Aspects of Conformal Gravity and Double Field Theory from a Double Copy of Yang-Mills Theory

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QCD Meets Gravity

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Outline:

- CK duality, BCJ relations and Double Copy **Donal's talk**
- T-duality
- Double Field Theory
- Double Field Theory as a Double Copy of Yang-Mills
- Higher-derivative Double Field Theory and Double Copy

Work with Eric Lescano and Jesús A. Rodríguez (arXiv:2307.14538)

- Outlook

Color-kinematics, BCJ relations and double copy

Bern, Carrasco, Chiodaroli, Johansson and Roiban, 2019

- In the mid-80s it was realized that tree-level closed string amplitudes can be written as sums of products of tree-level open-string amplitudes, the Kawai-Lewellen-Tye (KLT) relations. In the limit of infinite string tension, this becomes the field theory statement that the graviton tree amplitudes can be obtained as a sum of products of gluon scattering amplitudes:

$$\text{Gravity} = (\text{gauge theory})^2.$$

Kawai, Lewellen and Tye, 1986

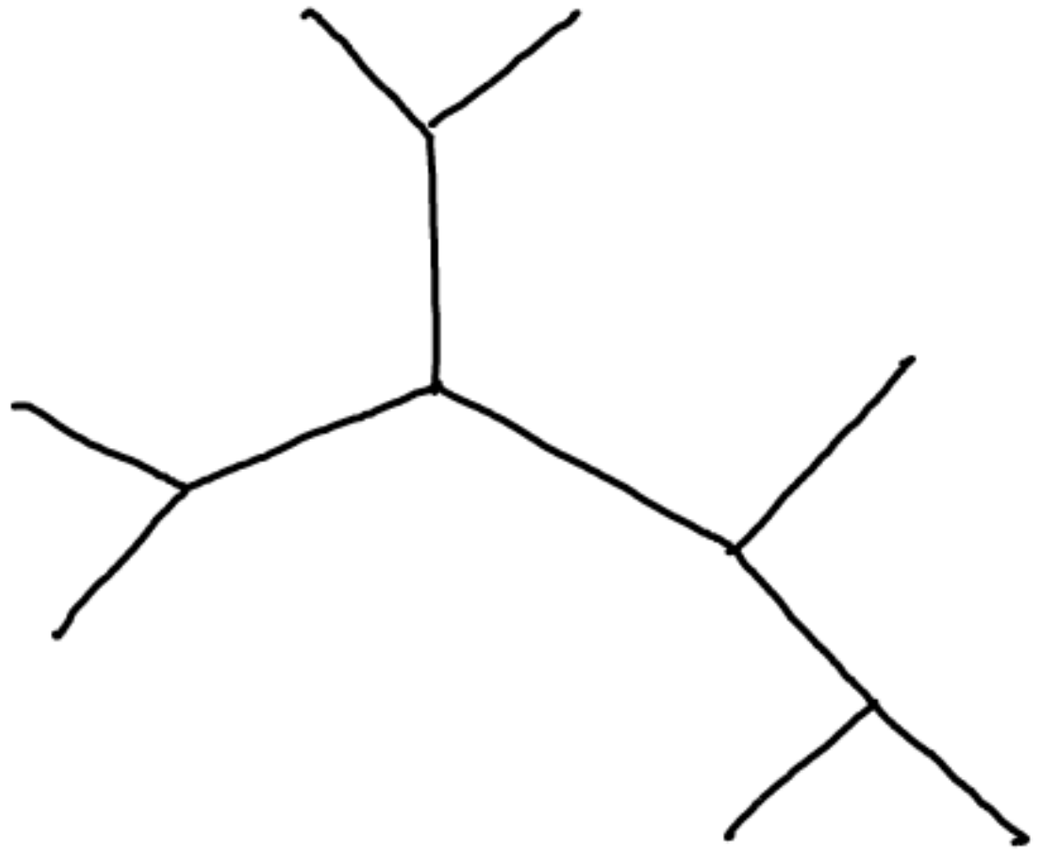
This is the so-called *double copy*.

- Tree-level gauge theory amplitudes of gluon scattering could be written in a form where certain kinematic numerators obey the same Jacobi identities as the algebraic color-factors of the non-abelian gauge group of the theory. This is called color-kinematics duality. Moreover, if one replaces the color factors in this representation of the amplitude with the kinematic factors of gauge theory, remarkably the result is the gravity tree amplitude! This is the BCJ (Bern, Carrasco, and Johansson) double copy.

Bern, Carrasco and Johansson, 2008

How does it work?

