Hidden zeros of string/particle amplitudes: the unity of colored scalars, pions & gluons

To appear (+ more to appear) with Nima Arkani-Hamed, Qu Cao, Jin Dong & Carolina Figueiredo

see Nima's talk @ Amplitudes 2023 here

Song He (ITP-CAS)
QCD Meets Gravity (CERN)
12.12. 2023

Introduction

Toy Models —> Real World via "numerators/zeros"

Combinatorial/Geometric structures: e.g. N=4 SYM/ABJM, or Tr ϕ^3 (simplest colored scalars)
Why? Amplitudes uniquely determined by long-distance sing. or "denominators" !

More realistic theories: need "pole @ infinity" or numerators: N=4 (no pole @ infinity by DCI) vs. N<4;
Tr ϕ^3 (projective inv.) vs. other scalar theories e.g. ϕ^p , derivative coup. such as pions, even gluons?

What are "zeros" of (tree) amplitudes? already highly non-trivial for Tr ϕ^3: pattern of zeros (some $s_{i,j} = 0$) & surprising factorizations near them; hidden in Feynman diagrams, manifest by geometries!

Amazingly, these zeros+ fact. generalize to non-linear sigma model (NLSM) & Yang-Mills (YM)

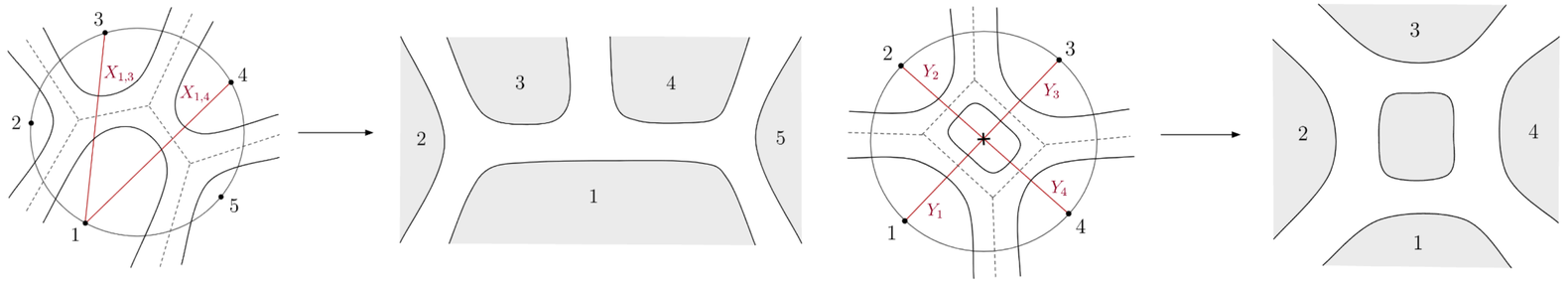
—> tree amps of Tr ϕ^3, pions & gluons given by one and same function at different kinematic points

***upcoming works:* all-loop NLSM & YM contained in all-loop stringy Tr ϕ^3**

Tr ϕ^3 amplitude [Arkani-Hamed, Bai, SH, Yan, '17; Arkani-Hamed, Frost, Salvatori, Plamondon, Thomas, '23]

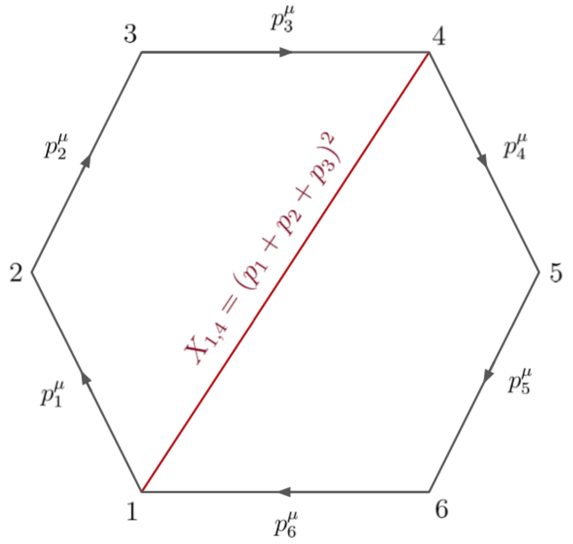
ϕ : N by N matrix -> fat graphs, genus expansion (only planar graphs for $N \to \infty$)

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$



planar variables: all poles of tree amps

$$X_{i,j} = (p_i + \cdots + p_{j-1})^2.$$



tree amp= sum over n-gon triangulations, e.g.

$$\mathcal{A}_4 = \frac{1}{X_{13}} + \frac{1}{X_{24}}\,,$$

$$\mathcal{A}_5 = \frac{1}{X_{1,3}X_{1,4}} + \frac{1}{X_{2,4}X_{2,5}} + \frac{1}{X_{1,3}X_{3,5}} + \frac{1}{X_{1,4}X_{2,4}} + \frac{1}{X_{2,5}X_{3,5}}.$$

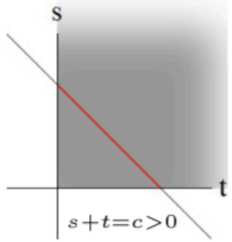
non-planar Mandelstam variables

$$c_{i,j} := -2p_i \cdot p_j = X_{i,j} + X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1}\,,$$

Tree amplitude from associahedron [ABHY; Arkani-Hamed, SH, Salvatori, Thomas '19]

$\mathscr{A}_{n-3}$: $\{X_{i,j} \geq 0\} \cap$ (n-3)-dim subspace

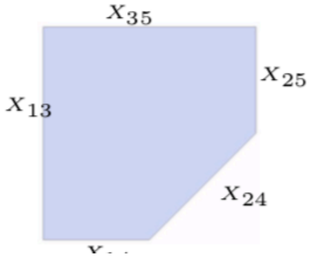
$c_{i,j} = \text{const.} > 0$ e.g. for $1 \leq i < j \leq n-1$

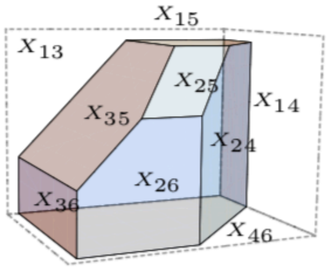


$$e.g.\ \mathcal{A}_1 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_2 = \{X_{13}, \cdots, X_{25} > 0\} \cap \{-s_{13} = c_{13}, -s_{14} = c_{14}, -s_{24} = c_{24}\}$$

$$\mathscr{A}_2 : (n=5) \begin{cases} X_{1,3} > 0 \\ X_{1,4} > 0 \\ X_{2,4} > 0 \Leftrightarrow c_{1,3} - X_{1,3} + X_{1,4} > 0 \\ X_{2,5} > 0 \Leftrightarrow c_{1,3} + c_{1,4} - X_{1,3} > 0 \\ X_{3,5} > 0 \Leftrightarrow c_{1,4} + c_{2,4} - X_{1,4} > 0 \end{cases}$$





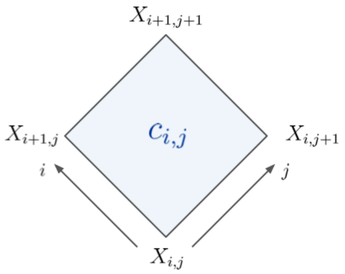
$$\Omega(\mathscr{A}_2) = d^2X \left(\frac{1}{X_{13}X_{14}} + \frac{1}{X_{13}X_{35}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{25}X_{24}} + \frac{1}{X_{24}X_{14}} \right)$$

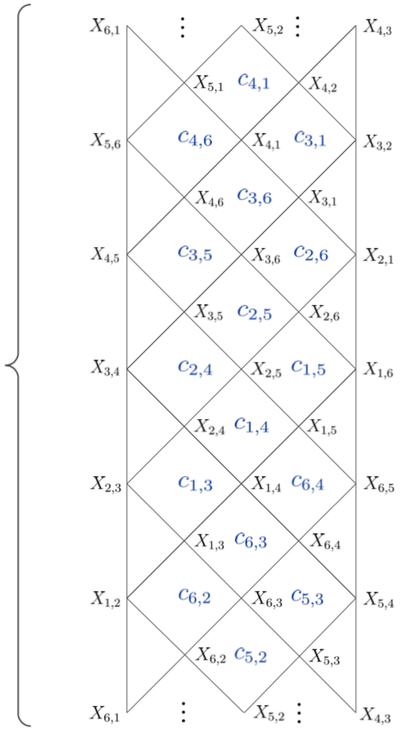
$$\Omega(\mathscr{A}_1) = \left(\frac{dX_{13}}{X_{13}} - \frac{dX_{24}}{X_{24}}\right)\Big|_{X_{13}+X_{24}=c_{13}} = dX_{13} \left(\frac{1}{X_{13}} + \frac{1}{c_{13} - X_{13}} \right)$$

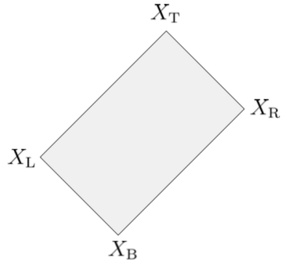
geometric picture: FD expansion = a special triangulation, others → new formula & recursion for ϕ^3 amps

hidden symmetry of ϕ^3 amps (invisible in FD's), analog of dual conformal symmetry, manifest by geometry!

