Black hole scattering: the self-force approach

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Talk outline

A) Introduction to self-force

B) Self-force in black hole scattering

C) Frequency-domain approach

D) Ongoing work
PART A: Introduction to self-force

Reviews:

The 2-body problem in GR: approaches

Image credit: L. Barack & A. Pound
Extreme mass ratio inspirals (EMRIs)

- Highly asymmetric compact binaries. Typical mass ratios

\[ q \sim \frac{10 M_\odot}{10^6 M_\odot} = 10^{-5} \ll 1 \quad (1) \]

- Inspiral slow compared to orbital periods:

\[ T_{RR} \sim \frac{T_{orb}}{q} \gg T_{orb}. \quad (2) \]

- Large number of gravitational wavecycles in LISA band before merger:

\[ N_{orb} \sim \frac{1}{q} \sim 10^5. \quad (3) \]

- Orbital dynamics complicated. Geodesics tri-periodic and generically ergodic.
- EMRIs offer a precision probe of strong-field geometry around black-holes.

Created using KerrGeodesics package from BHP toolkit.
Motion of small body given effective representation in background spacetime of the larger object.

No need for ad hoc regularisation procedures. EOM derived using \textit{matched asymptotic expansions} \cite{Mino, Sasaki & Tanaka 1997}.

No need to \textit{assume} a point-particle description; effective point-particle description is \textit{derived}.
1SF equation of motion

- Metric perturbation split into “direct” and “tail” contributions:

\[ g_{\alpha\beta}^{\text{phys}} = g_{\alpha\beta}^{\text{direct}} + h_{\alpha\beta}^{\text{tail}} \]

(4)

- Only \( h_{\alpha\beta}^{\text{tail}} \) contributes to the self-force:

\[ m \frac{D^2 z^\alpha}{d\tau^2} = m \nabla^\alpha \nabla^\beta \nabla^\gamma h_{\beta\gamma}^{\text{tail}} \bigg|_{z(\tau)} =: F_{\text{self}}^\alpha, \]

(5)

where

\[ \nabla^\alpha \nabla^\beta \nabla^\gamma h_{\beta\gamma} := -\frac{1}{2} \left( g^{\alpha\beta} + u^{\alpha} u^{\beta} \right) u^\gamma u^\delta \left( 2\nabla_\delta h_{\beta\gamma} - \nabla_\beta h_{\gamma\delta} \right). \]

(6)
Computational approach: mode-sum regularisation

- Singular field subtracted mode-by-mode in a spherical harmonic expansion around the large BH:

\[ F_{\text{self}}(\tau) = m \sum_{\ell=0}^{\infty} \left[ (\nabla h^\text{ret})^\ell - (\nabla h^\text{direct})^\ell \right]_z(\tau) \]

\[ = \sum_{\ell=0}^{\infty} \left[ m(\nabla h^\text{ret})^\ell \bigg|_{z(\tau)} - A(z)\ell - B(z) - C(z)/\ell \right] - D(z). \] (7)

- Regularization parameters: derived analytically for generic Kerr orbits.
  [Barack & Ori 2000-03]

- Numerical input: modes of $h^\text{ret}_{\alpha\beta}$ calculated numerically by solving perturbation equations with point-particle source and retarded BCs.

- Can accelerate convergence by subtracting additional parameters.
PART B: Self-force in black hole scattering


Scatter orbits

Particle starts at radial infinity at early times with velocity $v$ and *impact parameter* $b$:

$$b = \lim_{\tau \to -\infty} r_p(\tau) \sin |\varphi_p(\tau) - \varphi_p(-\infty)|.$$  \hspace{1cm} (8)

Provided $b > b_c(v)$, particle scatters off central black hole, approaching to within periapsis distance $r_{\text{min}}$. 
Why study scattering?

- **Theoretical grounds:**
  1. Can probe sub-ISCO region even at low velocities; down to light ring $r = 3M$ with large $v$.
  2. Scattering angle $\chi(b, v)$ defined unambiguously, even with radiation.

- Boundary-to-bound relations between scatter and bound orbit observables, derived using effective-field-theory. [Kalin & Porto 2020]

- $\chi_{1\text{SF}}$ determines **full** conservative dynamics to 4PM, valid at *any* mass ratio. Extend to 6PM with $\chi_{2\text{SF}}$ [Damour 2020]. PM expansion of $\chi$ can be used to calibrate effective-one-body models [Damour 2016].

- Can compare SF results with analytical PM for mutual validation; benchmark/calibrate PM in strong-field.
The self-force correction is defined by

$$\delta \chi := \chi(b, v) - \chi_0(b, v) = O(q),$$

(9)

where $$\chi_0 := \lim_{q \to 0} \chi$$ is the scatter angle of the geodesic with the same $$(b, v)$$.