- The full color-dressed n-point tree amplitude of Yang–Mills theory can be conveniently organized in terms of diagrams with only cubic vertices

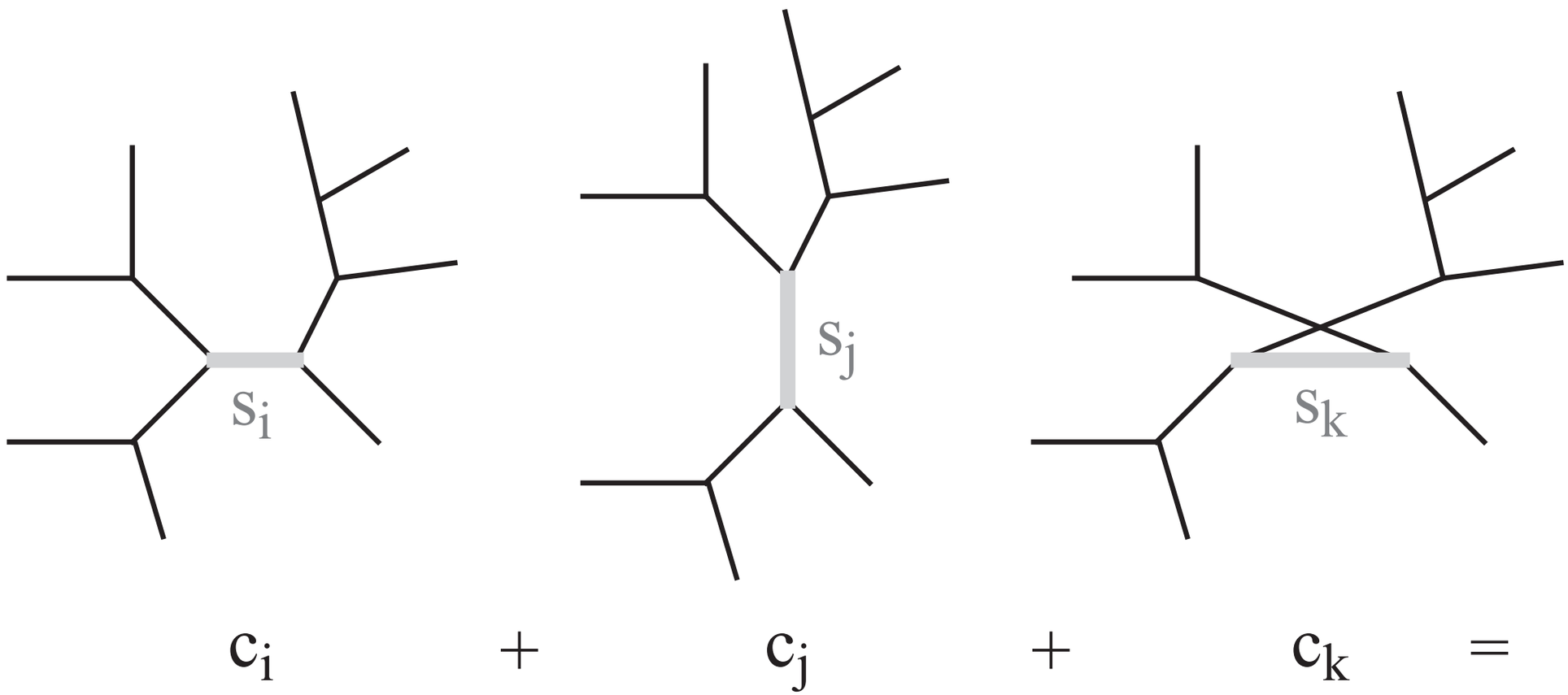


- The tree amplitude is then written as a sum over all distinct trivalent diagrams:

$$A_n^{\text{tree}} = \sum_i \frac{c_i n_i}{\Pi_{\alpha_i} p_{\alpha_i}^2}$$

- In general, for any set of three trivalent diagrams whose color factors are related through a Jacobi identity, $c_i + c_j + c_k = 0$, the following numerator-deformation leaves the amplitude invariant:

$$n_i \to n_i + s_i\Delta \quad n_j \to n_j + s_j\Delta \quad n_k \to n_k + s_k\Delta$$



Elvang and Huang, 2015

Color-kinematics duality

- The duality states that scattering amplitudes of Yang–Mills theory, and its supersymmetric extensions, can be given in a representation where the numerators have the same algebraic properties as the corresponding color factors:

$$c_i = -c_j \leftrightarrow n_i = -n_j$$

$$c_i + c_j + c_k = 0 \leftrightarrow n_i + n_j + n_k = 0$$

- The duality does not state that the numerator factors have to be local; they are allowed to have poles.

- Under color-kinematics duality, one can derive relations among color-ordered amplitudes, such as

$$tA_4^{\text{tree}}[1324] = sA_4^{\text{tree}}[1234].$$

This is an example of the BCJ relations.

Bern, Carrasco and Johansson, 2008

What about gravity?

