# The gravitational eikonal and the NLO scattering waveform

QCD Meets Gravity 2023

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14 December 2023

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# Based on:

2312.07452: NLO scattering waveform

in collaboration with:

Alessandro Georgoudis, Carlo Heissenberg

2306.16488: report on the gravitational eikonal

in collaboration with:

Paolo Di Vecchia, Carlo Heissenberg, Gabriele Veneziano

# Introduction

Amplitudes based techniques have revived the PM approach to the gravitational 2-body problem. Many new results for the impulse, total radiated (angular) momentum also in presence of spin, tidal ...

However a systematic analysis of the scattering NLO PM waveform was started only recently ... and for good reasons! See Mathieu's and Fei's talks

(i) One needs the $2 \to 3$ 1-loop amplitude which is very complicated
Brandhuber... 2303.06111, Herderschee ... 2303.06112, Elkhidir ... 2303.06211, Georgoudis... 2303.07006

(ii) The subtractions needed to isolate the classical part are subtler
Caron-Huot ... 2308.02125

(iii) The waveform depends heavily on the choice of frame

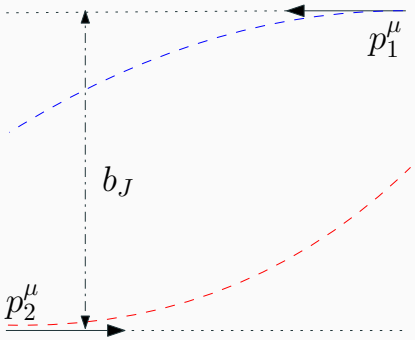
The aim of this talk is to clarify (ii)-(iii) above for the NLO waveform

As usual, the comparison with PN results is crucial both at the technical and at the conceptual level
Bini ... 2309.14925

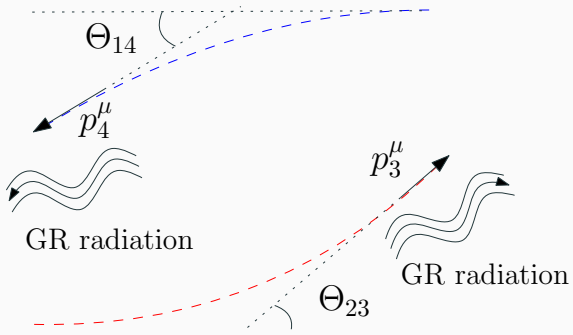
# The setup

Consider two "elementary" objects interacting gravitationally. What is the final state if they scatter with an initial relative Lorentz factor $\sigma = -\frac{p_1 p_2}{m_1 m_2}$ and a large impact parameter $b_J$?

We expect the following qualitative picture




In state (centre of mass frame)




Out state

Some key classical quantities:

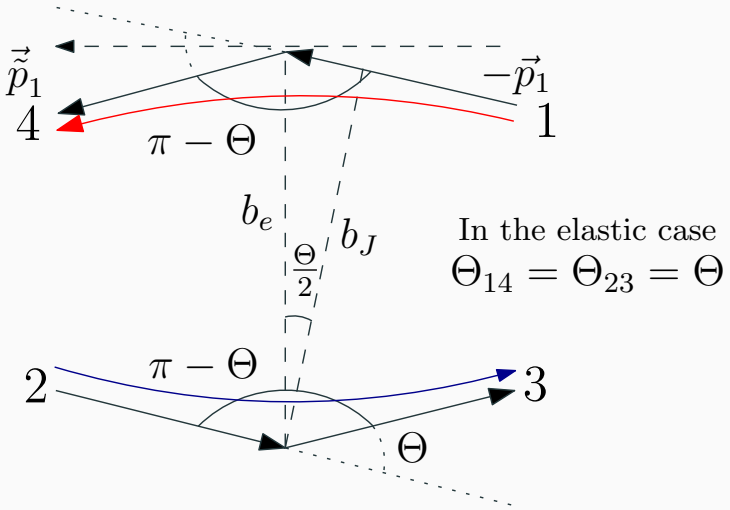
The centre-of-mass energy $E^2 = s = -(p_1 + p_2)^2 = (m_1^2 + m_2^2 + 2 m_1 m_2 \sigma)$.

