Improved Integral Reduction with Kira

(in collaboration with Fabian Lange and Zihao Wu)

QCD Meets Gravity

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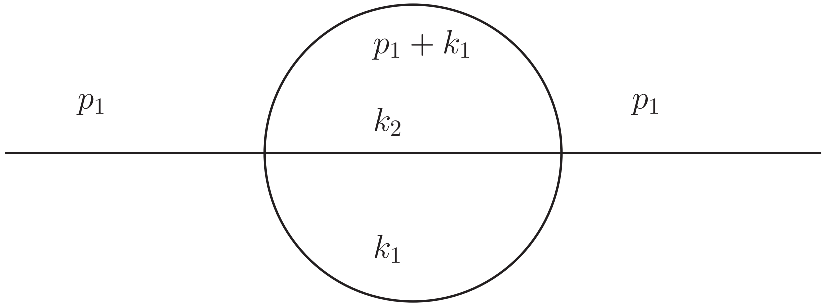
Integral family



$$I(\vec{s}, D|a_1, \ldots, a_5) = \frac{\int d^D k_1 d^D k_2}{\underbrace{[{k_1}^2 - m_1^2]^{a_1}}_{P_1^{a_1}} \underbrace{[(p_1 + k_1)^2]^{a_2}}_{P_2^{a_2}} \underbrace{[{k_2}^2]^{a_3}}_{P_3^{a_3}} \underbrace{[(p_1 + k_2)^2]^{a_4}}_{P_4^{a_4}} \underbrace{[(k_2 - k_1)^2]^{a_5}}_{P_5^{a_5}}}$$

- $q_j = k_1, \ldots, k_L, p_1, \ldots, p_E$
- $s_{ij} = q_i\, q_j, \quad i = 1, \ldots, L, \quad j = i, \ldots, L + E$
- $\vec{s} = (\{s_i\}, \{m_i^2\})$, dimensional regularization parameter $D = 4 - 2\epsilon$
- The integral family definition is complete, if all P_i are linearly independent in the s_{ij}
- $s_{11} = m_1^2 + P_1, \quad s_{12} = \frac{1}{2}(m_1^2 + P_1 + P_3 - P_5), \quad s_{22} = P_3,$
  $s_{13} = \frac{1}{2}(-m_1^2 - p_1^2 - P_1 + P_2), \quad s_{23} = \frac{1}{2}(-p_1^2 - P_3 + P_4)$

Integration-by-parts (IBP) identities

$$I(a_1,\ldots,a_5)=\int\frac{d^Dk_1d^Dl_2}{[k_1{}^2-m_1^2]^{a_1}[(p_1+k_1)^2]^{a_2}[k_2{}^2]^{a_3}[(p_1+k_2)^2]^{a_4}[(k_2-k_1)^2]^{a_5}}$$

$$\int d^D\boldsymbol{k_1}\ldots d^D\boldsymbol{k_L}\frac{\partial}{\partial(\boldsymbol{k_i})_\mu}\left((q_j)_\mu\frac{1}{[P_1]^{a_1}\ldots[P_N]^{a_N}}\right)=0$$

$$c_1(\{a_f\},\vec{s},D)I(a_1,\ldots,a_N\mathbf{-1})+\cdots+c_m(\{a_f\},\vec{s},D)I(a_1\mathbf{+1},\ldots,a_N)=0$$

m number of terms generated by one IBP identity [K. G. Chetyrkin, F. V. Tkachov, Nucl.Phys.B 192 (1981) 159-204] or by Lorentz-invariance identities [T. Gehrmann, E. Remiddi, arXiv:hep-ph/9912329]

Reduction: express all integrals with the same set of propagators but with different exponents a_f as a linear combination of some basis integrals (master integrals)

- Gives relations between the scalar integrals with different exponents a_f
- Number of $L(E+L)$ IBP equations, for each choice of $i=1,\ldots,L$ and $j=1,\ldots,E+L$
- a_f = symbols: Seek for recursion relations, `LiteRed` [R. N. Lee, arXiv:1212.2685]
- a_f = integers: Sample a system of equations, **Laporta algorithm** [S. Laporta, arXiv:hep-ph/0102033]