- Color-kinematics and BCJ relations allows us to write gravity amplitudes as

$$M_n^{\text{tree}} = \sum_i \frac{n_i^2}{\Pi_{\alpha_i} p_{\alpha_i}^2}$$

It can be generalized in terms of the product of two possibly distinct Yang–Mills numerators:

$$M_n^{\text{tree}} = \sum_i \frac{n_i \widetilde{n}_i}{\Pi_{\alpha_i} p_{\alpha_i}^2}$$

- The general idea is to start from the above formula, use color-kinematics duality and BCJ relations to show that we obtain the KLT relations, such as

$$\begin{aligned} M_4^{\text{tree}}(1,2,3,4) &= -is_{12} A_4^{\text{tree}}(1,2,3,4) \widetilde{A}_4^{\text{tree}}(1,2,4,3) \\ M_5^{\text{tree}}(1,2,3,4,5) &= is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5) \widetilde{A}_5^{\text{tree}}(2,1,4,3,5) \\ &+ is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5) \widetilde{A}_5^{\text{tree}}(3,1,4,2,5) \end{aligned}$$

Bern, Carrasco and Johansson, 2008, 2010

Bjerrum-Bohr, Donoghue and Vanhove, 2014

So indeed gravity = (gauge theory)^2!

- This is the famous double-copy formula. Only one copy of the numerators needs to satisfy the duality, not both.

- Assuming that there exists a duality-satisfying set of local numerators for the Yang– Mills tree amplitude, one can rigorously prove that the doubling-relation produces the correct gravity tree amplitude for any n. The proof is established inductively by showing that the difference between this equation and the gravity amplitude obtained from BCFW recursion vanishes if one assumes the doubling-relation to hold for all lower-point amplitudes.

Bern, Dennen, Huang and Kiermaier, 2010

Elvang and Huang, 2015

3-point graviton amplitudes:

$$M_3(1^{--},2^{--},3^{++}) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2} = A_3[1^-,2^-,3^+]^2$$

$$M_3(1^{++},2^{++},3^{--}) = \frac{[12]^6}{[13]^2[32]^2} = A_3[1^+,2^+,3^-]^2.$$

Caveats…

But see: Borsten, Jurco, Kim, Macrelli, Saemann, Wolf, arXiv 2102.11390 [hep-th]; arXiv 2108.03030 [hep-th]

Unlike at tree-level, there is currently not a formal proof of the existence of duality-satisfying numerators at loop level. Furthermore, the map does not produce a pure Einstein gravity. Typically the double copy of two vector fields contains more than just the graviton. For example, a gluon in four dimension has two helicities ± 1, so the square has four states: the ± 2 correspond to graviton, and the zero-helicity states which are identified as the dilaton scalar ϕ and a Kalb-Ramond two-form $B_{\mu\nu}$. This is a consequence of the fact that low-energy string theory produces

$$S = \frac{2}{\kappa^2}\int d^D x\sqrt{-g}\, e^{-2\Phi}\left[-\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + \mathcal{O}(\alpha')\right]$$

where $H_{\mu\nu\lambda}$ is the fully antisymmetric field strength for $B_{\mu\nu}$. The projection to pure gravity, which eliminates the dilaton and axion, can be performed in any case: When using the four-dimensional helicity method, we simply correlate the helicities of the two copies of gauge theory amplitudes. Moreover, we can simply apply the projection to graviton on each of the massless cut legs, which are external lines for tree-level blobs in a generalized unitarity cut.

In summary:

$$\text{4d axion-dilaton gravity} = \text{YM} \otimes \text{YM}$$

and similarly

$$\mathcal{N} = 8 \text{ SUGRA} = (\mathcal{N} = 4 \text{ SYM}) \otimes (\mathcal{N} = 4 \text{ SYM})$$

T-duality basics

- T-duality: a symmetry of string theory that relates winding modes in a given compact space with momentum modes in another compact space.

- Consider the mass spectrum of a closed string on a circle of radius R

$$M^2 = (N + \tilde{N} - 2) + p^2 \frac{\ell_s^2}{R^2} + \tilde{p}^2 \frac{\ell_s^2}{\tilde{R}^2}$$

where ℓ_s is the string length and $\tilde{R} = \ell_s^2/R$. We also have the level-matching condition $N - \tilde{N} = p\tilde{p}$.

- The mass spectrum is invariant under the following exchange:

$$\frac{R}{\ell_s} \leftrightarrow \frac{\tilde{R}}{\ell_s}, \quad P \leftrightarrow \tilde{p}.$$

One cannot distinguish between a closed string moving with a given momentum k on a circle of radius R, and a closed string winding k times on a circle of radius ℓ_s^2/R. This symmetry holds to any observable in the full theory!

T-duality basics (cont.)