The angular momentum $J = p\, b_J$, $p = |\vec{p}_1|$, $Ep = m_1 m_2 \sqrt{\sigma^2 - 1}$

The impulse $Q_1^\mu = p_1^\mu + p_4^\mu$, $Q_2^\mu = p_2^\mu + p_3^\mu$.

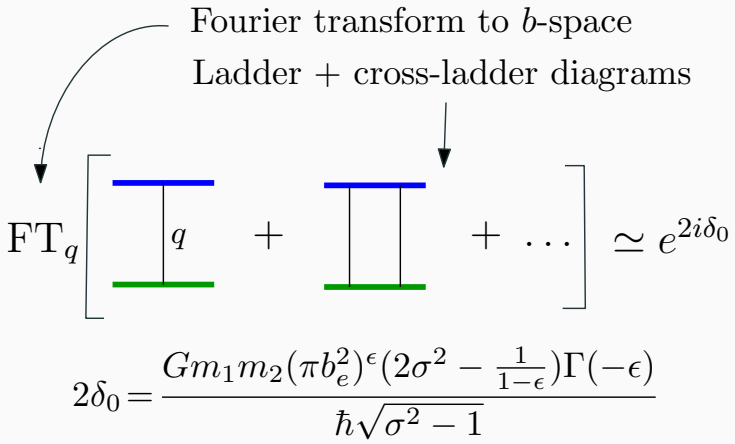
# The impulse: eikonal approach

Calculate the $2 \to 2$ scattering amplitude $\mathcal{A}(E, q^2)$ focusing on the non-analytic terms as $q \to 0$ ($q \sim \hbar/b$ is the typical momentum carried by a single graviton exchanged between $m_1$ and $m_2$). In pictures




In the elastic case
$\Theta_{14} = \Theta_{23} = \Theta$

A spacetime picture of the scattering




Diagrammatic picture

The relation between $b_e$ and $Q$ follows from a stationary phase argument

$$\mathcal{S}(s, Q^2) = 4Ep \int d^{D-2}b_e \, e^{-\frac{i}{\hbar}bQ + 2i\delta(s,b_e)} \ \Rightarrow \ Q_\mu = \hbar \frac{\partial\, 2\delta}{\partial b_e^\mu}\,, \ \ |Q| = 2p \sin\left(\frac{\Theta}{2}\right)$$

# The impulse: the KMOC approach

In the KMOC approach, instead of deriving a classical evolution operator, one calculates diagrammatically directly expectation values, such as

$$\langle \text{in}|S^\dagger Q^\mu S|\text{in}\rangle \quad \Rightarrow \quad \text{the leading term is automatically classical}$$

By using $S = 1 + iT$, and $-i(T - T^\dagger) = T^\dagger T$, the imaginary terms is

Real, classical part of the 1-loop amplitude

Imaginary contribution where **singular-classical terms cancel**

$$Q^\mu_{\rm 2PM} = i\,\mathrm{FT}_q\left[q^\mu\,\mathrm{Re}\,\mathcal{A}_1^{(4)[-1]} + is^\mu\right]$$

$-i(T-T^\dagger)$ $\quad$ $TT^\dagger$ $\quad$ Real, classical tree-level amplitude

$$s^\mu = \int \left(\overbrace{\frac{1}{2}(p_4^\mu - p_1^\mu)}^{-i(T-T^\dagger)} - \overbrace{(\ell^\mu - p_1^\mu)}^{TT^\dagger}\right) \mathcal{A}_0^{(4)}(p_1,p_2;\ell)\mathcal{A}_0^{(4)*}(p_3,p_4;q-\ell)dL(\ell)$$

Measure for a cut integral

However a classical contribution from $s^\mu$ survives! Performing the same $\mathrm{FT}_q$ as before one get an apparently different result for the impulse