Kinematic mesh: 1+1 dim wave eqs & causal diamonds

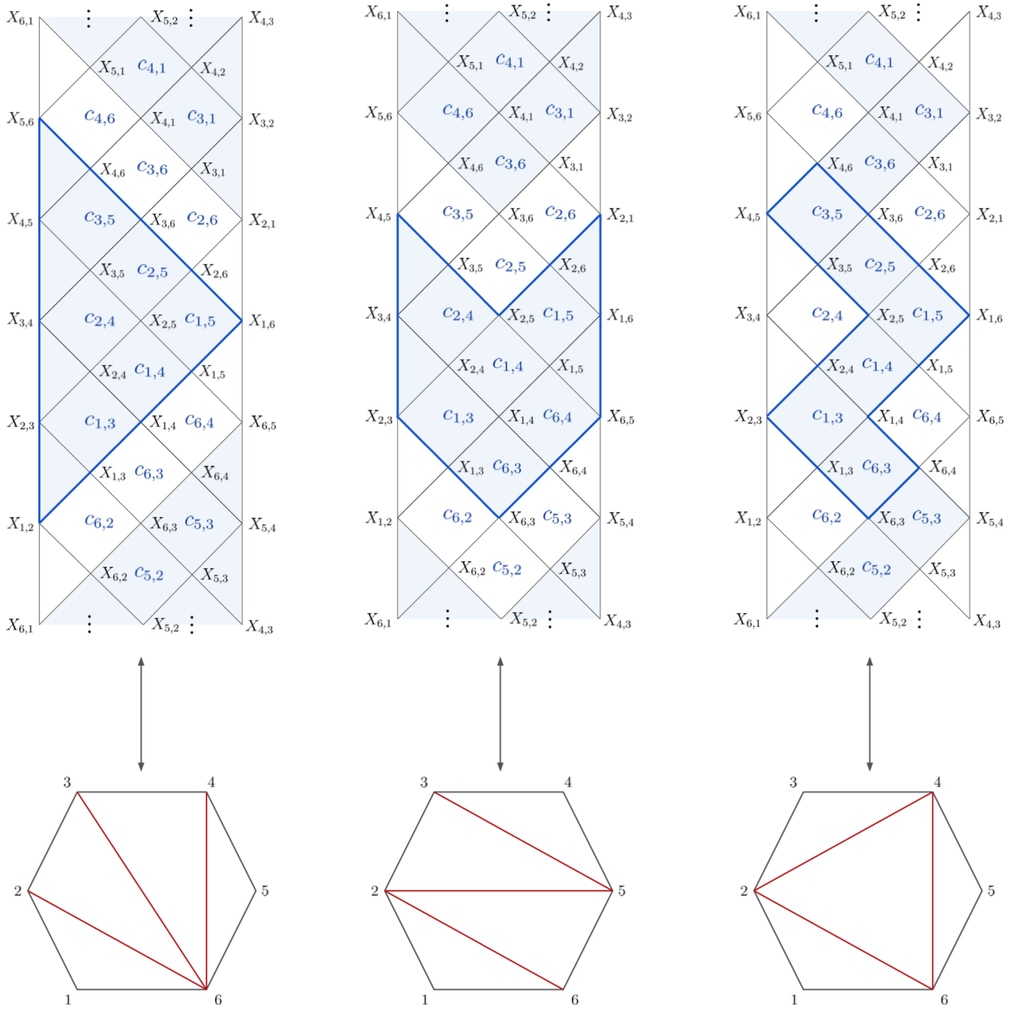






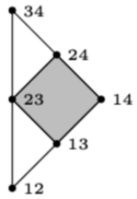
$$X_{\mathrm{B}} + X_{\mathrm{T}} - X_{\mathrm{L}} - X_{\mathrm{R}} = \sum_{c_{i,j} \in \diamond} c_{i,j}$$

mesh regions <=> initial triangulations (n=6)

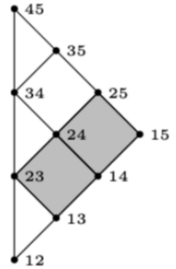


A surprise: zeros of Tr ϕ^3 on the mesh [Arkani-Hamed, Cao, Dong, Figuereido, SH, to appear]

$$n=4: c_{13}=0 \implies \frac{1}{X_{13}}+\frac{1}{X_{24}}=\frac{c_{13}}{X_{13}X_{24}}=0$$



$$n=5: c_{13}=c_{14}=0\,,\ \text{or}\ c_{14}=c_{24}=0\,,\text{etc}\,.$$



the big cubic polynomial $N^{(3)}=0$

Very difficult to see in Feynman diagrams:

$$\frac{1}{X_{13}X_{14}}+\frac{1}{X_{13}X_{35}}+\frac{1}{X_{25}X_{35}}+\frac{1}{X_{25}X_{24}}+\frac{1}{X_{24}X_{14}}=\frac{N^{(3)}(\{X\})}{X_{13}X_{24}X_{35}X_{14}X_{25}}$$

n-pt: highly non-trivial linear subspaces of the big numerator e.g.

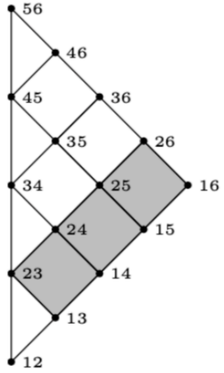
skinny rectangle ("soft limit")

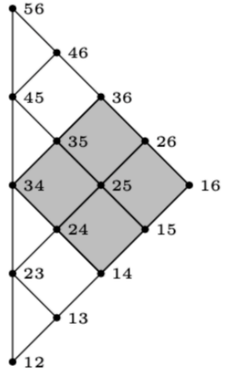
$$c_{13}=c_{14}=\cdots=c_{1,n-1}=0$$

2 by 2 square

$$n=6: \text{also}\ c_{14}=c_{15}=c_{24}=c_{25}=0\,;$$

generally any rectangle of the mesh





A lot more surprises for various particle/string amps [ACDFH]

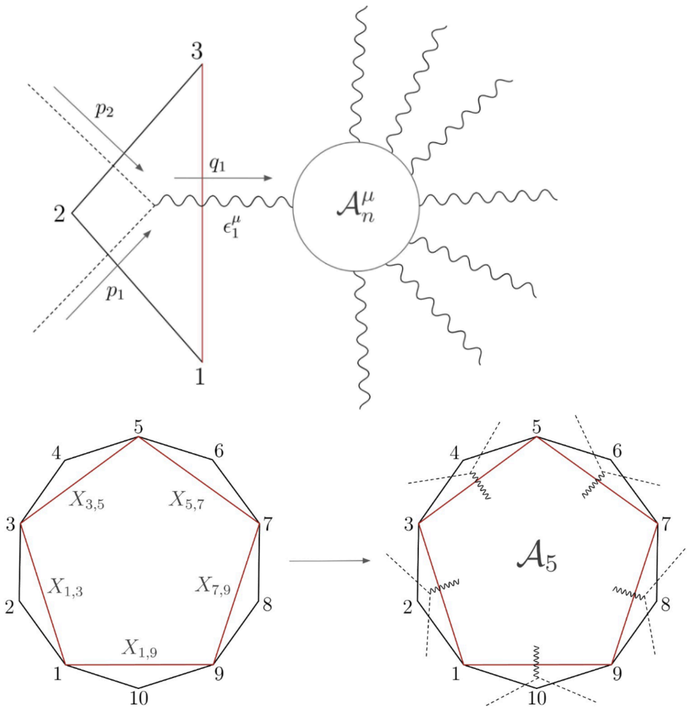
three extensions: factorizations, strings, pions/gluons:

- new factorizations near zero: turn on one $c_{i,j} \neq 0$: amp -> L * R * (4-pt = $\frac{c_{i,j}}{X_{i,j}(c_{i,j} - X_{i,j})}$)
- extend to stringy ϕ^3 amps: same patterns of zeros with $c_{i,j} = -n_{i,j}$ (non-positive integers)
- remarkably, same zero+fact. for 2n-pt NLSM amplitude or YMS amplitude (contains YM amp)!

$$\mathcal{L}_{\rm NLSM} = \frac{1}{8\lambda^2}\,{\rm Tr}\left(\partial_\mu {\rm U}^\dagger \partial^\mu {\rm U}\right), \quad \text{with} \quad {\rm U} = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$$

$$\mathcal{A}_4^{\rm NLSM} = X_{1,3} + X_{2,4} = c_{1,3} \xrightarrow{c_{1,3}=0} \mathcal{A}_4^{\rm NLSM} = 0,$$

$$\mathcal{L}_{\rm YMS} = -\,{\rm Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g_{\rm YM}^2}{4}\sum_{I\neq J}\left[\phi^I,\phi^J\right]^2\right)$$



2n-scalars (same zeros+fact.) -> scaffolded to n gluons: each $c_{i,j} = 0$

-> 4 conditions $p_i \cdot p_j = \epsilon_i \cdot p_j = p_i \cdot \epsilon_j = \epsilon_i \cdot \epsilon_j = 0$

Explaining Tr ϕ^3 zeros: $\mathscr{A}_{n-3}$ as Minkowski sum

key: ϕ^3 zero <=> associahedron collapses to lower dim.

ABHY = Minkowski sum of building blocks: $\sum_{i,j} c_{i,j}\Delta_{i,j}$

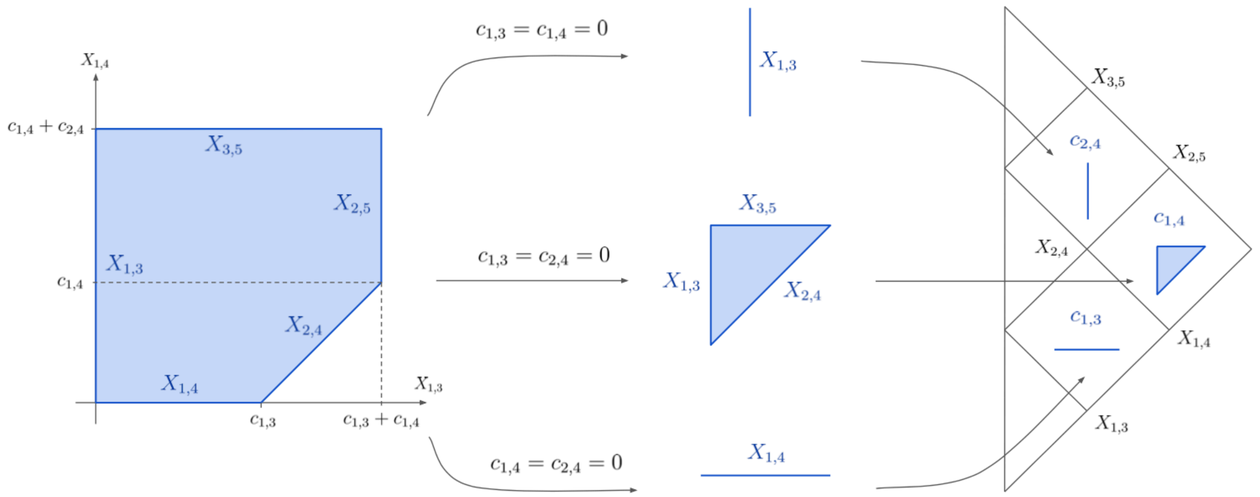
$\Delta_{i,j}$: $c_{i,j} = 1$, all other $c = 0$ —> lower dim. if enough c=0

n=5
$$\begin{cases} X_{1,3} > 0 \\ X_{1,4} > 0 \\ X_{2,4} > 0 \Leftrightarrow c_{1,3} - X_{1,3} + X_{1,4} > 0 \\ X_{2,5} > 0 \Leftrightarrow c_{1,3} + c_{1,4} - X_{1,3} > 0 \\ X_{3,5} > 0 \Leftrightarrow c_{1,4} + c_{2,4} - X_{1,4} > 0 \end{cases}$$