- The T-duality symmetry of circle compactifications is generalized to $O(D,D)$ toroidal compactifications with constant background metric and antisymmetric field. The elements of the infinite discrete group $O(D,D)$ can be defined as the set of $2D \times 2D$ matrices h_{mn} that preserve the $O(D,D)$ invariant metric, $h_m^{\ p}\, \eta_{pq}\, h_n^{\ q} = \eta_{mn}$, where

$$\eta_{mn} = \begin{pmatrix} 0 & \delta^i_{\ j} \\ \delta_i^{\ j} & 0 \end{pmatrix}$$

  raises and lowers all the $O(D,D)$ indices $m, n = 1, \ldots, 2D$.

- Momentum and winding modes are now D-dimensional objects which can be arranged into a generalized momentum

$$\mathcal{P}^M = \begin{pmatrix} \tilde{p}_i \\ p^i \end{pmatrix}$$

- Mass operator: $M^2 = (N + \tilde{N} - 2) + \mathcal{P}^p\, \mathcal{H}_{pq}\, \mathcal{P}^q$, where the generalized metric is

$$\mathcal{H}_{mn} = \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{pmatrix}$$

  T-duality invariant dilaton: $e^{-2d} = \sqrt{|g|}\, e^{-2\phi}$.

- Any element of $O(D,D)$ can be decomposed as successive products of diffeomorphisms, shifts and factorized T-dualities.

- Summarizing, the T-duality symmetry of the circle compactification is generalized in toroidal compactifications to $O(D,D)$ acting as $\mathcal{H}_{mn} \to h_m^{\ p}\, \mathcal{H}_{pq}\, h_n^{\ q}$, $\mathcal{P}^m \to h^m_{\ n}\, P^n$, with $h \in O(D,D)$ on constant backgrounds.

Overview of Double Field Theory

Aldazabal, Marqués and Núñez, 2013

- Double Field Theory is a proposal to incorporate T-duality as a symmetry of a field theory.

- Traditionally, DFT restricts to the modes of the string that are massless in the decompactified limit, $N + \tilde{N} = 2$, but usually considers them on a compact space. These modes correspond to the levels $N = \tilde{N} = 1$, corresponding to a symmetric metric g_{ij}, an antisymmetric 2-form b_{ij} and a dilaton ϕ.

- For DFT states, the generalized momenta must be orthogonal with respect to the $O(D, D)$ metric.

- T-duality in DFT is allowed in non-isometric directions!

- Everything must be organized in T-duality representations, including coordinates $\rightarrow$ one must introduce new coordinates $\tilde{x}_i$ such that $X^m = (\tilde{x}_i, x^i)$.

- Transformation of coordinates and fields:

$$\begin{aligned} X^m &\rightarrow h^m{}_n\, X^n \\ \mathcal{H}_{mn}(X) &\rightarrow h^p_m\, h^q_n\, \mathcal{H}_{pq}(hX) \\ d(X) &\rightarrow d(hX). \end{aligned}$$

- DFT: a T-duality invariant formulation of supergravity with compact dimensions.

Constraints in DFT

Aldazabal, Marqués and Núñez, 2013

- There must be some mechanism to constrain the coordinate dependence. Usually one employs the strong constraint:

$$\eta^{mn}\partial_m\partial_n(\cdots) = 0$$

which in components takes the form $\tilde{\partial}^i\partial_i(\cdots) = 0 \rightarrow$ only possible configurations allowed depend on a D-dimensional section of the space!

- But, a weakly constrained version exists containing genuine massive string modes and perhaps exhibits an improved UV behavior.

- Also, since the metric and the 2-form field are unified into a generalized metric, diffeomorphism and gauge symmetry of the 2-form can also be unified in generalized diffeomorphisms and gauge transformations. But in order to have a closed group for the generalized diffeomorphisms, one must impose a consistency condition – DFT is a constrained or restricted theory!

Kerr-Schild DFT and double copy

- Generalized Kerr-Schild ansatz:

Lee, 2018 (See also Lee, 2016)

$$\mathcal{H}_{mn} = H_{mn} + \kappa\varphi(\bar{K}_m K_n + K_m \bar{K}_n)$$

where $K_m, \bar{K}_m$ are generalized null vectors. Also:

$$d = d_0 + \kappa f, \quad f = \sum_{n=0}^{\infty} \kappa^n f_n.$$

- Can be used to explore a double Yang-Mills formulation of heterotic DFT, where the fundamental fields are in representations of $O(D, D)$ and a generalized gauged field/symmetry explicitly appears in the double geometry.

Lescano and Roychowdhury, 2022

- Can be used to exploit heterotic formulation of DFT when higher-derivative terms are included. Essentially, one adapts the formalism to the flux formulation of DFT and consider a higher-derivative extension for DFT with multiplets of $O(D, D+K)$, by choosing free parameters to match with the heterotic case. A generalized Green-Schwarz transformation is also used.