Kosower . . . 1811.10950, Herrmann . . . 2104.03957

# Choice of frame (baby version)

The eikonal and the KMOC results agree if interpreted in different frames

$$Q^\mu = -\frac{b_e^\mu}{b_e}\left(\overbrace{\frac{2Gm_1m_2\,(2\sigma^2-1)}{b_e\sqrt{\sigma^2-1}}}^{Q_{1\text{PM}}} + \overbrace{\frac{3\pi G^2 m_1 m_2(m_1+m_2)(5\sigma^2-1)}{4b_e^2\sqrt{\sigma^2-1}}}^{Q_{2\text{PM}}}\right) + \mathcal{O}(G^3) \quad \text{Eikonal result}$$

$$Q^\mu = -\frac{b_J^\mu}{b_J}\left(\underbrace{\overbrace{\frac{2Gm_1m_2\,(2\sigma^2-1)}{b_J\sqrt{\sigma^2-1}}}^{Q_{1\text{PM}}+\mathcal{O}(G^3)} + \overbrace{\frac{3\pi G^2 m_1 m_2(m_1+m_2)(5\sigma^2-1)}{4b_J^2\sqrt{\sigma^2-1}}}^{Q_{2\text{PM}}+\mathcal{O}(G^3)}}_{\text{Re}\mathcal{A}_1}\right) + \underbrace{\frac{(\vec{p}_1)^\mu}{p}\,\frac{Q_{1\text{PM}}^2}{2p}}_{-s^\mu} + \mathcal{O}(G^3) \quad \text{KMOC result}$$

This is due to a different way of extracting the classical terms. It is convenient to introduce the operator

$$\bar\partial = -\frac{b_J^\alpha}{b_J}\left(\frac{1}{2m_1}\frac{\partial}{\partial v_1^\alpha} - \frac{1}{2m_2}\frac{\partial}{\partial v_2^\alpha}\right) - b_J\left(\frac{\check v_1^\alpha}{2m_1} - \frac{\check v_2^\alpha}{2m_2}\right)\frac{\partial}{\partial b_J^\alpha}$$

$$p_1^\mu = -m_1 v_1^\mu\,,\quad p_2^\mu = -m_2 v_2^\mu \quad \check v_1^\mu = \frac{\sigma v_2^\mu - v_1^\mu}{\sigma^2-1}\,,\quad \check v_2^\mu = \frac{\sigma v_1^\mu - v_2^\mu}{\sigma^2-1}$$

so we have $s^\mu = (Q_{1\text{PM}}\bar\partial)Q^\mu$. Note that $\bar\partial(p_1 b_J) = \bar\partial(p_2 b_J) = 0$.

# Derivation

A sketch of an instructive derivation

$$\tilde{s}^\mu = \mathrm{FT}_q \left[ \int \left( \tfrac{1}{2}\, q^\mu - \ell^\mu \right) \mathcal{A}_0^{(4)}(p_1, p_2; \ell) \mathcal{A}_0^{(4)*}(p_3, p_4; q - \ell) dL(\ell) \right]$$

It is convenient to use $\bar{p}_1^\mu = \frac{p_4^\mu - p_1^\mu}{2} = -p_1^\mu + \frac{q^\mu}{2}$ , $\bar{p}_2^\mu = \frac{p_3^\mu - p_2^\mu}{2} = -p_2^\mu - \frac{q^\mu}{2}$

We have $\mathcal{A}_0^{(4)[-2]}(\bar{p}_1, \bar{p}_2; q) + 0 A_0^{(4)[-1]}$ so we can focus in $dL(\ell)$

$$\delta(2\bar{p}_1 \cdot \ell + \ell \cdot (\ell - q))\delta(2\bar{p}_2 \cdot \ell - \ell \cdot (\ell - q)) = \delta(2\bar{p}_1 \cdot \ell)\delta(2\bar{p}_2 \cdot \ell)$$
$$+ \ell \cdot (\ell - q) \left( \delta'(2\bar{p}_1 \cdot \ell)\delta(2\bar{p}_2 \cdot \ell) - \delta(2\bar{p}_1 \cdot \ell)\delta'(2\bar{p}_2 \cdot \ell) \right) + \cdots$$