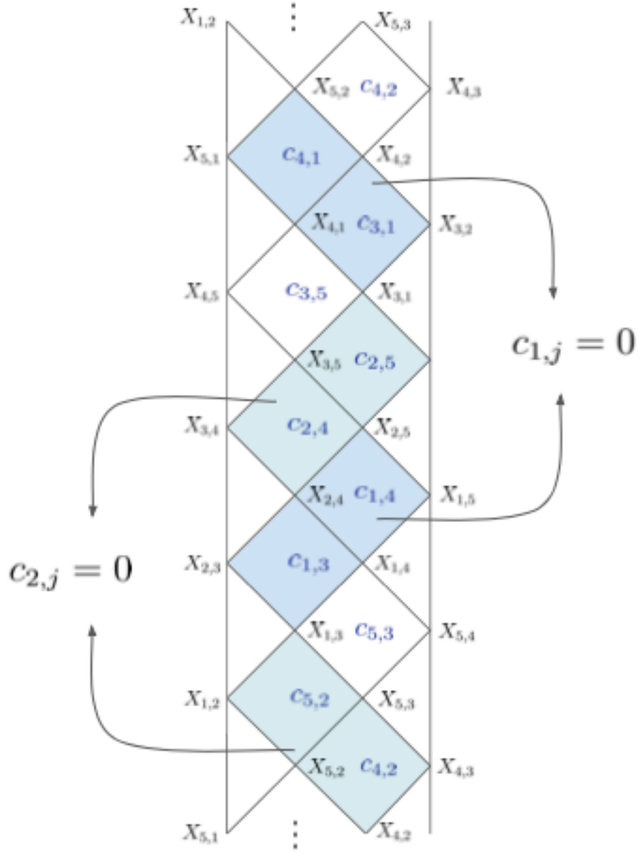
n=5: 5 zeros $c_{1,3} = c_{1,4} = 0$ & cyclic

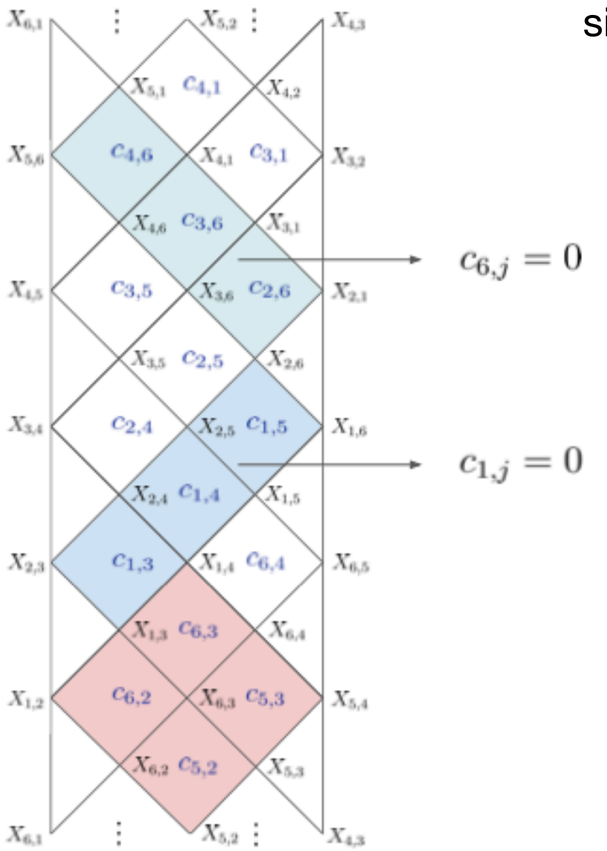
n=6: 6 “skinny” zeros $c_{1,3} = c_{1,4} = c_{1,5} = 0$ & cyclic +
3 “square” zeros $c_{1,4} = c_{2,4} = c_{1,5} = c_{2,5} = 0$ & cyclic

general n: $\frac{n(n-3)}{2}$ zeros; same # as poles!



Zeros on the mesh: n=5,6 examples





e.g. n=5: $c_{13} = c_{24} = 0$ not a zero

n=6: $c_{13} = c_{14} = c_{24} = c_{25} = 0$ not a zero
since they are still top dimensional!

Remarkably, already "skinny" zeros suffice to uniquely determine ϕ^3amps (checked up to n=7)

byproduct: "skinny" zeros imply Adler zeros for NLSM (not YMS or ϕ^3) amps

$p_1 \to 0 \implies c_{1,i} = 0$

+ no poles at $s_{12} = 0$, $s_{1n} = 0$!

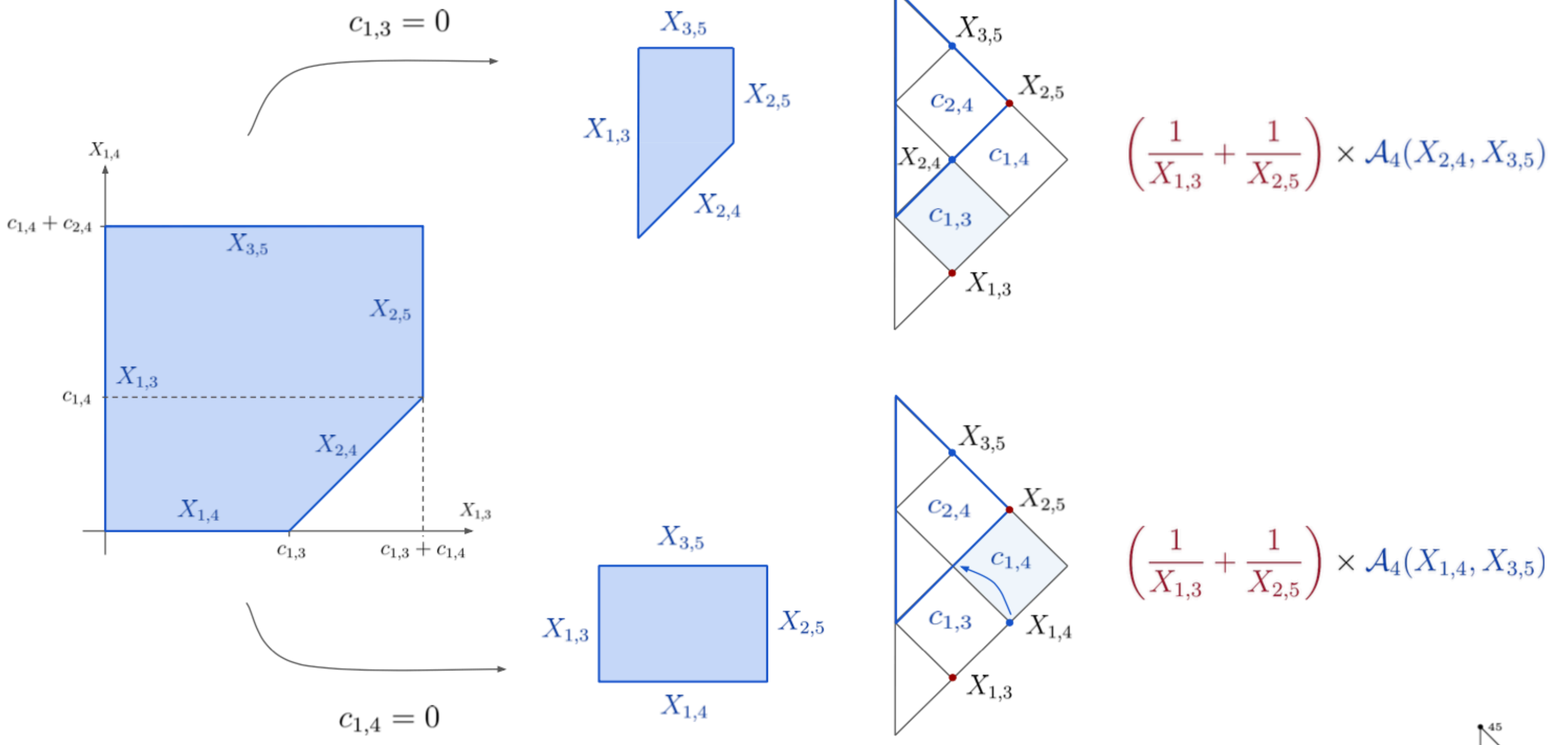
Factorization of 5-pt amplitude

the last step: turn on one $c \neq 0$

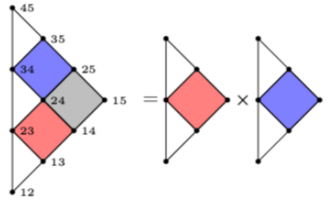
$$\mathcal{A}_5(X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}) \xrightarrow{c_{1,3}=0} \left(\frac{1}{X_{1,3}} + \frac{1}{X_{2,5}}\right) \times \left(\frac{1}{X_{2,4}} + \frac{1}{X_{3,5}}\right),$$

$c_{1,4} \neq 0$:
Mink. sum interval + triangle

$c_{1,3} \neq 0$:
Mink. sum interval + interval

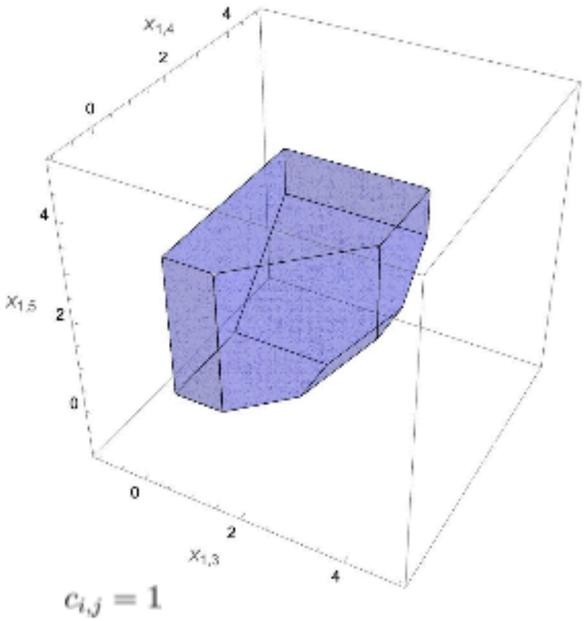


$$\mathcal{A}_5(X_{1,3}, X_{1,4}, X_{2,4}, X_{2,5}, X_{3,5}) \xrightarrow{c_{1,4}=0} \left(\frac{1}{X_{1,3}} + \frac{1}{X_{2,5}}\right) \times \left(\frac{1}{X_{1,4}} + \frac{1}{X_{3,5}}\right)$$



