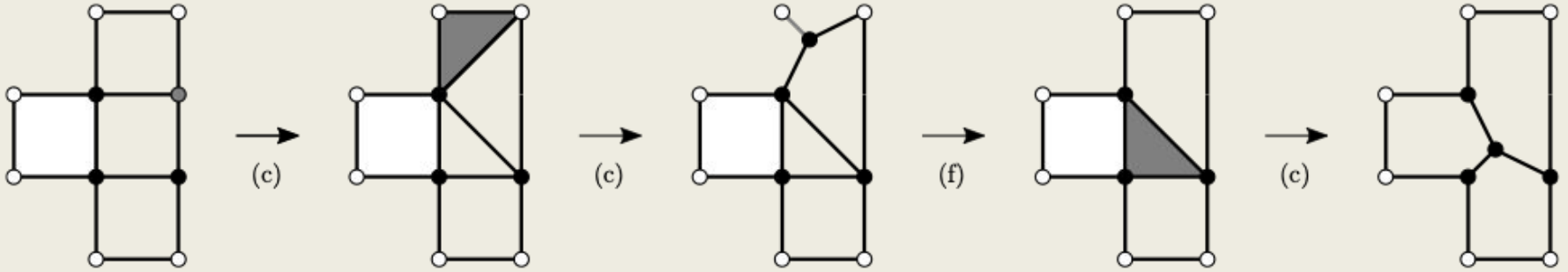
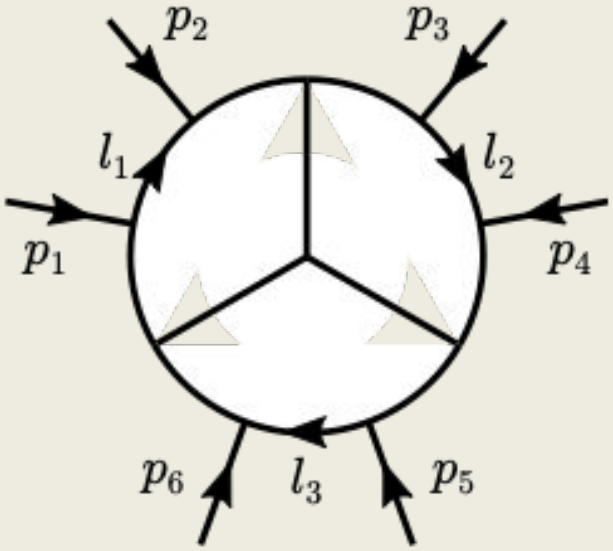
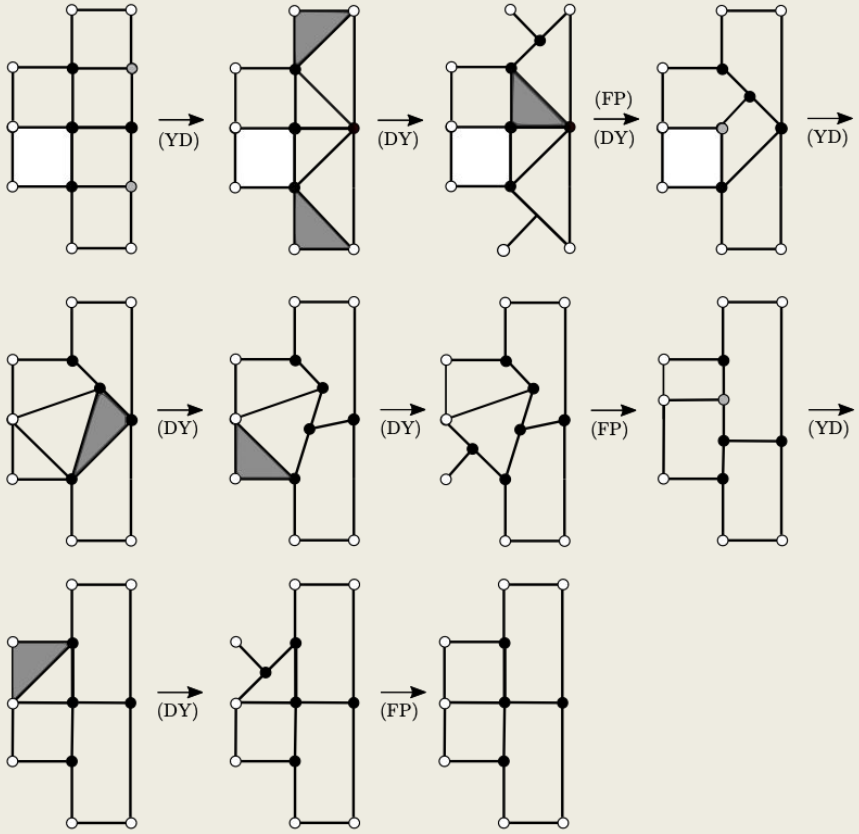


FIG. 3. The six-terminal ziggurat graph can be reduced to a three loop graph by a sequence of three Y-Δ reductions and one FP assignment. In each case the vertex, edge, or face to be transformed is highlighted in gray.