The leading term vanishes by symmetry

By using the relation below, one can integrate by parts and the derivatives act only on $\ell^\mu$

$$\delta'(2p_1 \cdot \ell) = \frac{\check{v}_1^\rho}{2m_1} \frac{\partial}{\partial \ell^\rho} \delta(2p_1 \cdot \ell)\,, \quad \delta'(2p_2 \cdot \ell) = \frac{\check{v}_2^\rho}{2m_2} \frac{\partial}{\partial \ell^\rho} \delta(2p_2 \cdot \ell)$$

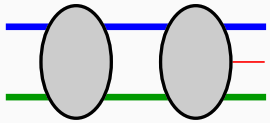
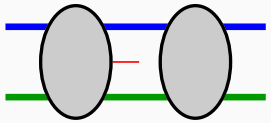
The Fourier transform is factorised!

We obtain the extra term on the previous page without the need of evaluating explicitly the 1-loop integral and directly in impact parameter space! (So we don't need to worry about local terms)

# The classical 5-point 1-loop amplitude (in pictures)

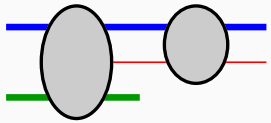
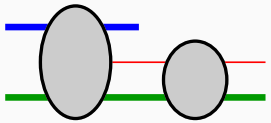
The first ingredient is the $2 \to 3$ 1-loop amplitude

Brandhuber. . . 2303.06111, Herderschee . . . 2303.06112, Elkhidir . . . 2303.06211, Georgoudis. . . 2303.07006

$s =$  , $s' =$ 

$$\mathcal{A}_1^{\mu\nu} = \mathcal{B}_1^{\mu\nu} + \frac{i}{2}(s^{\mu\nu} + s'^{\mu\nu}) + \frac{i}{2}(c_1^{\mu\nu} + c_2^{\mu\nu})$$

Real, "irreducible" part of $\mathcal{A}_1^{\mu\nu}$

$c_1 =$  , $c_2 =$ 

In the classical limit up to quantum corrections

$$\frac{i}{2}(s^{\mu\nu} + s'^{\mu\nu}) \sim i s_+^{[-2]\mu\nu} + 0\,\mathcal{O}([-1])\,, \quad \frac{i}{2}\,c_{1,2}^{\mu\nu} \sim \frac{i}{2}\,c_{1,2}^{[-1]\mu\nu}\,, \quad \mathcal{B}_1^{\mu\nu} \sim \mathcal{B}_1^{[-1]\mu\nu}$$

$$\mathcal{A}_1^{\mu\nu} \sim i s_+^{[-2]\mu\nu} + \left[\mathcal{B}_1^{[-1]\mu\nu} + \frac{i}{2}\left(c_1^{[-1]\mu\nu} + c_2^{[-1]\mu\nu}\right)\right]$$

It is tempting to interpret $\mathrm{FT}_q\big[\,\ldots\big]$ and the waveform, but this is not fully correct . . .

See Mathieu's talk

# The NLO waveform: general considerations

The waveform is obtained from the expectation value of the metric fluction $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ in the final state: $\langle \text{in}|S^\dagger h_{\mu\nu} S|\text{in}\rangle$

Caron-Huot . . . 2308.02125

retarded time $\quad k^\mu = \omega\, n^\mu$

$$h_{\mu\nu}(x) \sim \frac{4G}{r}\int_0^\infty e^{-i\omega U}\,\frac{\tilde{W}_{\mu\nu}(\omega\, n)}{\sqrt{8\pi G}}\,\frac{d\omega}{2\pi} + (\text{c.c.})\,, \qquad \langle \text{in}|S^\dagger a_{\mu\nu}(k) S|\text{in}\rangle = i\,\text{FT}[W_{\mu\nu}(k)] \equiv i\tilde{W}_{\mu\nu}$$

$$\text{FT}_q\left[\mathcal{A}_1^{\mu\nu} - i s'^{\mu\nu}\right] = \text{FT}_q\left[\mathcal{B}_1^{[-1]\mu\nu} + i s_-^{[-1]\mu\nu} + \tfrac{i}{2}(c_1^{[-1]\mu\nu} + c_2^{[-1]\mu\nu})\right]$$

The new constribution $i s_-^{[-1]\mu\nu} = \frac{i}{2}(s^{\mu\nu} - s'^{\mu\nu})$ can be calculated by following the same steps discussed for $s^\mu$