Factorization of 6-pt amplitude

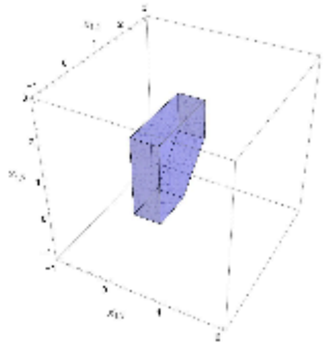
$$\mathcal{A}_6 \xrightarrow{c_{1,3}=0,c_{1,5}=0} \left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,6}}\right)\times\left(\frac{1}{X_{2,4}X_{1,5}}+\frac{1}{X_{1,5}X_{3,5}}+\frac{1}{X_{3,5}X_{3,6}}+\frac{1}{X_{3,6}X_{4,6}}+\frac{1}{X_{4,6}X_{2,4}}\right)$$



$c_{i,j}=1$

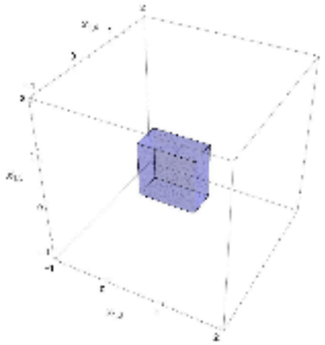
$c_{1,3}=0.5$

$c_{1,4}=c_{1,5}=0$

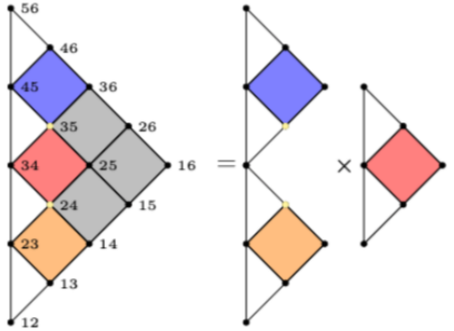


$c_{2,4}=0.5$

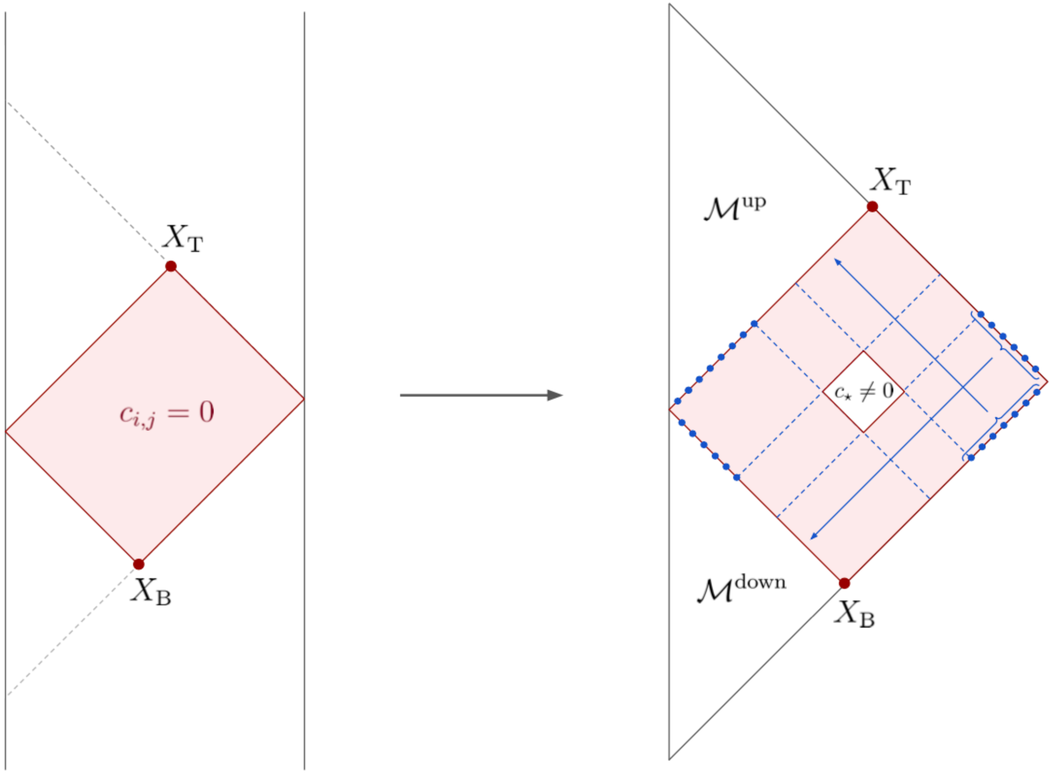
$c_{1,4}=c_{1,5}=c_{2,5}=0$



$$\mathcal{A}_6 \xrightarrow{c_{1,4}=0,c_{1,5}=0,c_{2,5}=0} \left(\frac{1}{X_{1,4}}+\frac{1}{X_{3,6}}\right)\times\left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,6}}\right)\times\left(\frac{1}{X_{1,5}}+\frac{1}{X_{4,6}}\right)$$



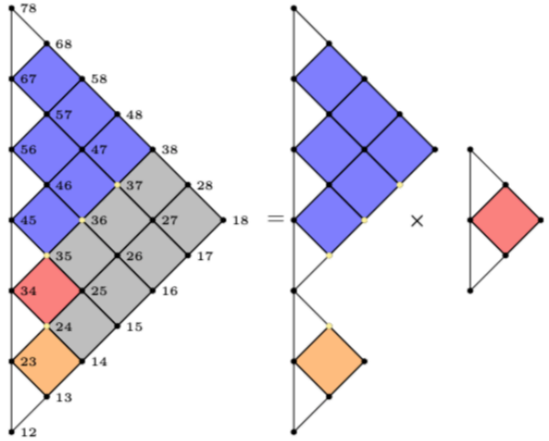
General zeros & factorizations



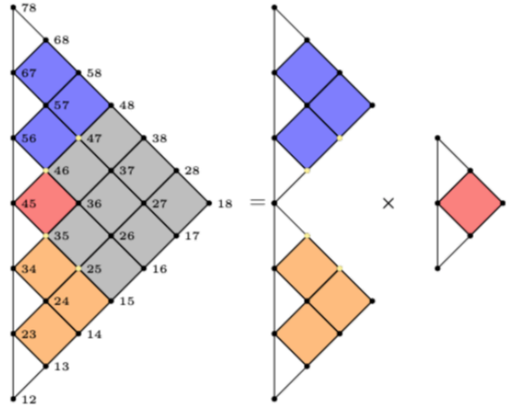
$$\mathcal{M}_n(c_\star \neq 0) = \left(\frac{1}{X_{\mathrm{B}}} + \frac{1}{X_{\mathrm{T}}}\right) \times \mathcal{M}^{\mathrm{down}} \times \mathcal{M}^{\mathrm{up}}.$$

$$\tilde{X} = X_{\mathrm{T}} + X, \quad X = X_B + \tilde{X}.$$

8-> 6*4*4



8-> 5*4*5



hints: same zeros & fact. for “shifted” amps with all the c preserved

Stringy Tr ϕ^3 amplitude [Arkani-Hamed, SH, Lam, 19']

known as "stringy integrals for ABHY" or Z integrals [Carrasco, Mafra, Schlotterer]

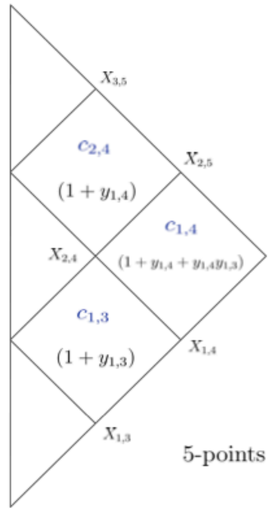
$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3}(1,2,\ldots,n) = \int_{D(1\ldots n)} \frac{\mathrm{d}z_1\ldots\mathrm{d}z_n}{\mathrm{vol}\ \mathrm{SL}(2,\mathbb{R})} \underbrace{\frac{1}{z_{1,2}z_{2,3}\ldots z_{n,1}}}_{\mathrm{PT}(1,2,\ldots,n)} \times \underbrace{\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Koba-Nielsen factor}}$$

u variables & positive para. y

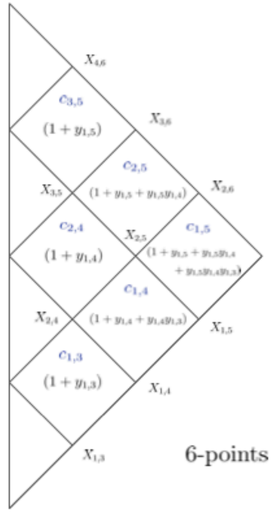
$$u_{i,j} = \frac{z_{i-1,j}z_{i,j-1}}{z_{i,j}z_{i-1,j-1}}.$$

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{I=1}^{n-3} \frac{\mathrm{d}y_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}}(y) = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{I=1}^{n-3} \frac{\mathrm{d}y_I}{y_I} y_I^{\alpha' X_I} \prod_{i,j} F_{i,j}(y)^{-\alpha' c_{i,j}}$$

ray-like triangulation => $F_{i,j} = 1 + y_{1,j} + y_{1,j}y_{1,j-1} + \cdots + y_{1,j}\ldots y_{1,i+2}.$



5-points



6-points

$$\mathcal{I}_5^{\mathrm{Tr}\,\phi^3} = \int_0^\infty \prod_{i=3}^{4} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}} \prod_{i<j} F_{i,j}(\mathbf{y})^{-\alpha' c_{i,j}}$$

$$= \int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}} \frac{\mathrm{d}y_{1,4}}{y_{1,4}} y_{1,3}^{\alpha' X_{1,3}} y_{1,4}^{\alpha' X_{1,4}} (1+y_{1,3})^{-\alpha' c_{1,3}} (1+y_{1,4})^{-\alpha' c_{2,4}} (1+y_{1,4}+y_{1,3}y_{1,4})^{-\alpha' c_{1,4}}.$$

Zeros of string amplitude

$$c_{1,3} = c_{1,4} = 0\,, \qquad \mathcal{I}_5^{\mathrm{Tr}\,\phi^3} \to \left(\int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}}\right) \int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}} y_{1,4}^{\alpha' X_{1,4}} \left(1+y_{1,4}\right)^{-\alpha' c_{2,4}} = 0.$$

any non-positive integer works! e.g.