Lescano and Rodríguez, 2021

A little disclaimer : Lee-Wick theories are unitary!

- Lee-Wick type theories have propagators of the form:

$$iD(q) = \frac{i}{q^2 + i\epsilon - \frac{q^4}{M^2} + \Sigma(q)}.$$

The pole at $q^2 = 0$ is the stable particle of the theory.

- At one-loop order, the self-energy typically has the form

$$\Sigma(q) = -\frac{\gamma}{\pi} \log\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) = \left[-\frac{\gamma}{\pi} \log\left(\frac{|q^2|}{\mu^2}\right) + i\gamma\theta(q^2)\right]$$

for some calculable quantity γ with dimensions of mass squared.

- A massive resonance for timelike values of q^2 appears. Expanding near that resonance:

$$iD(q)\Big|_{q^2 \sim M^2} \sim \frac{-i}{q^2 - M^2 - i\gamma} \quad .$$

Merlin modes!

Observe the minus sign in the numerator – a ghost like resonance. In addition, the width also comes with the opposite sign from normal.

Formal discussion of unitarity

- Unitarity:

$$\langle f|T|i\rangle - \langle f|T^\dagger|i\rangle = i\sum_j \langle f|T^\dagger|j\rangle\langle j|T|i\rangle$$

- In processes that involve loop diagrams, the sum over real intermediate states can by accomplished by the Cutkosky cutting rules.

- What we usually do: look first at the free field theory to identify the free particles.

- Turn on interactions: some of these particles become unstable and no longer appear as the asymptotic states of the theory. *The free field limit has mislead us.*

- Should one include such unstable particles in the sums over states required for unitarity? Veltman says *no*!

- Veltman: unitarity is indeed satisfied by the inclusion of *only* the asymptotically stable states. Cuts are not to be taken through the unstable particles, and unstable particles are not to be included in unitarity sums.

- *However*, in the narrow-width approximation, the off-resonance production becomes small and only resonance production is important. In this limit a cut taken through the unstable particle with its width set to zero reproduces the same result as a cut through the decay products.

UNITARITY AND CAUSALITY IN A RENORMALIZABLE FIELD THEORY WITH UNSTABLE PARTICLES

M. VELTMAN *)

Donoghue and GM, PRD 100, 105006 (2019)

Donoghue and GM, PRD 99, 065017 (2019)

Causality in higher-derivative theories

Donoghue and GM, PRL 123, 171601 (2019) (Editor's suggestion)

Donoghue and GM, Progress in Particle and Nuclear Physics 115, 103812 (2020)

Donoghue and GM, JHEP 11, 010 (2021)

- Causality in quantum field theory is defined by the vanishing of field commutators for space-like separations.

- However, this does not imply a direction for causal effects. Hidden in our conventions for quantization is a connection to the definition of an arrow of causality.

- Mixing quantization conventions within the same theory, we get a violation of microcausality. In such a theory with mixed conventions the dominant definition of the arrow of causality is determined by the stable states.

- In some quantum gravity theories, such as quadratic gravity and possibly asymptotic safety, such a mixed causality condition occurs.

DFT as a Double Copy of Yang-Mills

Díaz-Jaramillo, Hohm and Plefka, 2021

- Weakly Constrained DFT naturally arises from the double copy of Yang-Mills theory, after the duality invariant dilaton has been integrated out.

- Works fine at quadratic and cubic order. Quartic vertex is more subtle, but can be done by using the framework of homotopy algebras (homotopy Lie or L_∞ algebras).

Bonezzi, Chiaffrino, Díaz-Jaramillo and Hohm, 2023

- Quadratic action (in momentum space):

$$S_{\rm YM}^{(2)} = -\frac{1}{2}\int_k \kappa_{ab}\, k^2\, \Pi^{\mu\nu}(k) A_\mu{}^a(-k) A_\nu{}^b(k)$$

  where $\Pi^{\mu\nu}(k) = \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$.

- Double-copy recipe:

$$A_\mu{}^a(k) \;\longrightarrow\; e_{\mu\bar\mu}(k,\bar k), \quad \kappa_{ab} \;\longrightarrow\; \frac{1}{2}\,\bar\Pi^{\bar\mu\bar\nu}(\bar k)$$

- The action becomes:

$$S_{\rm DC}^{(2)} = -\frac{1}{4}\int_{k,\bar k} k^2\, \Pi^{\mu\nu}(k)\bar\Pi^{\bar\mu\bar\nu}(\bar k) e_{\mu\bar\mu}(-k,-\bar k) e_{\nu\bar\nu}(k,\bar k)\,.$$

- In doubled position space:

Level-matching constraint: $\bar k^2 = k^2$!