Feynman integral reduction

- The number of master integrals is finite [A. V. Smirnov, A.V. Petukhov, arXiv:1004.4199], however the proof is non-constructive
- neatIBP syzygy approach, [J. Gluza, K. Kajda, D. A. Kosower arXiv:1009.0472, Z. Wu, J. Boehm, R. Ma, H. Xu, Y. Zhang, arXiv:2305.08783]
- Linear relations from syzygies [B. Agarwal, S. P. Jones, A. von Manteuffel, arXiv:2011.15113]
- Reconstruct the rational coefficients on-the-fly, directly in a form which is decomposed in partial fractions [S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H. B. Hartanto, J. Henn, M. Marcoli, R. Moodie, T. Peraro, S. Zoia, arXiv:2106.08664]
- Block triangular form [X. Liu, Y.-Q. Ma, arXiv:1801.10523]
- Reduction to master integrals via intersection numbers [S. Mizera, arXiv:1711.00469, G. Fontana, T. Peraro, arXiv:2304.14336]

Reduction programs

- Laporta algorithm implementations you should try:
  - FIRE 6.5 [A. V. Smirnov, M. Zeng, arXiv:2311.02370]
  - FiniteFlow [T. Peraro, arXiv:1905.08019]
  - Reduze 2 [A. von Manteuffel, C. Studerus, arXiv:1201.4330]
  - First mover Laporta integral implementation: AIR [C. Anastasiou, A. Lazopoulos, arXiv:hep-ph/0404258]
  - Kira [P. Maierhöfer, J. Usovitsch, P. Uwer, arXiv:1705.05610, J. Klappert, F. Lange, P. Maierhöfer, J. Usovitsch, arXiv:2008.06494] + Firefly [J. Klappert, F. Lange, arXiv:1904.00009, J. Klappert, S. Y. Klein, F. Lange, arXiv:2004.01463]

Laporta algorithm

Object of interest: $I(a_1, \ldots, a_N)$

- The variable t counts the number of propagators with a positive propagator power a_i.
- The sector number is given by $S = \sum_{i=1}^{N} \theta(a_i - \frac{1}{2}) 2^{i-1}$
- $\mathbf{r} = \{r_1, \ldots, r_t\}$ is a vector of all positive propagator powers a_i
- $\mathbf{s} = \{s_1, \ldots, s_{N-t}\}$ is a vector of all negative propagator powers a_i
- Sums of indices $r = \sum_{i=1}^{t} r_i$, $s = -\sum_{i=1}^{N-t} s_i$, $d = \sum_{i=1}^{t} (r_i - 1)$

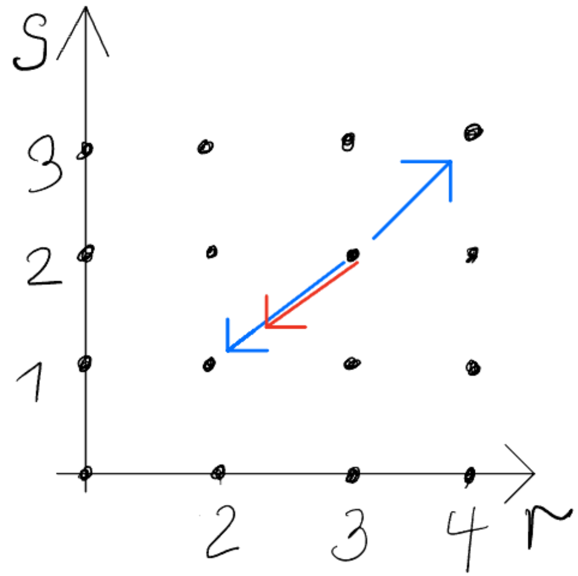
Examples:

- $I(1,1,1,1,1,1,-3)$, $t = 6$, $N - t = 1$, $s = 3$, $r = 6$, $d = 0$
- $I(2,1,2,-1,1,0,-1)$, $t = 4$, $N - t = 3$, $s = 2$, $r = 6$, $d = 2$
- $I(0,1,2,-1,3,0,0)$, $t = 3$, $N - t = 4$, $s = 1$, $r = 6$, $d = 2$

Can you spot an error in one of the three examples?