The effect of the $s_-^{\mu\nu}$ cut is to implement a change of frame a rotation and a $U$ shift

Rotation from $(-\vec{p}_1, b_J)$ to $(\vec{\tilde{p}}_1, b_e)$ frame (up to $\mathcal{O}(G^3)$)

Shapiro time delay

Impulse

Tree-level result

$$i\,\tilde{s}_-^{[-1]\mu\nu} = Q_{1\text{PM}}\bar{\partial}\tilde{W}_0^{\mu\nu} + \frac{i}{2}\,\omega\left(T_{1\text{PM}} + b\,Q_{1\text{PM}}\,\frac{E_1 E_2}{Ep^2}\right)\tilde{W}_0^{\mu\nu}\,.$$

(retarded) time-origin shift $U_0^{(1)}$

We can write the result for $i s_-^{[-1]\mu\nu}$ in terms of the operator $Q_{1\text{PM}}\bar{\partial}$

# The NLO waveform: limits

The NLO waveform is the classical part of $\mathcal{A}_1^{\mu\nu}$ in the $(\vec{\tilde{p}}_1, b_e)$ frame. This is what is obtained directly in the eikonal approach!

Cristofoli ... 2112.07556, Di Vecchia 2306.16488

From now on we work in the "eikonal" frame

We choose the (retarded) time origin $U_0$ to absorb the IR div. part and make the finite part as "simple" as possible ... however the expressions are still very complicated! Let's try some limits:

- The PN limit $\sigma = \sqrt{1+p_\infty^2}\,,\ p_\infty \ll 1,\ u_{\rm KT} = \frac{\omega b}{p_\infty}$ fixed

  Too difficult for us now; Bini ... 2309.14925

- Soft limit $u_{\rm KT} \ll 1$

$$\tilde{W} = \tilde{W}^{[\omega^{-1}]} + \tilde{W}^{[\log\omega]} + \tilde{W}^{[\omega^0]} + \tilde{W}^{[\omega(\log\omega)^2]} + \tilde{W}^{[\omega\log\omega]} + \cdots$$

  (Universal terms: $\tilde{W}^{[\omega^{-1}]}$, $\tilde{W}^{[\log\omega]}$, $\tilde{W}^{[\omega(\log\omega)^2]}$; Non universal terms: $\tilde{W}^{[\omega^0]}$, $\tilde{W}^{[\omega\log\omega]}$)

  We derived the highlighted terms from the NLO waveform. The universal ones match expectations from soft theorems

  Weinberg 1964-65, Laddha ... 1806.01872, Sahoo ... 2105.08739

# Puzzles in the small velocity limit

Let us focus on the term $\tilde{W}^{[\omega \log \omega]}$. I'll not discuss/write it explicitly.

see Fei's talk or 2312.07452

This term is basically determined by the "Compton" cuts

$$\tilde{W}_1^{[\omega \log \omega]} = \tilde{\mathcal{B}}_{1O}^{[\omega \log \omega]} + \frac{i}{2}(\tilde{c}_1 + \tilde{c}_2)_{\text{reg}}^{[\omega \log \omega]} , \quad \tilde{\mathcal{B}}_{1O}^{[\omega \log \omega]} = \left[1 - \frac{\sigma(\sigma^2 - \frac{3}{2})}{(\sigma^2 - 1)^{3/2}}\right] \pi G E \omega \, \tilde{W}_0^{[\log \omega]}$$

Recall that the waveform is a tensor. I'll use the parametrisation

$$k^\mu = \omega n^\mu = \omega(1, \overbrace{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta}^{\vec{n}}) , \qquad \tilde{W}(\omega, \theta, \phi) = \varepsilon^i \varepsilon^j \tilde{W}_{ij}(\omega\, n)$$

$$\vec{\varepsilon} = \frac{1}{\sqrt{2}}(\cos\theta\cos\phi + i\sin\phi, \cos\theta\sin\phi - i\cos\phi, -\sin\theta) , \quad \hat{n}\cdot\vec{\varepsilon} = \vec{\varepsilon}\cdot\vec{\varepsilon} = 0$$