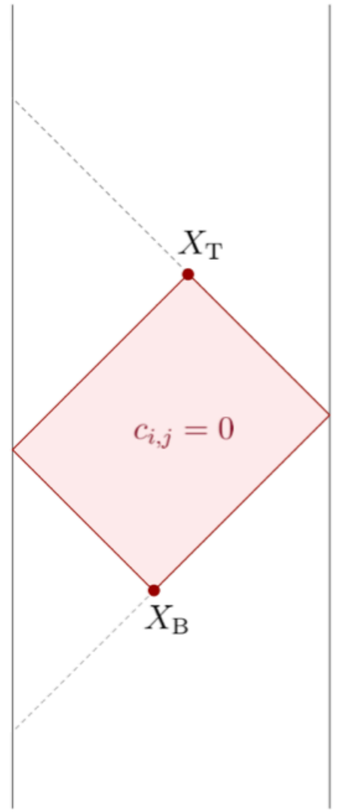
by setting $\alpha' c_{1,3} = -n,$

$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} \to \sum_{k=0}^{n} \underbrace{\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}+k}}_{=0} = 0.$$

$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \sum_{k_{a_1,b_1},\ldots,k_{a_N,b_N}=0}^{n_{a_1,b_1},\ldots,n_{a_N,b_N}} (\text{remaining integrals}) \times \underbrace{\int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}+k_{a_1,b_1}+\cdots+k_{a_N,b_N}}}_{=0} = 0$$

$$c_{i,j} = -\,n_{ij}\,, \quad 1 \le i < a-1,\ a \le j < n$$

n(n-3)/2 infinite families of zeros

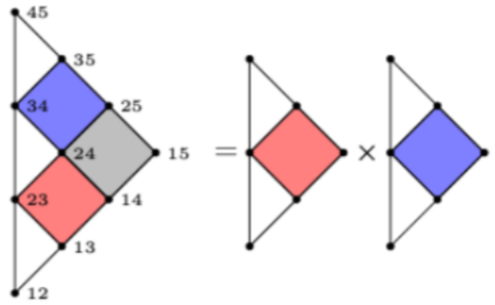


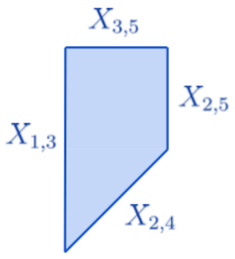
Factorizations of string amplitude

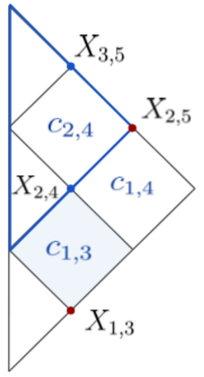
$$\mathcal{I}_4^{\mathrm{Tr}(\phi^3)} = \int_{\mathbb{R}_{>0}} \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} (1+y_{1,3})^{-\alpha' c_{13}} = \frac{\Gamma[\alpha' X_{1,3}]\Gamma[\alpha'(c_{1,3}-X_{1,3})]}{\Gamma[\alpha' c_{1,3}]}$$

$$c_{1,3}=0\,, \quad \text{but} \quad c_{1,4}\neq 0\,, \qquad \mathcal{I}_5^{\mathrm{Tr}\,\phi^3} \to \int_0^\infty \frac{\mathrm{d}y_{1,3}}{y_{1,3}} y_{1,3}^{\alpha' X_{1,3}} \int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}} y_{1,4}^{\alpha' X_{1,4}} \, (1+y_{1,4})^{-\alpha' c_{2,4}} \, (1+y_{1,4}+y_{1,3}y_{1,4})^{-\alpha' c_{1,4}}$$

$$\begin{aligned}
\mathcal{I}_5^{\mathrm{Tr}\,\phi^3} &\to \int_0^\infty \frac{\mathrm{d}\tilde{y}_{1,3}}{\tilde{y}_{1,3}} \tilde{y}_{1,3}^{\alpha' X_{1,3}} \, (1+\tilde{y}_{1,3})^{-\alpha' c_{1,4}} \int_0^\infty \frac{\mathrm{d}y_{1,4}}{y_{1,4}} y_{1,4}^{\alpha'(X_{1,4}-X_{1,3})} \, (1+y_{1,4})^{-\alpha'(c_{2,4}+c_{1,4}-X_{1,3})} \\
&= \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3}, \alpha'(c_{1,3}-X_{1,3})) \times \mathcal{I}_4^{\mathrm{up,Tr}\,\phi^3}(\alpha'(X_{1,4}-X_{1,3}), \alpha'(c_{2,4}+c_{1,4}-X_{1,4})) \\
&= \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3}, \alpha' X_{2,5}) \times \mathcal{I}_4^{\mathrm{up,Tr}\,\phi^3}(\alpha' X_{2,4}, \alpha' X_{3,5}).
\end{aligned}$$

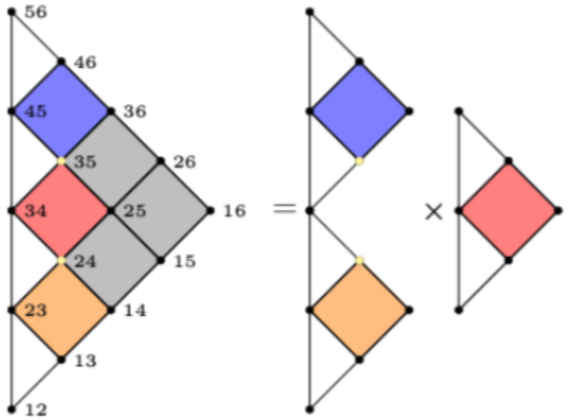




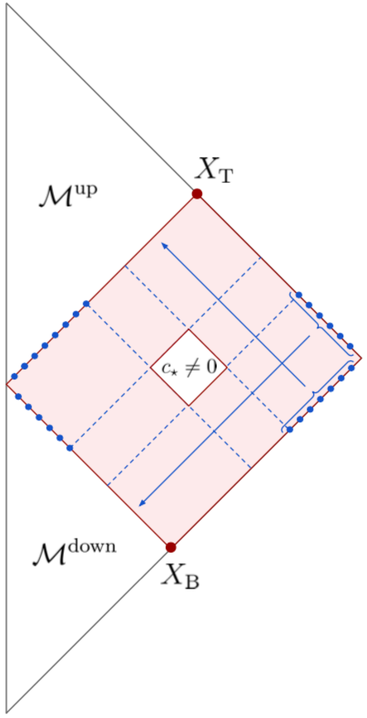


$$\left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,5}}\right) \times \mathcal{A}_4(X_{2,4}, X_{3,5})$$

$$c_{1,4} = c_{2,4} = c_{2,5} = 0\,, \quad \text{but} \quad c_{1,5} \neq 0\,,$$



$$\begin{aligned}\mathcal{I}_6^{\mathrm{Tr}\,\phi^3} &\to \mathcal{I}_4^{\mathrm{down},\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3}, \alpha'(c_{1,3} + X_{1,4} - X_{1,3})) \times \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,4}, \alpha'(c_{1,5} - X_{1,4})) \\ &\quad \times \mathcal{I}_4^{\mathrm{up},\mathrm{Tr}\,\phi^3}(\alpha'(X_{1,5} - X_{1,4}), \alpha'(c_{3,5} + c_{1,5} - X_{1,5})) \\ &= \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,4}, \alpha' X_{3,6}) \times \mathcal{I}_4^{\mathrm{down},\mathrm{Tr}\,\phi^3}(\alpha' X_{1,3}, \alpha' X_{2,4}) \times \mathcal{I}_4^{\mathrm{up},\mathrm{Tr}\,\phi^3}(\alpha' X_{3,5}, \alpha' X_{4,6}).\end{aligned}$$



$$\mathcal{I}_n^{\mathrm{Tr}\,\phi^3} \to \mathcal{I}_i^{\mathrm{down},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_{n-i+2}^{\mathrm{up},\mathrm{Tr}\,\phi^3} \times \mathcal{I}_4^{\mathrm{Tr}\,\phi^3}(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i})).$$

$$X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n}, \text{for} \quad l = 2, \ldots, k.$$
$$X_{i-1,j} \to X_{i-1,j} - X_{i-1,n} = X_{1,j}, \text{for} \quad j = m, \ldots, n-1.$$

near non-positive integers: sum of such factorizations

Stringy δeformation (even pt)

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}, \qquad \mathcal{I}_{2n}^{\delta} = \mathcal{I}_{2n}^{\mathrm{Tr}\,\phi^3}[\alpha' X_{e,e} \to \alpha'(X_{e,e}+\delta), \alpha' X_{o,o} \to \alpha'(X_{o,o}-\delta)]\,.$$

key: all $c_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$ are preserved, thus all zero + fact. unchanged!