Matches six-point symbol letters!

Seven-point Ziggurat



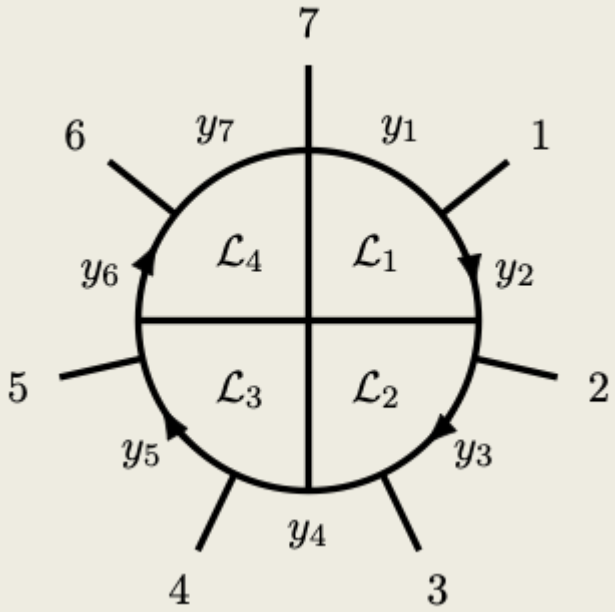


Figure 3: A sequence of graphical moves (see [15]) that transforms the 7-point ziggurat graph Fig. 1(d) into the wheel graph Fig. 2(b). YD indicates wye-delta transformation(s) on the node(s) shaded in grey; DY indicates delta-wye transformations(s) on the triangle(s) shaded in grey, and FP indicates a trivial contraction of external edges.

Seven-point Ziggurat

- **We computed leading Landau singularities and found agreement with all but 7 letters.**

  Lippstreu, Spradlin, AV

- **Next, we analyzed subleading Landau singularities.**

Lippstreu, Spradlin, Yelleshpur Srikant, AV

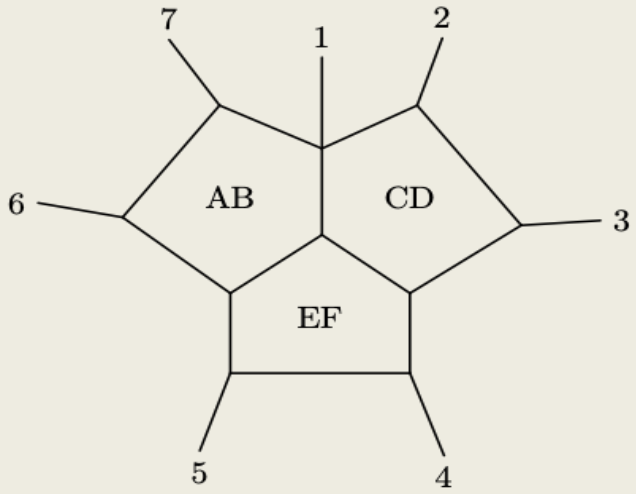


Figure 2: Graph 1. The Landau equations contain the letter $\langle 6(17)(25)(34)\rangle$, which is outside the heptagon symbol alphabet.

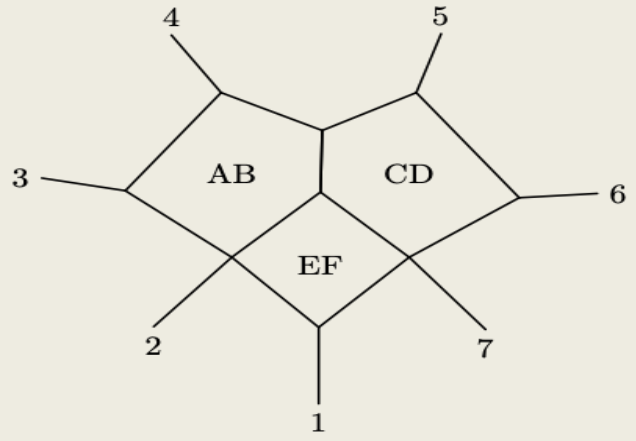


Figure 3: Graph 2. The Landau equations contain the letter $\langle 1(27)(36)(45)\rangle$, which is outside the heptagon symbol alphabet.