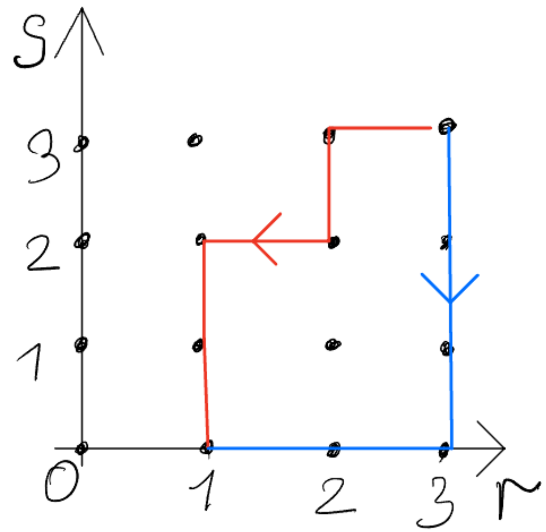
Laporta algorithm

$IBP_1(I(2,1,0,-2)) = 0 = c_1(d)I(2,1,0,-2)+$
$c_2(d)I(1,2,0,-2) + c_3(d)I(1,1,0,-1)+$
$c_4(d)I(3,1,0,-3) + c_5(d)I(3,1,-1,-2)$
$IBP_2(I(2,1,0,-2)), \ldots, IBP_N(I(2,1,0,-2))$
Linear combination of IBP_1, ... , IBP_N gives desired direction depicted in red.



For a fast implementation it is decisive which linear combination of equations to take.

Kira's reduction strategies



- We start with a huge system of equations which is
  - very sparse
  - has very simple coefficients
- After each Gaussian elimination step
  - A intermediate equations grow in size
  - B coefficients grow in size
- Kira strategy Fermat
  - Forward elimination while minimizing A and B using Fermat for intermediate coefficient calculation
  - In backward substitution we avoid expression swell when we combine a huge number of terms. The trick is to add simple terms only and to reorder the terms in complexity before adding more terms. Complexity is identified by the string length.
- Kira strategy FireFly
  - Evaluate the system of equations as a black box numerically over a finite field, with one rule: first equation comes, first serves.
  - Reconstruct the results at the end with FireFly