$\alpha'\delta = 0$

$0 < \alpha'\delta < 1 \quad (\text{or } \mathbb{R}/\mathbb{Z})$

$\alpha'\delta = \pm 1$

$\alpha' \to 0$

$$\mathcal{L}_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3),$$

$$\mathcal{L}_{\mathrm{NLSM}} = \frac{1}{8\lambda^2}\,\mathrm{Tr}\left(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U}\right), \quad \text{with} \quad \mathrm{U} = (\mathbb{I} + \lambda\Phi)(\mathbb{I} - \lambda\Phi)^{-1}$$

$$\mathcal{L}_{\mathrm{YMS}} = -\,\mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\phi^I D_\mu\phi^I - \frac{g_{\mathrm{YM}}^2}{4}\sum_{I\neq J}\left[\phi^I,\phi^J\right]^2\right)$$

2n-pt δeformed string amps -> stringy pions or stringy 2n-scalar YMS (contains n gluons)

How are (stringy) Tr ϕ^3, NLSM & YMS unified?

e.g. n=4:

$$\mathcal{I}_4^{\delta} = \frac{\Gamma[\alpha'(X_{1,3}-\delta)]\Gamma[\alpha'(X_{2,4}+\delta)]}{\Gamma[\alpha'(X_{1,3}+X_{2,4})]}$$

stringy ϕ^3 at $\delta = 0$: $\alpha' \to 0 \implies \frac{1}{X_{13}} + \frac{1}{X_{24}}$, and massive spectrum at higher level,

e.g. + massive spin-1 since residue at $s := \alpha' X_{13} = -1$ is $1 - \alpha' X_{24} = \frac{1+\cos(\theta)}{2}$

NLSM at small $\alpha'\delta \neq 0$: $\alpha' \to 0 \implies \alpha'\Gamma(-\alpha'\delta)\Gamma(+\alpha'\delta) \times (X_{1,3} + X_{2,4})$, massive level: residue =1 at $X_{1,3} = \delta$,

an extra massive spin-0 particle-> softens power-law UV behavior of NLSM-> string UV completion at intermediate $\alpha'\delta$

as $\alpha'\delta \to 1$: exchange both spin-0 and spin-1 particles at light state $\alpha' X_{1,3} = 1 - \alpha'\delta \equiv \epsilon$ (residue=$\alpha' X_{2,4} - \epsilon$)
this spin-1 softens UV of NLSM, like ρ meson for unitarization of pions!

this light state becomes massless at $\alpha'\delta = 1$-> has to be Yang-Mills (coupled to pairs of scalars):

$$\mathcal{I}_4^{\delta=1} = \frac{\Gamma(\alpha' X_{13} - 1)\Gamma(\alpha' X_{24} + 1)}{\Gamma(\alpha'(X_{13} + X_{24}))} \to M(1^A 2^A 3^B 4^B) = \frac{X_{24}}{X_{13}}$$

massless gluon pole at $X_{13} = 0$ but not at $X_{24} = 0$

NLSM directly from Tr ϕ^3

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{2n-3}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$

$$\to \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

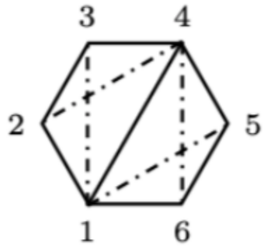
(note: if both $+\delta$ => tr ϕ^4 amplitude)

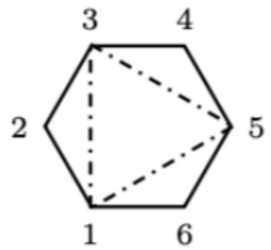
Field-theory directly take $\delta \to \infty$:

$$A_{2n}^{\mathrm{NLSM}} = \lim_{\delta\to\infty} \delta^{n-1} A_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

$$\mathcal{A}_4^{\mathrm{Tr}\,\phi^3}(X_{1,3}-\delta, X_{2,4}+\delta) \xrightarrow{\delta\to\infty} \frac{1}{\delta}(1-1) - \frac{1}{\delta^2}\underbrace{(X_{1,3}+X_{2,4})}_{\mathcal{A}_4^{\mathrm{NLSM}}} + \mathcal{O}(1/\delta^3),$$

$$\mathcal{A}_6^{\mathrm{Tr}\,\phi^3}(X \to X \pm \delta) \xrightarrow{\delta\to\infty} -\frac{1}{\delta^4}\left(-\frac{(X_{1,3}+X_{2,4})(X_{1,5}+X_{4,6})}{X_{1,4}} + (\text{cyclic}, i \to i+2)\right.$$
$$\left. + X_{1,3} + X_{3,5} + X_{1,5} + X_{2,4} + X_{4,6} + X_{2,6}\right) + \mathcal{O}(1/\delta^5),$$





Proof: 1. correct pole structure (only $X_{o,e} = 0$) 2. correct factorization 3. Adler zero from “skinny zero”

NLSM zeros and factorizations

$$\mathcal{I}_{2n}^{\delta} = \int_{\mathbb{R}_{>0}^{n-3}} \prod_{I=1}^{n-3} \frac{dy_I}{y_I} \prod_{(e,e)} u_{e,e}^{\alpha'(X_{e,e}+\delta)} \times \prod_{(o,o)} u_{o,o}^{\alpha'(X_{o,o}-\delta)} \times \prod_{(o,e)} u_{o,e}^{\alpha' X_{o,e}}$$
$$\to \mathcal{A}_{2n}^{\mathrm{Tr}\,\phi^3}(X_{e,e} \to X_{e,e}+\delta, X_{o,o} \to X_{o,o}-\delta),$$

All zeros directly follow; factorizations?

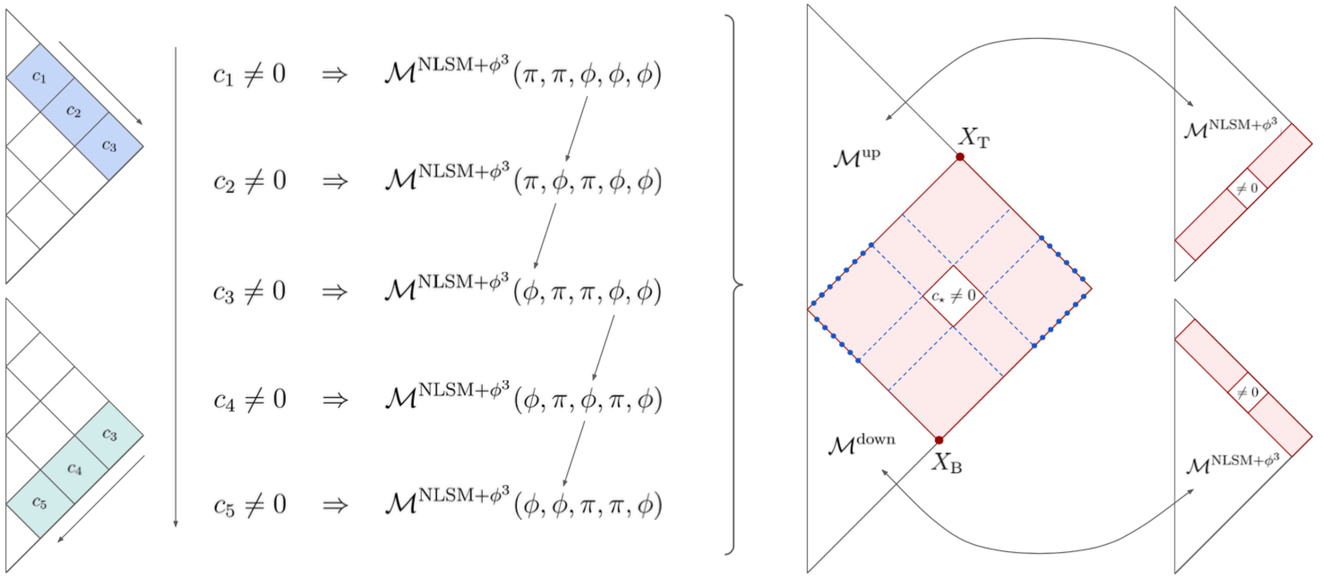
1. even-even-> NLSM* 4pt ϕ^3 *NLSM:
2. odd-odd-> mixed* 4pt NLSM* mixed:

$$\mathcal{A}_6(c_{1,4}=c_{1,5}=c_{2,5}=0) = \frac{c_{1,3}c_{2,4}c_{3,5}}{X_{1,4}(X_{1,4}-c_{2,4})} = \left(\frac{1}{X_{1,4}}+\frac{1}{X_{3,6}}\right)\cdot c_{1,3}\cdot c_{3,5}$$
$$= \left(\frac{1}{X_{1,4}}+\frac{1}{X_{3,6}}\right)\cdot \underbrace{(X_{1,3}+X_{2,6})}_{\mathcal{M}_4^{\text{down,NLSM}}}\cdot \underbrace{(X_{1,5}+X_{4,6})}_{\mathcal{M}_4^{\text{up,NLSM}}},$$

$$\mathcal{A}_6(c_{1,3}=c_{1,4}=0) = -c_{1,5}\cdot\left(\frac{c_{3,5}}{X_{3,6}}+\frac{c_{2,4}}{X_{2,5}}+1\right)$$
$$= -(X_{1,3}+X_{2,6})\cdot \underbrace{\left(\frac{X_{3,5}+X_{4,6}}{X_{3,6}}+\frac{X_{2,4}+X_{3,5}}{X_{2,5}}-1\right)}_{\mathcal{A}_5^{\text{NLSM}+\phi^3}(\phi,\pi,\pi,\phi,\phi)},$$

$$\mathcal{A}_6(c_{1,3}=c_{1,5}=0) = c_{1,4}\cdot\frac{c_{3,5}}{X_{3,6}}$$
$$= (X_{1,3}+X_{2,6})\cdot \underbrace{\left(\frac{X_{3,5}+X_{4,6}}{X_{3,6}}-1\right)}_{\mathcal{A}_5^{\text{NLSM}+\phi^3}(\phi,\pi,\phi,\pi,\phi)},$$

$$\mathcal{A}_6(c_{1,4}=c_{1,5}=0) = c_{1,3}\cdot\left(\frac{c_{3,5}}{X_{3,6}}+\frac{c_{3,5}+c_{2,5}}{X_{1,4}}\right)$$
$$= (X_{1,3}+X_{2,6})\cdot \underbrace{\left(\frac{X_{3,5}+X_{4,6}}{X_{3,6}}+\frac{X_{4,6}+X_{1,5}}{X_{1,4}}-1\right)}_{\mathcal{A}_5^{\text{NLSM}+\phi^3}(\phi,\phi,\pi,\pi,\phi)},$$

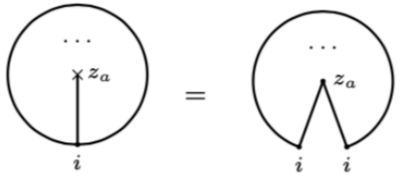


All-loop NLSM integrands contained in Tr ϕ^3 [ACDFH, to appear]

Same shift works for planar integrand of NLSM: $X_{e,e} \to X_{e,e} + \delta\,, \quad X_{o,o} \to X_{o,o} - \delta$ (inc. loop punctures)

$$\lim_{\delta\to\infty} \sum_{z_{a=1,\cdots,L}}^{2^L} \sum_{\text{even/odd}} (\delta)^{n+2L-2} A^{\delta}_{n,L} = A^{\text{NLSM}}_{n,L}\,.$$

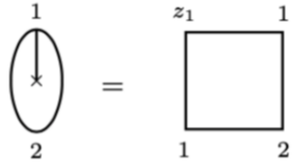
Proof: satisfies all-loop single-cuts + fact. (follow from cuts/fact. of ϕ^3)



e.g. 1-loop 2-pt:

$$A^{\delta}_{2,1} = \frac{1}{(X_{1,z_1}-\delta)(X_{2,z_1})} + \frac{1}{(X_{1,z_1}-\delta)(X_{1,1}-\delta)} + \frac{1}{(X_{2,z_1})(X_{2,2}-\delta)} + (1 \leftrightarrow 2, \delta \to -\delta)$$
$$\to \delta^{-2}\left(2 - \frac{X_{1,1}+X_{2,z_1}}{X_{1,z_1}} - \frac{X_{2,2}+X_{1,z_1}}{X_{2,z_1}}\right) = \delta^{-2} A^{\text{NLSM}}_{2,1},$$

single-cut=forward-limit of tree 4-pt



$$\operatorname*{Res}_{X_{1,z_1}=0} A^{\text{NLSM}}_{2,1} = -(X_{1,1}+X_{2,z_1}) = A^{\text{NLSM}}_{4,0}(1,z_1,1,2).$$

"Adler zero": soft limit -> scaleless integrals! Very practical, e.g. 3-loop 4-pt NLSM integrand

Scaffolded gluons: combinatorial origin of YM

$\alpha'\delta = 1$ gives 2n-scalar stringy amplitude = 2n-scalar in bosonic string!