Correction expressed as integral over the worldline, [Barack & Long 2022]

$$\delta \chi = \int_{-\infty}^{+\infty} A_{\alpha}(\tau; b, v) F_{\text{self}}^\alpha(\tau) d\tau.$$  

(10)

At $O(q)$, integral may be evaluated along limiting geodesic.
Conservative and dissipative effects

We can split the self-force into conservative and dissipative pieces,

\[ F^\alpha_{\text{cons}} = \frac{1}{2} \left[ F^\alpha_{\text{self}}(h^{\text{ret}}) + F^\alpha_{\text{self}}(h^{\text{adv}}) \right], \quad (11) \]

\[ F^\alpha_{\text{diss}} = \frac{1}{2} \left[ F^\alpha_{\text{self}}(h^{\text{ret}}) - F^\alpha_{\text{self}}(h^{\text{adv}}) \right], \quad (12) \]

and consider their effects separately, [Barack & Long 2022]

\[ \delta \chi_{\text{cons}} = \int_0^{+\infty} A^\alpha_{\text{cons}}(\tau; b, v) F^\alpha_{\text{cons}}(\tau) d\tau, \quad (13) \]

\[ \delta \chi_{\text{diss}} = \int_0^{+\infty} A^\alpha_{\text{diss}}(\tau; b, v) F^\alpha_{\text{diss}}(\tau) d\tau. \quad (14) \]
Scalar-field toy model in Schwarzschild

- **Toy model**: scalar charge $Q$ with mass $m$ moving in a background Schwarzschild spacetime of mass $M$:

$$\nabla^\mu \nabla_\mu \Phi = -4\pi Q \int_{-\infty}^{+\infty} \frac{\delta^4(x - z(\tau))}{\sqrt{-g(x)}} \, d\tau,$$

(15)

$$\frac{Dp^\alpha}{d\tau} = Q \nabla^\alpha \Phi^{\text{tail}} =: F_\text{self}^\alpha.$$

(16)

- Scalar-field calculation captures the main challenges of gravitational self-force calculations, in a simpler overall framework.

- Parameter $q_s := Q^2/(mM) \ll 1$ takes the role of the mass ratio. Integral formulae for $\delta\chi$ essentially unchanged.

- First numerical calculations by Long & Barack using their (1+1)D time-domain code for the self-force.
Early scatter angle results [Barack & Long 2022]

\[ v = 0.2 \]
PM comparisons [Barack et al 2023]

Conservative

Conservative $v = 0.5$

$r_{\text{min}}/M$

$b/M$

$\delta \chi$

$\delta \chi_{\text{cons}}$  
$\delta \chi_{\text{2 cons}}$  
$\delta \chi_{\text{3 cons}}$

$\delta \chi_{\text{cons}} - \delta \chi_{\text{2 cons}} - \delta \chi_{\text{3 cons}}$  
$\propto 1/b^4$
PM comparisons [Barack et al 2023]

Conservative $v = 0.5$

\[ \frac{r_{\text{min}}}{M} \]

\[ \delta \chi \]

$|\delta \chi_{\text{cons}}|$

$|\delta \chi_2^{\text{cons}}|$

$|\delta \chi_2^{\text{cons}} + \delta \chi_3^{\text{cons}}|$

$|\delta \chi_2^{\text{cons}} + \delta \chi_3^{\text{cons}} + \delta \chi_4^{\text{cons}}|$

$|\delta \chi_2^{\text{cons}} + \delta \chi_3^{\text{cons}} + \delta \chi_4^{\text{cons}}|_{c_1=0}$

$|\delta \chi_2^{\text{cons}} + \delta \chi_3^{\text{cons}} + \delta \chi_4^{\text{cons}}|_{c_1=c_2=0}$
PART C: Frequency-domain approach

Frequency-domain methods

- Fields are additionally decomposed into Fourier harmonics, e.g.

\[ \psi_{\ell m}(t, r) = \int_{-\infty}^{+\infty} \psi_{\ell m\omega}(r) e^{-i\omega t}. \]  

(17)

- Many frequency-domain (FD) self-force codes in existence for bound orbits. Valued for their accuracy and efficiency.

- FD methods expected to retain these advantages when moving to unbound orbits, but challenges must be overcome:
  - Continuous spectrum.
  - Source with non-compact radial support.
  - Cancellation during TD reconstruction.

We use a scalar-field toy model in Schwarzschild to investigate and manage these problems.
Extended homogeneous solutions [Barack, Ori & Sago 2008]

- **Gibbs phenomenon**: impractical to reconstruct SF modes from physical inhomogeneous solution $\psi_{\ell m \omega}(r)$.

- Method of **Extended Homogeneous Solutions** restores exponential, uniform convergence.

![VoP and EHS graphs](image)
Extended homogeneous solutions: unbound orbits

- Physical time-domain field is reconstructed piecewise from **homogeneous** solutions.