It is interesting to compare $\tilde{W}_1^{[\omega \log \omega]}(\omega, \theta, \phi)$ at small velocities with what has been obtained in the PN (MPM) approach Bini . . . 2309.14925

The surprise is that there are mismatches at 2.5PN order ($\mathcal{O}(G^3/c^5)$) between the amplitudes and PN-MPM results for this non-universal term

# The MPM approach (a sketchy/imprecise review)

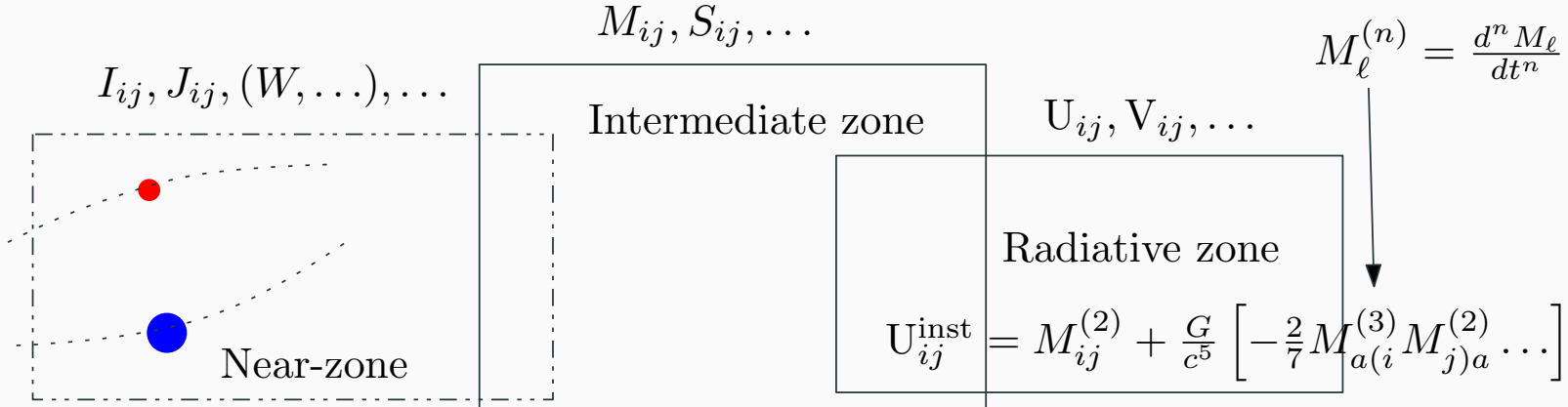
To understand the situation let me attempt a 1-slide summary of the PN-MPM approach

Very challenging: 40 years of activity and many, many results! Blanchet 1310.1528

At low velocities we can expand the radiative field in $SO(3)$ multipoles

$$\frac{\tilde{W}_{ij}(\omega\, n)}{\kappa} = \sum_{\ell=2}^{\infty} \frac{1}{\ell!}\, n^{i_1} \cdots n^{i_{\ell-2}} \mathrm{U}_{iji_1\cdots i_{\ell-2}} - \frac{\ell}{\ell+1} n^k\, n^{i_1} \cdots n^{i_{\ell-2}} \left(\epsilon_{khi} \mathrm{V}_{jhi_1\ldots i_{\ell-2}} + i \leftrightarrow j\right)$$

$\mathrm{U}_{i_1\ldots i_{k+2}}$ is $\mathcal{O}\left(\frac{1}{c^k}\right)$ ($(\mathrm{V}_{i_1\ldots i_{k+2}})$ is $\mathcal{O}\left(\frac{1}{c^{k+1}}\right)$) relative to the leading term in $\mathrm{U}_{ij}$




The near field is derived from the source motion example: $I_{ij} \sim x_{(i} x_{j)}$