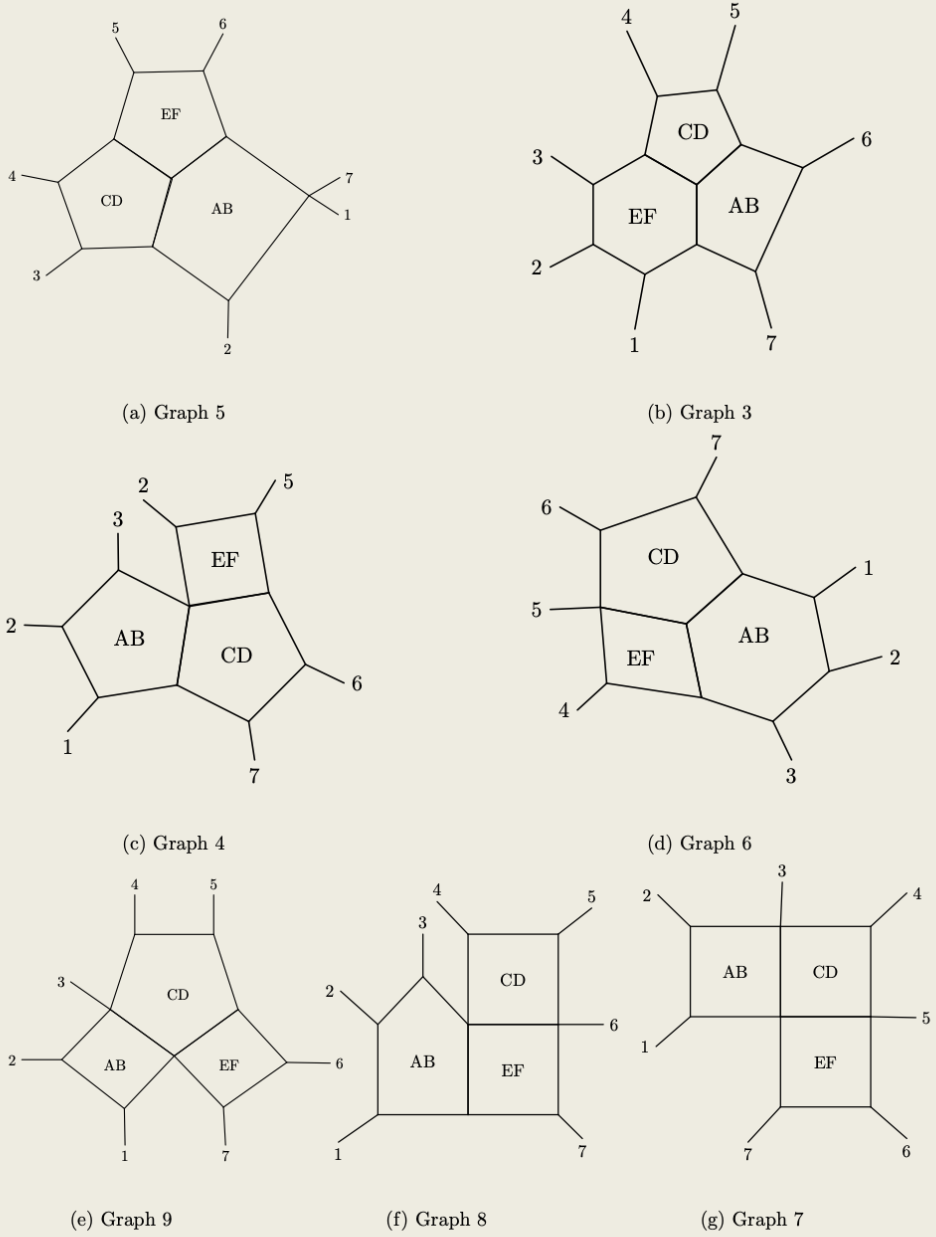


Figure 6: Seven other three-loop graphs we analyzed and did not find any letters beyond the heptagon symbol alphabet.