Seeding IBP with 'old' Kira

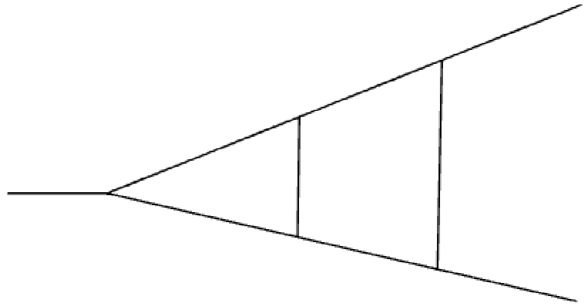
L(L+E)=8 IBP identities

E(E-1)/2=1 LI identities

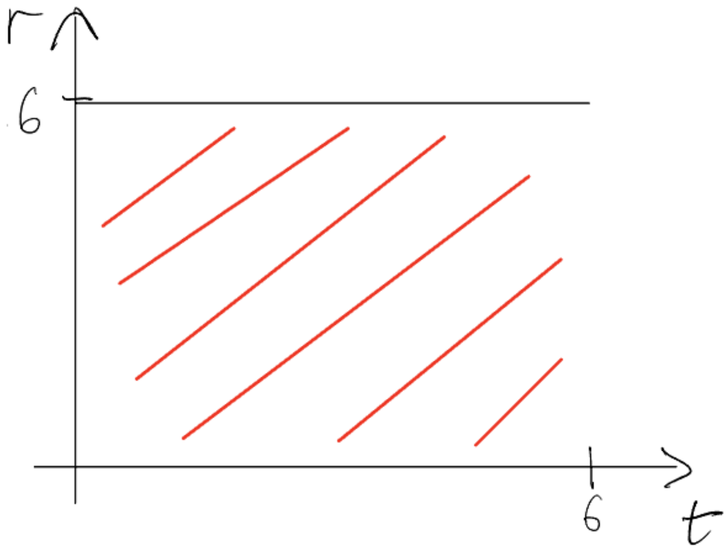
N=E L+L(L+1)/2=7 propagators

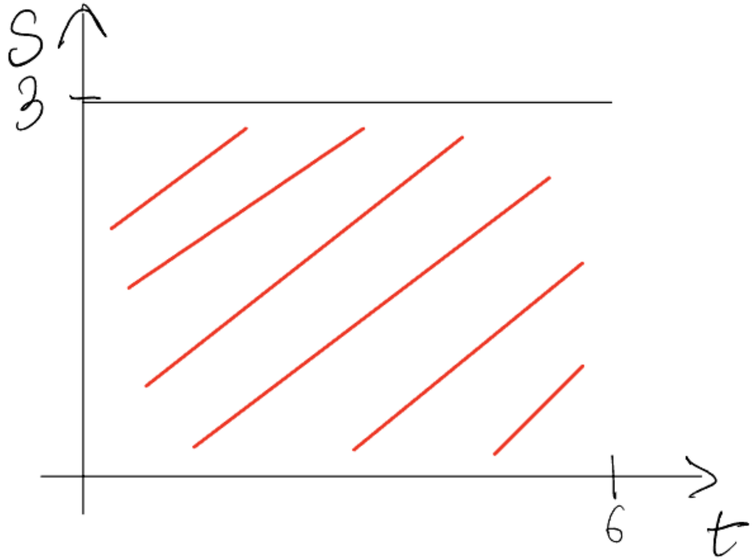
t=6 edges

Goal, reduce integral I(1,1,1,1,1,1,-3)



Old seeding strategy

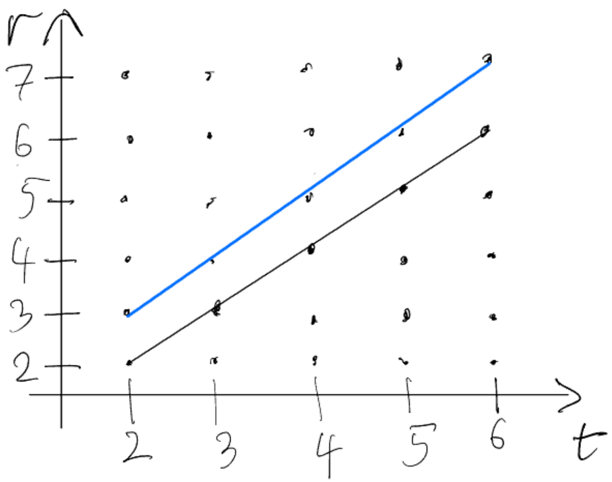


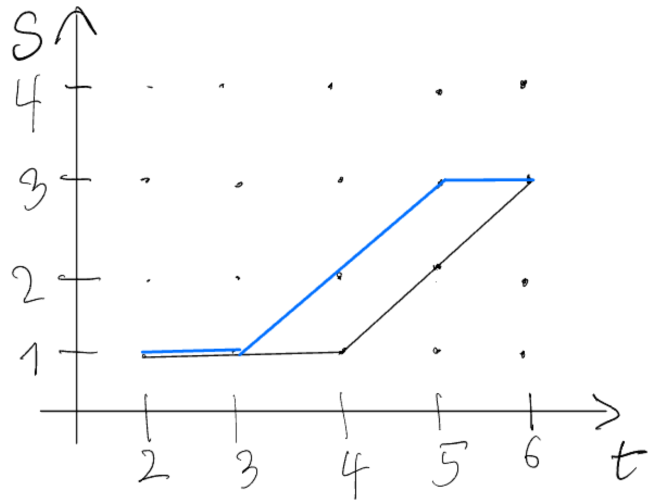


Subset of integrals: I(1,1,4,-2,-1,0,0), I(1,1,2,-2,2,0,-1), I(1,1,1,1,2,0,-3)

Improved IBP reduction with Kira

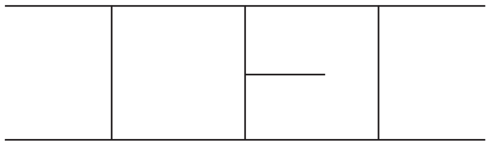
- Keep number of dots constant for all integrals for different values of t
- Decrease s proportional to the value of t for all integrals