Non-linear redefinition at order $1/c^5$

It is clearly interesting to get to $\mathcal{O}(1/c^5)$

as done in Bini . . . 2309.14925

# The self-force limit

Beside the non-linear definition there are other half-PN effects. The one important for us is the tail terms: $\mathrm{U}_{ij} = \mathrm{U}_{ij}^{\rm inst} + \mathrm{U}_{ij}^{\rm tail} + \dots$

$$\mathrm{U}_L^{\rm tail}(U) = 2\frac{GE}{c^3}\int_0^{+\infty} d\tau\, M_L^{(\ell+2)}(U-\tau)\left[\ln\left(\frac{\tau}{2b_0}\right) + \kappa_\ell\right], \quad \kappa_\ell = \frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)} + \sum_{k=1}^{\ell-2}\frac{1}{k}$$

Tail contribution to the $\ell$-order multiple: time domain  "Harmonic" numbers

$$\mathrm{U}_L^{\rm tail}(U) = 2GEi\omega I_L^{(\ell)}(\omega)\Big[\ln(2b_0\omega) - \gamma - \kappa_\ell - i\,\frac{\pi}{2}\Big]$$  Tail in the frequency domain

In the leading "self-force" limit, these are the only half-PN corrections

The leading order of $\tilde{W}$ is $\mathcal{O}(\nu)$

$$\nu = \frac{m_1 m_2}{m^2}, \qquad m = m_1 + m_2, \qquad \Delta = \frac{m_1 - m_2}{m}$$

A non-trivial part of the 1-loop 5-point amplitude follows from black-hole perturbation theory (and so is $\sim \tilde{W}_0^{\mu\nu}$)

See for instance, Fucito, Morales 2311.14637

In the spirit of the talks by Nabha's and Jordan's talks on Monday

# Comparison at $\mathcal{O}(\nu)$

We have

$$\tilde{W}_1^{[\omega\log\omega]} = \mathcal{B}_{1O}^{[\omega\log\omega]} + \tilde{W}_C^{[\omega\log\omega]} + 2iGE\,\log\tilde{b}_0\,\omega\tilde{W}_0^{[\log\omega]}\,, \quad \log(2b_0) = \log\tilde{b}_0 + \gamma$$

$$\begin{aligned}
&\frac{\tilde{W}_C^{[\omega\log\omega],\mathrm{MPM}}}{\kappa\omega\log\omega} \\
&= -\frac{1}{24\,p_\infty} iG^2m^3\nu\left(35(\cos(2\theta)+3)\cos(2\phi) + 140i\cos(\theta)\sin(2\phi) + 22\sin^2(\theta)\right) \qquad 1/c^3 \\
&+ \frac{1}{60} G^2m^3\Delta\,\nu\sin(\theta)(\cos(\phi)+i\cos(\theta)\sin(\phi))(\cos(\theta)(307\cos(2\phi)-67)+614i\sin(\phi)\cos(\phi)) \\
&+ \frac{1}{120}\,p_\infty iG^2m^3\nu\Big[\Big(79\cos(2\theta)+57\cos(4\theta)+420\Big)\cos(2\phi) \\
&+ 2\sin^2(\theta)\Big(51(\cos(2\theta)+3)\cos(4\phi) + 204i\cos(\theta)\sin(4\phi) + 63\cos(2\theta)+400\Big) \\
&+ 4i\cos(\theta)(42\cos(2\theta)+97)\sin(2\phi)\Big] + \mathcal{O}(p_\infty^2) + (\text{quadratic in }\nu)\,. \qquad 1/c^5
\end{aligned}$$

It matches the amplitudes based result if the the IR cutoffs are related by $\log\mu_{\mathrm{IR}} = -\ln\tilde{b}_0 + \frac{1}{2}$ and $U \mapsto U + 2Gmp_\infty^2$ (matching the time origin)!