Seven-point Ziggurat

- We computed leading Landau singularities and find agreement with all but 7 letters.
- Next, we analyzed subleading Landau singularities.
- We found the missing 7, and extra letters

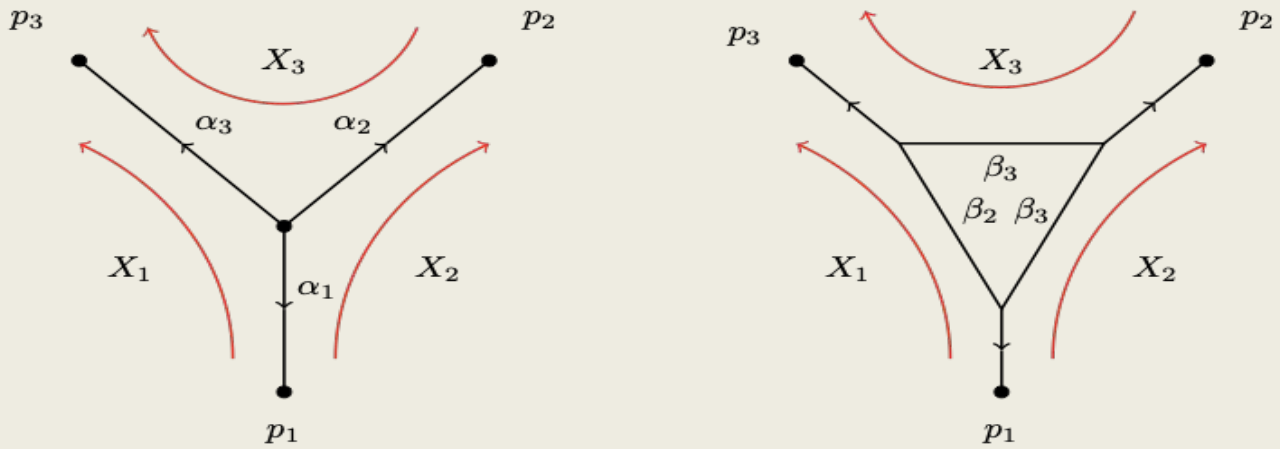
  <6(17)(25)(34)>and<1(27)(36)(45)>
- They appear in certain 3-loop Feynman integrals and it will be interesting to see if they appear in higher loop amplitudes.

Lippstreu, Spradlin, Yelleshpur Srikant, AV

Violation of Y-Delta

Lippstreu, Spradlin, Yelleshpur Srikant, AV

- **Along the way, we found that there are some solutions that violate Y-Delta equivalence.**



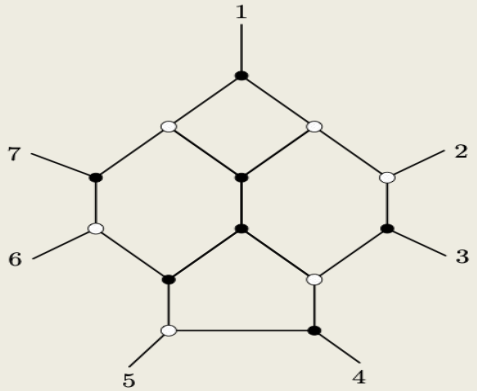
- **When $X_1 = X_2 = X_3 = 0$, the Y has extra solutions that are not present in the Delta.**

Scalar vs N=4 leading singularities

We compared a particular leading singularity of the scalar integral associated to the graph and the corresponding N=4 leading singularity computed via its on-shell diagram.

$$LS_1^{\text{scalar}} = \frac{1}{\langle 3467\rangle \langle 1567\rangle \langle 1367\rangle \langle 1234\rangle \langle 1467\rangle \langle 3456\rangle \langle 6(17)(34)(25)\rangle}$$

$$LS_1^{\mathcal{N}=4} = \frac{\langle 2567\rangle^3 \langle 3456\rangle^3}{\langle 1256\rangle \langle 1267\rangle \langle 1567\rangle \langle 2345\rangle \langle 2356\rangle \langle 2456\rangle \langle 2(17)(34)(56)\rangle}$$



Non-heptagon letter <6(17)(34)(25)> is absent, indicating cancellations occurring in N=4 Yang-Mills.

Conclusions

We solved Landau equations for nine 3-loop 7-point graphs that arise as relaxations of "ziggurat" graph. We found two graphs with singularities outside heptagon alphabet.

Future: all n, all L, beyond polylogs, beyond planar, beyond N=4….

Thank you!