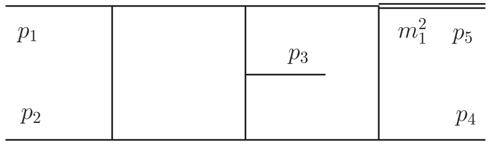


- Subset of integrals: I(1,1,1,0,-1,0,0), I(1,1,1,0,1,0,-1), I(1,1,1,1,1,0,-2)
- Subset of integrals: I(1,1,2,0,-1,0,0), I(2,1,1,0,1,0,-2), I(1,1,1,1,2,0,-3)
- Assumption for each value of t there is exactly one sector
  - Original seeding introduces 504 integrals
  - Black line in the left and right figure introduces 25 integrals
  - Blue line left, black line right introduces 122 integrals
  - Black line left, blue line right introduces 35 integrals

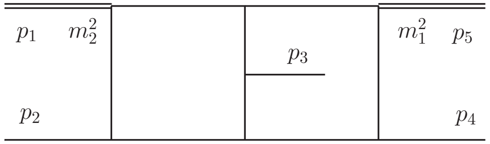
Two-loop examples



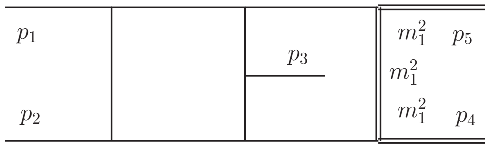
(a)



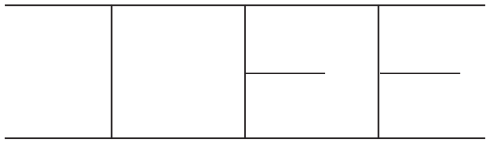
(b)



(c)



(d)



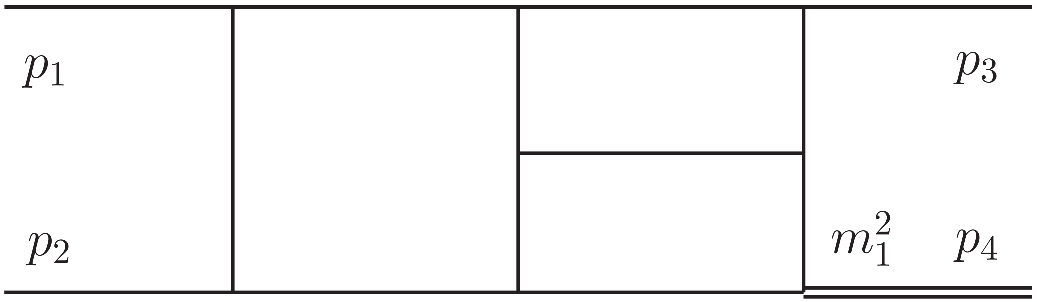
(e)

Single numerical evaluation over a finite field of all possible integrals with $s = 5$ and $t = 8$ for (a)-(d) and with $s = 5$ and $t = 9$ for (g). Furthermore timing is improved with RatRacer [Vitaly Magerya, arXiv:2211.03572]

problem	Kira/+RatRacer	problem	Kira/+RatRacer
(a) 108 MI	1.7/0.3 sec	(b) 142 MI	1.8/0.6 sec
(c) 185 MI	2.7/0.9 sec	(d) 172 MI	2.2/0.8 sec
(e) 313 MI	17/9 sec		

Current public version of Kira runs reduction with a) in 40 secs for one single numerical evaluation over a finite field.

Three-loop example



r.s.d	old	improved Kira
10.3.0	410 sec	2 sec
10.4.0		10 sec
10.5.0		41 sec
10.6.0		360 sec

- Single numerical evaluation over a finite field
- Number of master integrals is 166
- Two orders of magnitude improvement

Summary and Outlook

- How to truncate a system of equations is introduced
- Keep number of dots d for all sectors constant
- Decrease number of numerators s proportional to the number of propagators t
- The strategy is general and applicabale to more loops, more legs, and more internal scales
- New team member Zihao Wu: improve seeding by combining neatIBP approach with Kira's Laporta implementation
- Improved Kira reductions run time is comparable with the run time of reducing the system of equations generated with current version of neatIBP
- Integration of Flint or/ and Symbolica [Ben Ruijl], see FIRE 6.5 [A. V. Smirnov, M. Zeng, arXiv:2311.02370] to replace Fermat, is in discussion, may awaken Kira's strategy: simplify coefficients after each Gaussian elimination step
- Kira version release, 3.0, is in production