$$S_{\rm DC}^{(2)} = \frac{1}{4}\int d^Dx\, d^D\bar x \Big(e^{\mu\bar\nu}\Box e_{\mu\bar\nu} + \partial^\mu e_{\mu\bar\nu}\,\partial^\rho e_\rho{}^{\bar\nu} + \bar\partial^{\bar\nu} e_{\mu\bar\nu}\,\bar\partial^{\bar\sigma} e^\mu{}_{\bar\sigma} - \phi\Box\phi + 2\phi\partial^\mu\bar\partial^{\bar\nu} e_{\mu\bar\nu}\Big).$$

Reproduces the standard quadratic DFT action after Fourier transforming to a local action in doubled position space

- Cubic term: Díaz-Jaramillo, Hohm and Plefka, 2021

$$S^{(3)}_{\rm YM} = -\frac{ig_{\rm YM}}{6\,(2\pi)^{D/2}} \int_{k_i} \delta\left(k_1+k_2+k_3\right) f_{abc}\pi^{\mu\nu\rho} A^a_{1\mu} A^b_{2\nu} A^c_{3\rho}$$

where $\pi^{\mu\nu\rho}(k_1,k_2,k_3) = \eta^{\mu\nu}k_{12}^{\rho} + \eta^{\nu\rho}k_{23}^{\mu} + \eta^{\rho\mu}k_{31}^{\nu}$.

- Recipe: $f_{abc} \longrightarrow \frac{i}{4}\bar{\pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}$. The action becomes:

$$\begin{aligned} S^{(3)}_{\rm DC} &= \frac{1}{48(2\pi)^{D/2}} \int dK_1 dK_2 dK_3 \delta(K_1+K_2+K_3) \\ &\times \bar{\pi}^{\bar{\mu}\bar{\nu}\bar{\rho}} \pi^{\rho\mu\nu} e_{1\mu\bar{\mu}} e_{2\nu\bar{\nu}} e_{3\rho\bar{\rho}} \end{aligned}$$

where $K=(k,\bar{k})$, $dK = d^{2D}K$ and $e_{i\mu\bar{\mu}} = e_{\mu\bar{\mu}}(K_i)$.

- In doubled position space:

$$\begin{aligned} S^{(3)}_{\rm DC} &= \frac{1}{8}\int d^Dx\, d^D\bar{x}\; e_{\mu\bar{\mu}}\Big[\, 2\partial^{\mu} e_{\rho\bar{\rho}}\, \bar{\partial}^{\bar{\mu}} e^{\rho\bar{\rho}} - 2\partial^{\mu} e_{\nu\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e^{\nu\bar{\mu}} \\ &- 2\partial^{\rho} e^{\mu\bar{\rho}}\, \bar{\partial}^{\bar{\mu}} e_{\rho\bar{\rho}} + \partial^{\rho} e_{\rho\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e^{\mu\bar{\mu}} + \bar{\partial}_{\bar{\rho}} e^{\mu\bar{\rho}}\, \partial_{\rho} e^{\rho\bar{\mu}} \,\Big]. \end{aligned}$$

Agrees with the cubic DFT action by a gauge fixing condition (Siegel gauge), integrating out the dilaton

L_∞ structure of DFT

Bonezzi, Chiaffrino, Díaz-Jaramillo and Hohm, 2023

- Consistency of Yang-Mills theory: C_∞ algebra structure on kinematic vector space (graded vector space given by the direct sum of subspaces of homogeneous degree).

- Non-trivial C_∞ relations: Nilpotency ($m_1^2(\psi) = 0$); differential acting as derivative ($m_1 m_2(\psi_1, \psi_2) = m_2(m_1\psi_1, \psi_2) + (-1)^{\psi_1} m_2(\psi_1, m_1\psi_2)$); associative (up to homotopy):

$$
\begin{aligned}
& m_2(m_2(\psi_1,\psi_2),\psi_3) - m_2(\psi_1, m_2(\psi_2,\psi_3)) \\
& = m_1 m_3(\psi_1,\psi_2,\psi_3) + m_3(m_1\psi_1,\psi_2,\psi_3) \\
& +(-1)^{\psi_1} m_3(\psi_1, m_1\psi_2,\psi_3) + (-1)^{\psi_1+\psi_2} m_3(\psi_1,\psi_2,m_1\psi_3)
\end{aligned}
$$

m_n have to vanish under so-called shuffle permutations.

- Two copies of the kinematic algebra $\mathcal{K}$ produce an algebra $\mathrm{BV}_\infty^\triangle$ on the tensor product $\mathcal{K} \otimes \bar{\mathcal{K}}$, with $\triangle = (\Box - \bar{\Box})/2$.

- Weakly constrained DFT: Homotopy transfer! Vector space carrying the L_∞ algebra: $\ker\triangle \cap \mathrm{b}^-$, where $b^\pm = (1/2)(b \otimes \mathbf{1} \pm \mathbf{1} \otimes \bar{b})$ is a second differential, of opposite degree to m_1.