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,2n) = \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \left(\prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}\right) \exp\left(\sum_{i\neq j} 2\frac{\epsilon_i\cdot\epsilon_j}{z_{i,j}^2} - \frac{\sqrt{\alpha'}\epsilon_i\cdot p_j}{z_{i,j}}\right)\Big|_{\text{multi-linear in } \epsilon_i},$$

$$p_i\cdot\epsilon_j = 0, \quad \forall\ (i,j)\in(1,\ldots,2n),$$

special component $\epsilon_1\cdot\epsilon_2\ \ \ldots\ \ \epsilon_{2n-1}\cdot\epsilon_{2n}$

$$\epsilon_i\cdot\epsilon_j = \begin{cases} 1 & \text{if } (i,j)\in\{(1,2);(3,4);(5,6);\ldots;(2n-1,2n)\},\\ 0 & \text{otherwise.}\end{cases}$$

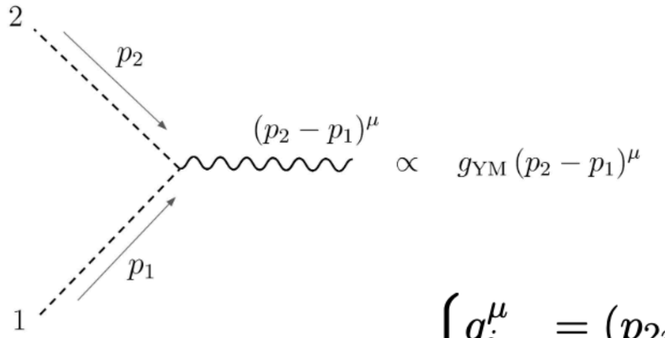
$$\mathcal{A}_{2n}(1,2,\ldots,2n) \xrightarrow{\text{special kinematics}} \int \frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j} \frac{1}{z_{1,2}^2 z_{3,4}^2 z_{5,6}^2 \cdots z_{2n-1,2n}^2}$$

$$= \int \underbrace{\frac{\mathrm{d}^{2n} z_i}{\mathrm{SL}(2,\mathbb{R})} \frac{1}{z_{1,2}z_{2,3}z_{3,4}\cdots z_{2n,1}} \prod_{i<j} z_{i,j}^{2\alpha' p_i\cdot p_j}}_{\text{Stringy Tr}\,\phi^3} \frac{z_{2,3}z_{4,5}z_{6,7}\cdots z_{2n,1}}{z_{1,2}z_{3,4}z_{5,6}\cdots z_{2n-1,2n}} \qquad \left(\prod u_{e,e}/\prod u_{o,o}\right)$$

exactly corresponds to $\alpha'\delta = 1$: n pairs of scalars in bosonic string, $(1,2)(3,4)\cdots(2n-1,2n)$

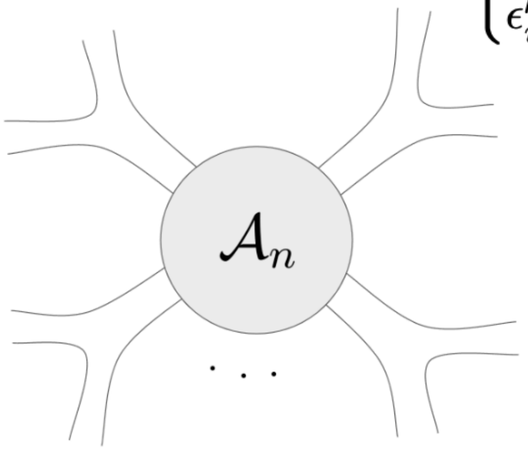
note $\alpha'\delta = -1$ => $(2,3)(4,5)\cdots(2n,1)$

by taking n ``scaffolding residues” -> n-gluon bosonic string amp in X (scalar) language!

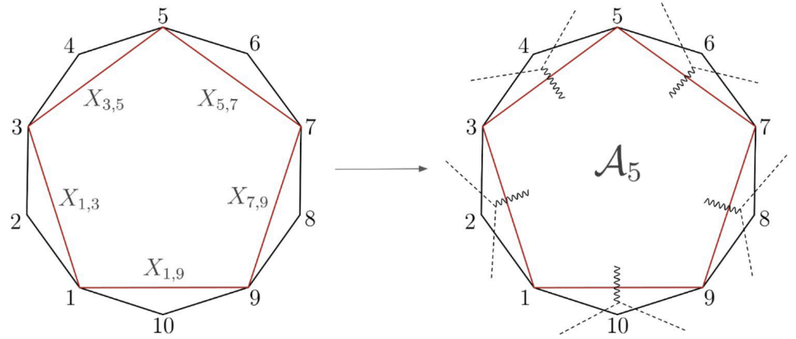


$$\mathcal{I}_{2n}^{\delta}=\int_{\mathbb{R}_{>0}^{2n-3}}\underbrace{\prod_{i=1}^{n}\frac{dy_{2i-1,2i+1}}{y_{2i-1,2i+1}^2}\prod_{I\in\mathcal{T}'}\frac{dy_I}{y_I^2}\prod_{(a,b)}u_{a,b}^{\alpha' X_{a,b}}}_{\Omega_{2n}},$$

$$\begin{cases} q_i^\mu &= (p_{2i}+p_{2i-1})^\mu \\ \epsilon_i^\mu &\propto (p_{2i}-p_{2i-1})^\mu \end{cases},$$



$$X_{1,3}=X_{3,5}=\cdots=X_{1,2n-1}=0$$



$$\mathcal{I}_n^{\rm gluon}=\int_{\mathbb{R}_{>0}^{n-3}}\mathrm{Res}_{y_{1,3}=0}\left(\mathrm{Res}_{y_{3,5}=0}\left(\ldots\left(\mathrm{Res}_{y_{1,2n-1}=0}\left(\Omega_{2n}\right)\right)\ldots\right)\right)\Big|_{X_{2i-1,2i+1}=0}$$

$$A_3^{\rm gluon}=\alpha'^2\left(c_{1,3}c_{1,5}+c_{1,3}c_{2,5}+c_{1,3}c_{3,5}+c_{1,4}c_{3,5}+c_{1,5}c_{3,5}+c_{1,5}c_{3,6}\right)-\alpha'^3\left(X_{1,4}X_{2,5}X_{3,6}\right)$$

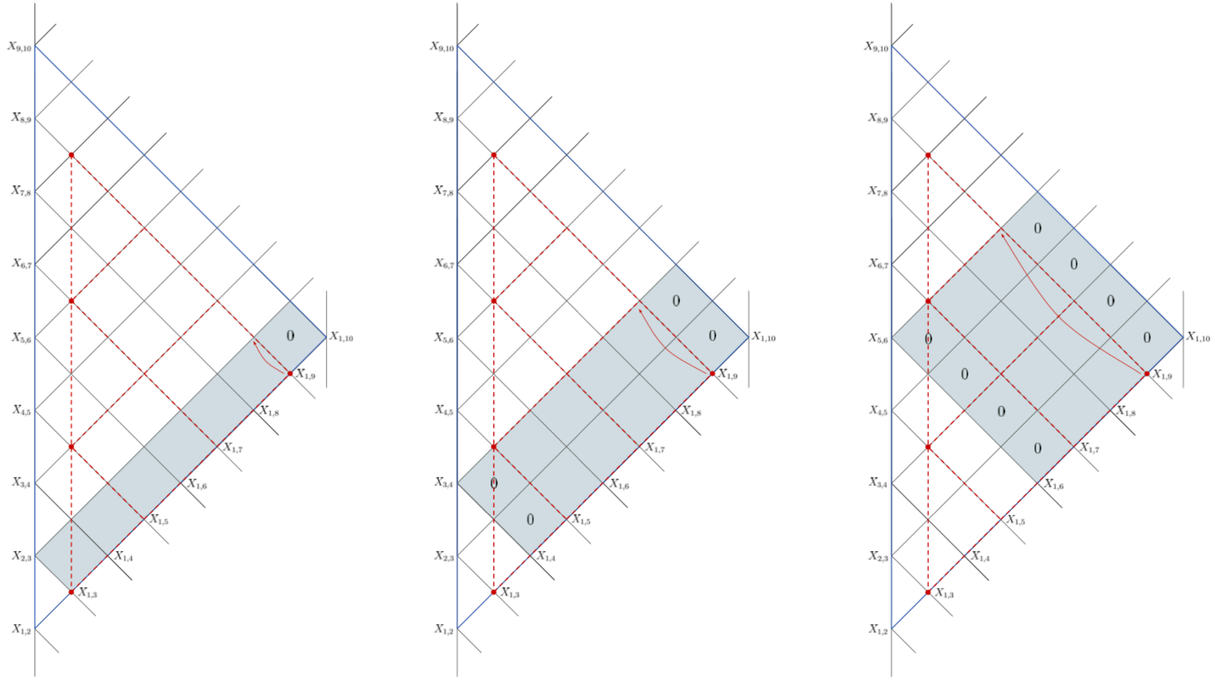
$$A_3^{\rm YM}(1,2,3)=\frac{1}{2}(\epsilon_1\cdot\epsilon_3\;\epsilon_2\cdot q_1-\epsilon_1\cdot q_2\;\epsilon_2\cdot\epsilon_3+\epsilon_1\cdot\epsilon_2\;q_2\cdot\epsilon_3).$$

$$A_3^{F^3}(1,2,3)=\epsilon_1\cdot q_3\;\epsilon_2\cdot q_1\;\epsilon_3\cdot q_2,$$