Comment: in this check the "harmonic" numbers $\kappa_\ell$ up $\ell = 4$ are used, but the analogue terms in the $\mathrm{V}_\ell$ are washed away by the soft limit

# Comparison at $\mathcal{O}(\nu^2)$

At order $\nu^2$ the result don't seem to match ...

$$\frac{\tilde{W}_C^{[\omega\log\omega],\mathrm{MPM}}}{\kappa\omega\log\omega} = (\text{linear in } \nu)$$
$$- \frac{1}{160} iG^2m^3 \Big[842i\sin(2\theta)\sin(\theta)\sin(4\phi) + (102\cos(2\theta) + 241\cos(4\theta) + 1025)\cos(2\phi)$$
$$+ \sin^2(\theta)(421(\cos(2\theta)+3)\cos(4\phi) + 543\cos(2\theta) + 705)$$
$$+ 4i\cos(\theta)(181\cos(2\theta) + 161)\sin(2\phi)\Big] + \mathcal{O}(p_\infty^2)\,.$$

$$\frac{\tilde{W}_C^{[\omega\log\omega]}}{\kappa\omega\log\omega} = (\text{linear in } \nu)$$
$$+ \frac{G^2m^3\nu^2}{160} p_\infty \Big[822\sin(2\theta)\sin(\theta)\sin(4\phi) - i(142\cos(2\theta) + 231(\cos(4\theta)+5))\cos(2\phi)$$
$$- i\sin^2(\theta)(\cos(2\theta)(411\cos(4\phi)+513) + 1233\cos(4\phi) + 775)$$
$$+ 4\cos(\theta)(171\cos(2\theta) + 211)\sin(2\phi)\Big] + \mathcal{O}(p_\infty^2)\,.$$

However, the $\mathcal{O}(\nu^2)$ mismatch is not that bad ... but what does it mean?

$$\frac{\tilde{W}_C^{[\omega\log\omega]} - \tilde{W}_C^{[\omega\log\omega],\mathrm{MPM}}}{\kappa\omega\log\omega} \equiv mis_{\nu^2} \qquad 1/c^5$$
$$= -iG^2m^3\nu^2p_\infty\left(\sin^2(\theta)\sin^2(\phi)+1\right)(\cos(\phi) + i\cos(\theta)\sin(\phi))^2 + \mathcal{O}(p_\infty^2)\,.$$

# The role of the BMS frame

The non-linear terms contribute at $\mathcal{O}(G^2, \nu^2)$ thanks to the static part of the near-zone multipoles ($I_{ij}^{(2)} \sim p_{(i}p_{j)}$ with no $G$!)

$G(M_{a(i}^{(3)} M_{j)a}^{(2)}) \sim G^2$ since $M_{ij}^{(2)} \sim I_{ij}^{(2)} \sim G^0$ and $M_{ij}^{(3)} \sim I_{ij}^{(3)} \sim G^1$

But the static part of the waveform depends on the choice of BMS frame

So the time-dependent part of $\tilde{W}_1^{\rm MPM}$ depends on the BMS frame!

The BMS choice already played a role in the comparison between MPM and amplitudes results for the static angular momentum

Damour 2010.01641, Veneziano ... 2201.11607, ...

Following that discussion a natural guess is that the (current) amplitude result is in the "canonical" frame, while we know that the MPM is in the "intrinsic" frame. The two are related by the same $\mathcal{O}(G)$ supertranslation

Veneziano ... 2201.11607, ...

$\alpha_i$ are simple $\theta$-dependent quantities, see 2312.07452

$$U \mapsto U - T(n)\,, \qquad T(n) = 2G(m_1\alpha_1 \log\alpha_1 + m_2\alpha_2 \log\alpha_2)$$

$$\delta_T h_{AB} = -T(n)\,\partial_u h_{AB} + r\,[2D_A D_B - \gamma_{AB}\Delta]\,T(n)$$

One can check that

$$mis_{\nu^2} = i\omega T(n)\tilde{W}_0^{[\log\omega]}$$

# Conclusions and outlook

I presented a first step in the analysis of the scattering PM waveform

- Both the eikonal and the KMOC approaches agree in a non-trivial way once the subtractions are properly calculated and interpreted
- The comparison with the PN results is interesting both technically and conceptually. I focused on the radiation-reaction effects up to 2.5PN: there is full agreement once the amplitudes and the MPM results are written in the same frame

Outlook Many open questions:

- Is the choice of a BMS frame relevant in other comparisons (PN versus NR, PN versus NRGR-EFT)? Is it relevant for bound orbits?
- Does the agreement at $\mathcal{O}(G^2)$ survive beyond the soft limit?
- When does the naive eikonal exponentiation break down? (If it does)
- NNLO waveform? ...