Conformal Gravity and Double Field Theory

Lescano, GM and Rodríguez, arXiv:2307.14538 (Accepted for PRD)

- The construction of double field theory as the double copy of a Yang-Mills theory can be extended to include higher-derivative terms.

- Starting from a four-derivative extension of Yang-Mills whose double copy is known to correspond to a conformal-gravity theory, a four-derivative theory can be formulated in double space, which in the pure gravity limit reduces to conformal gravity at quadratic order.

- Cubic action with additional contributions beyond the Weyl square action. Physical interpretation in terms of symmetry arguments?

- Our construction is a first step in the program of consider conformal symmetry in the context of Double Field Theory – important step further in the construction of a non-perturbative conformal double geometry with applications to scattering amplitudes.

- Higher-derivative Lagrangian:

$$\mathcal{L} = \frac{1}{2}\kappa_{ab} D_\mu F^{\mu\nu a} D_\rho F^{\rho}{}_{\nu}{}^{b}$$

with $D_\rho F_{\mu\nu}{}^{a} = \partial_\rho F_{\mu\nu}{}^{a} + g_{\rm YM} f^{a}{}_{bc} A_\rho{}^{b} F_{\mu\nu}{}^{c}$.

Quadratic term

- Quadratic action in momentum space:

$$S^{(2)}_{\rm HD} = -\frac{1}{2}\int_k \kappa_{ab}\, k^4\, \Pi^{\mu\nu}(k) A_\mu{}^a(-k) A_\nu{}^b(k).$$

- After imposing the double copy relations:

$$S^{(2)}_{\rm HD/DC} = -\frac{1}{4}\int_{k,\bar{k}} k^4\, \Pi^{\mu\nu}(k)\bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k}) e_{\mu\bar{\mu}}(-k,-\bar{k}) e_{\nu\bar{\nu}}(k,\bar{k}).$$

- In doubled position space

$$\begin{aligned} S^{(2)}_{\rm HD/DC} = -\frac{1}{4}\int d^D x d^D \bar{x} \Big[& \Box e^{\mu\bar{\nu}} \Box e_{\mu\bar{\nu}} - \Box e^{\mu\bar{\nu}} \partial_\mu \partial^\rho e_{\rho\bar{\nu}} \\ & - \Box e^{\mu\bar{\nu}} \bar{\partial}_{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\sigma}} + \partial^\mu \bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}} \partial^\rho \bar{\partial}^{\bar{\sigma}} e_{\rho\bar{\sigma}} \Big]. \end{aligned}$$

- Expand the conformal gravity action $S_{\rm CG} = \int d^D x\, \sqrt{-g}\, C_{\mu\nu\rho\lambda} C^{\mu\nu\rho\lambda}$ (C is the Weyl tensor) up to quadratic order for $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and impose the gauge fixing condition $\Box h = \partial_\mu \partial_\nu h^{\mu\nu}$, related to dilatation symmetry $\delta h_{\mu\nu} = -2\lambda_D \eta_{\mu\nu}$. After setting $x = \bar{x}$ and properly rescaling the metric, in the pure gravity case, both results match.

A higher-derivative extension of DFT with conformal symmetry in the double space (in a classical sense)

Cubic term

- Cubic action in momentum space:

$$S^{(3)}_{\rm HD} = \frac{i g_{YM}}{3(2\pi)^{D/2}} \int_{k_i} \delta(k_i) f_{abc} k_1^2 \Pi^{\mu\nu\rho} A_{1\mu}{}^a A_{2\nu}{}^b A_{3\rho}{}^c .$$

where $\Pi^{\mu\nu\rho}(k_1,k_2,k_3) = \Pi^{\mu\nu} k_{12}^{\rho} + \Pi^{\nu\rho} k_{23}^{\mu} + \Pi^{\rho\mu} k_{31}^{\nu}$.

$$f_{abc} \longrightarrow \frac{i}{4} \bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}$$

- Applying the double copy prescription:

$$\begin{aligned} S^{(3)}_{\rm HD/DC} &= -\frac{1}{24(2\pi)^{D/2}} \int dK_1 dK_2 dK_3 \delta(K_1+K_2+K_3) \\ &\times \; k_1^2 \bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}} \Pi^{\rho\mu\nu} e_{1\mu\bar{\mu}} e_{2\nu\bar{\nu}} e_{3\rho\bar{\rho}} . \end{aligned}$$