YM zeros and factorizations

$$\mathcal{A}_{2n}^{\alpha'\delta=1}(c_\star \neq 0) \to \frac{\Gamma(\alpha' X_{\rm B})\Gamma(\alpha' X_{\rm T})}{\Gamma(\alpha'(X_{\rm B}+X_{\rm T}))} \times \mathcal{A}_{2n_1}^{\alpha'\delta=1,\rm up} \times \mathcal{A}_{2n_2}^{\alpha'\delta=1,\rm down},$$

scaffolded mesh: need zeros+ fact. compatible with residues $X_{1,3} = X_{3,5} = \cdots = X_{1,2n-1} = 0$



$$\mathcal{A}_{10}(c_\star \neq 0) \to \left(\frac{1}{X_{1,6}} + \frac{1}{X_{5,10}}\right) \times \mathcal{A}_6^{\rm down} \times \mathcal{A}_6^{\rm up}. \qquad \mathcal{A}_5^{\rm gluons}(c_\star \neq 0) \to \left(\frac{1}{X_{1,7}} + \frac{1}{X_{5,9}}\right) \times \mathcal{A}_4^{\rm down,\ gluons\ +\ \phi} \times \mathcal{A}_3^{\rm up,\ gluons},$$

$$\mathcal{A}_{10}(c_\star \neq 0) \to \left(\frac{1}{X_{1,4}} + \frac{1}{X_{3,10}}\right) \times \mathcal{A}_4^{\rm down} \times \mathcal{A}_8^{\rm up}, \qquad \mathcal{A}_5^{\rm gluon}(c_\star \neq 0) \to \left(\frac{1}{X_{1,5}} + \frac{1}{X_{3,9}}\right) \times \mathcal{A}_2^{\rm down,\ gluons} \times \mathcal{A}_4^{\rm up,\ gluons}.$$

$$\mathcal{A}_{10}(c_\star \neq 0) \xrightarrow{\alpha' X \ll 1} \frac{X_{2,10}}{X_{1,3}} \times \mathcal{A}_9^{\rm up}; \quad \mathcal{A}_5^{\rm gluon}(c_\star \neq 0) \to X_{2,10} \times \mathcal{A}_9^{\rm up,scaffolded}$$

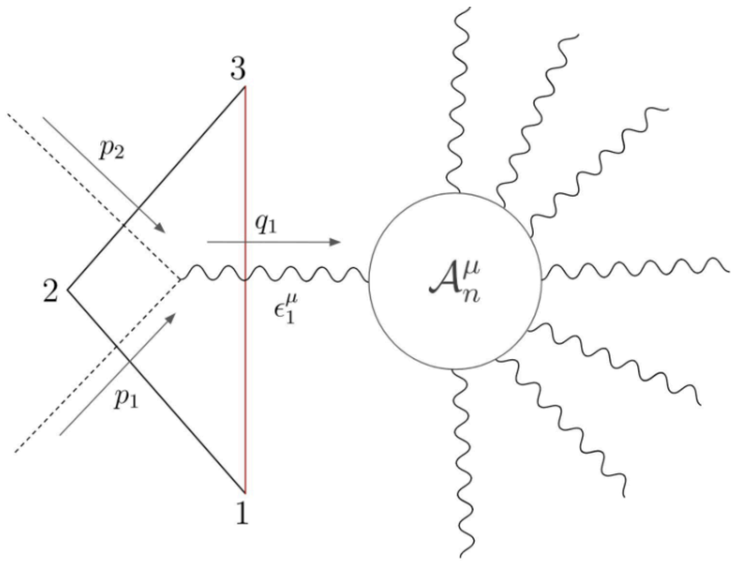
All-loop YM contained in stringy Tr ϕ^3 [ACDFH, to appear]

Generalize stringy tree amp (disk) to loops (higher-genus surfaces):

$$A_n^{\text{gluon}} = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \, \text{Res}_{y_{s_1}=0} \left(\text{Res}_{y_{s_2}=0} \left(\dots \left(\text{Res}_{y_{s_n}=0} \Omega_{2n} \right) \dots \right) \right) .$$

e.g. 1-loop w. self-intersecting curves & closed curve Δ (absent for scalars)

$$\mathcal{I}_{2n}^{\text{1-loop}}(1,2,\dots,2n) = \int_0^\infty \prod_i \frac{\mathrm{d}y_i}{y_i^2} \prod_C u_C^{\alpha' X_C} \times \prod_{C' \in \text{ s.i.}} u_{C'}^{\alpha' X_{C'}} \times u_\Delta^\Delta$$



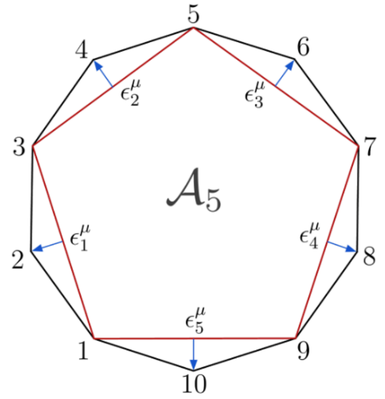
$$\epsilon_1^\mu \propto (X_3 - X_2)^\mu - (X_2 - X_1)^\mu \propto X_2^\mu - \frac{(X_3 + X_1)^\mu}{2}$$

contour strictly defined (similar to $i\epsilon$) , but still highly non-trivial to take $\alpha' \to 0$:

not bosonic string beyond tree, but conjecturally all-loop YM integrands!

first evidence by leading singularities (residues only): checked up to 2 loops;

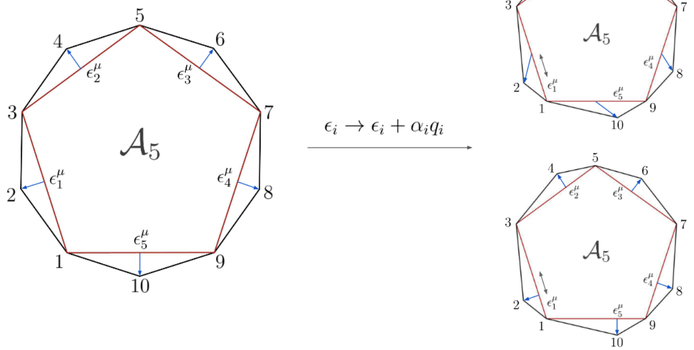
LS = residue of $\int \prod \frac{dy}{y^2} \prod u^X$ = gluing of 3pt (in X space) iff $\Delta = 1 - D$.

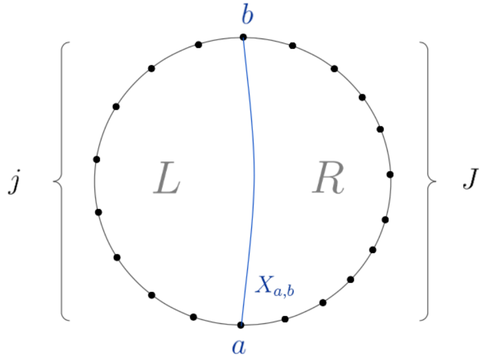


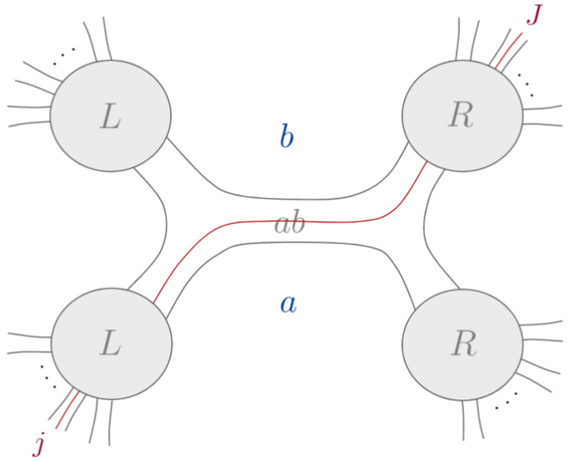
Gauge invariance, factorization & cuts [ACDFH, to appear]

strong evidence beyond LS: surfaceology -> gauge inv., tree factorization & gluon single cuts (all in X)

$$\mathcal{A}_n^{\text{gluon}} = \sum_{j\neq\{2i-1,2i,2i+1\}} (X_{2i,j} - X_{2i-1,j}) \times \mathcal{Q}_j$$
$$= \sum_{j\neq\{2i-1,2i,2i+1\}} (X_{2i,j} - X_{2i+1,j}) \times \tilde{\mathcal{Q}}_j \quad ,$$







$$Q_j^L = \frac{\partial \mathcal{A}_L(a,\ldots,b,x)}{\partial X_{x,j}},$$
$$Q_J^R = \frac{\partial \mathcal{A}_R(b,\ldots,a,x)}{\partial X_{x,J}}.$$

$$\text{Res}_{X_{a,b}=0}\,\mathcal{A}_n = -\sum_{j,J}(X_{j,J} - X_{j,b} - X_{a,J}) \underbrace{\prod_{X_L\in\mathcal{L}_j} \tilde{u}_{X_L}}_{Q_j^L} \cdot \underbrace{\prod_{X_R\in\mathcal{R}_J} \tilde{u}_{X_R}}_{Q_J^R}$$

Summary & outlook

- Stringy Tr ϕ^3 (binary/Z integrals) has (infinite) hidden zeros & new factorizations near them
- Same is true for NLSM & YM (2n-scalar) tree amps (& extends to mixed amps)
- String δeformation: same function (with kinematic shifts) describe them all!
- Generalize to all loops: NLSM in field-theory Tr ϕ^3; YM in stringy Tr ϕ^3 (binary integrals on surfaces)

- Physical origin of zeros+factorizations? extensions (related to 3-split [Cachazo et al] from CHY)
- Loop-level zeros/factorizations: similar for integrands (surfacehedra), but integrated results?
- Mixed amps -> class of shifted amplitudes with factorizations; General Lagrangians?
- Gravity: Tree = closed string = scaffolded scalars on complex binary geometry —> all loops?
- ……

Thank you!