- Transforming back to coordinate space:

$$\begin{aligned} S^{(3)}_{\rm HD/DC} &= -\frac{1}{4} \int d^D x \; d^D \bar{x} \\ &\times \Big[\Box \partial^\rho \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} e^{\mu\bar{\nu}} e_{\rho\bar{\sigma}} - \Box \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \partial^\rho e^{\mu\bar{\nu}} e_{\rho\bar{\sigma}} + \Box \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \partial^\mu e^{\rho\bar{\nu}} e_{\rho\bar{\sigma}} - \Box \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} e^{\rho\bar{\nu}} \partial^\mu e_{\rho\bar{\sigma}} \\ &- \Box e_{\mu\bar{\nu}} \partial^\mu \bar{\partial}^{\bar{\sigma}} e^{\rho\bar{\nu}} e_{\rho\bar{\sigma}} + \Box e_{\mu\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e^{\rho\bar{\nu}} \partial^\mu e_{\rho\bar{\sigma}} - \partial^\mu \partial^\rho \partial^\lambda \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} e_\rho{}^{\bar{\nu}} e_{\lambda\bar{\sigma}} + \partial^\mu \partial^\rho \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \partial^\lambda e_\rho{}^{\bar{\nu}} e_{\lambda\bar{\sigma}} \\ &- \partial^\rho \partial^\lambda \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \partial^\mu e_\rho{}^{\bar{\nu}} e_{\lambda\bar{\sigma}} + \partial^\rho \partial^\lambda \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} e_\rho{}^{\bar{\nu}} \partial^\mu e_{\lambda\bar{\sigma}} + \partial^\rho \partial^\lambda e_{\mu\bar{\nu}} \partial^\mu \bar{\partial}^{\bar{\sigma}} e_\rho{}^{\bar{\nu}} e_{\lambda\bar{\sigma}} - \partial^\rho \partial^\lambda e_{\mu\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e_\rho{}^{\bar{\nu}} \partial^\mu e_{\lambda\bar{\sigma}} \\ &- \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} \bar{\partial}^{\bar{\kappa}} \partial^\rho e_{\mu\bar{\nu}} e^\mu{}_{\bar{\sigma}} e_{\rho\bar{\kappa}} + \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} \bar{\partial}^{\bar{\kappa}} e_{\mu\bar{\nu}} \partial^\rho e^\mu{}_{\bar{\sigma}} e_{\rho\bar{\kappa}} + \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \bar{\partial}^{\bar{\kappa}} \partial^\mu e^\rho{}_{\bar{\sigma}} e_{\rho\bar{\kappa}} - \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \bar{\partial}^{\bar{\kappa}} e^\rho{}_{\bar{\sigma}} \partial^\mu e_{\rho\bar{\kappa}} \\ &+ \partial^\mu \partial^\rho \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} \phi e_{\mu\bar{\nu}} e_{\rho\bar{\sigma}} - \partial^\mu \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} \phi \partial^\rho e_{\mu\bar{\nu}} e_{\rho\bar{\sigma}} + \partial^\mu \partial^\rho \phi \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} e_{\rho\bar{\sigma}} - \partial^\rho \phi \bar{\partial}^{\bar{\nu}} \bar{\partial}^{\bar{\sigma}} e_{\mu\bar{\nu}} \partial^\mu e_{\rho\bar{\sigma}} \Big] . \end{aligned}$$

Set $x = \bar{x}$ for the pure gravity case in order to compare with the cubic contributions coming from the Weyl action. This cubic action contains further contributions apart from the Weyl square action.

Conformal DFT?

- We conjecture that our results are part of contributions to a would-be conformal Double Field Theory defined in a suitable generalized Siegel gauge. Proposals that could account for this relation:

  1. *Generalized Kerr-Schild ansatz* – Previously mentioned analysis can be extended to heterotic DFT at order α' and from classical double copy a higher-derivative corrections to the Maxwell equations at order κ in the perturbations can be found. One can continue this path by extending the order of the perturbations to κ^2 which match with the order of gauge fields expected for the equations of motion. Lescano and Rodríguez, 2021

  2. *Conformal symmetry in DFT* – Consideration of this kind of transformation leads to a particular parameterization of the generalized metric as well as a deformation of the $O(D,D)$ metric. As a consequence, the theory requires a coupling with an extra scalar field to have a conformal invariant action at leading order.

  3. *Generalized Weyl tensor* – One can look for the existence of a two-derivative combination of the DFT fields $\mathcal{C}_{mnpq}$, mimicking the usual Weyl tensor, such that an action with a term $\mathcal{C}_{mnpq}\mathcal{C}^{mnpq}$ could be constructed and reduced to conformal gravity.

Outlook

A natural double copy prescription applied to the Yang- Mills Lagrangian produces a weakly constrained double field theory

Some conclusions and perspectives:

- Important non-trivial examples of features of the double copy naturally emerging at the level of an action.
- A potential higher derivative extension is promising (must understand cubic terms).
- Explorations of conformal symmetry in the context of Double Field Theory.
- Explore the double copy prescription for the DF^2 theory and its extensions?