- For example, SF modes in the “internal” region $r \leq r_p(t)$ reconstructed from

  $\tilde{\psi}_{\ell m \omega}(r) := C_{\ell m \omega} \psi_{\ell \omega}(r),$  

  \[ (18) \]

  where normalisation the factor $C_{\ell m \omega}$ is such that EHS and physical field coincide in $r \leq r_{\text{min}}$.

- For unbound orbits, EHS **cannot** be used to reconstruct field in the “external” region $r > r_p(t)$.

  We use EHS and one-sided mode-sum regularisation.
Truncation problem

- Normalisation factor $C_{\ell m \omega}$ can be expressed as an integral over the (unbounded) radial extent of the orbit:

$$C_{\ell m \omega} = \int_{r_{\text{min}}}^{+\infty} \frac{\psi_{\ell \omega}^+(r') S_{\ell m \omega}(r')}{W_{\ell \omega} f(r')} dr'.$$

- Slow, oscillatory convergence: problems truncating at finite $r_{\text{max}}$.

Developed solutions:

1. Tail corrections: use large-$r$ approximation to integrand to derive analytical estimates to the neglected tail.
2. Integration by parts (IBP): use IBP to increase decay rate of integrand.
$C_{\ell m\omega}$ spectra

Example $C_{\ell m\omega}$ spectra for orbit $E = 1.1$, $r_{\text{min}} = 4M$. Note QNM features.
Self-force: regularisation tests

FD code agrees better with regularisation parameters at this radius

\[ F_{\text{self}}(\tau) = \sum_{\ell=0}^{\infty} \left[ q (\nabla \Phi^{\text{ret}})^{\ell} \bigg|_{z(\tau)} - A(z)\ell - B(z) - C(z)/\ell - H.O.P \right] - D(z) \]
Cancellation problem

- Significant cancellation between low-frequency modes at large $\ell$ and $r$.

- Caused by unphysical growth of the EHS field.

- Problem intrinsic to EHS approach. Afflicts scatter calculations more severely than bound orbit case.

Partially mitigate using dynamic $\ell$-truncation in the mode-sum.
Self-force: along orbit

Gradual loss of accuracy along orbit due to progressive loss of $\ell$-modes.
PART D: Ongoing work
Analytical calculation at large $r$ (preliminary)

- Supplement FD code with analytic expansion of the SF in $1/r$.

- Makes use of a hierarchical expansion, [Barack 1998]

$$
\psi_{\ell m}(u, v) = \sum_{N=0}^{\infty} \psi_N(u, v),
$$

$$
\psi_{0, uv} + V_0(r)\psi_0 = S(u, v),
$$

$$
\psi_{N, uv} + V_0(r)\psi_N = -\delta V(r)\psi_{N-1} \quad (N > 0),
$$

where $V_0(r)$ approximates asymptotic behaviour of exact potential $V(r)$, and $\delta V(r) := V(r) - V_0(r)$.

- $\psi_0$ (complete) does not contribute to SF; $\psi_1$ (underway) gives leading large-$r$ behaviour.
PM resummation (preliminary)

- As \( b \to b_c(v) \),

\[
\chi_0 \sim A(v) \log \left(1 - \frac{b_c(v)}{b}\right) + \text{const}(v), \quad \delta \chi_{1\text{SF}} \sim q_s B(v) \frac{b_c(v)}{b - b_c(v)}.
\]

\( A(v) \) known analytically; \( B(v) \) inferred from SF calculations.

- Consider the function

\[
\psi^{n\text{PM}} = A \left[ \log \left(1 - \frac{b_c(1 - q_s B/A)}{b}\right) + \sum_{k=1}^{n} \frac{1}{k} \left(\frac{b_c(1 - q_s B/A)}{b}\right)^k \right].
\]

- We define the resummed scatter angle \( \tilde{\chi}^{n\text{PM}} := \chi^{n\text{PM}} + \psi^{n\text{PM}} \).
  - Agrees with \( \chi^{n\text{PM}} \) through \( n\text{PM} \) order.
  - Matches the 0SF and 1SF divergences near separatrix.
PM resummation (preliminary)

\[ v = 0.2 \]

\[ \delta \chi_{1SF} \]

\[ \delta \chi^{3\text{PM}} \]

\[ \delta \chi^{-3\text{PM}} \]

\[ \frac{(b - b_c)}{M} \]

\[ \frac{b}{M} \]

\[ \text{Rel diff} \]
PM resummation: additional developments (preliminary)

- **High velocities**: large-$\ell$ modes become more important at higher velocities.
  - Possibly related to relativistic beaming of radiation.
  - Effect strongest near periapsis.
  - FD code can get $\ell \geq 15$ modes near periapsis.
  - Developing FD/TD hybrid method.

- **Direct approach**: express $B(\nu)$ as integral over critical orbit, $b = b_c(\nu)$.
  - Only need to calculate SF along critical orbit. More accurate and efficient than fitting.
  - FD approach needs to handle a discrete, distributional piece of the spectrum arising from asymptotic circular orbit.
Prospects

- Analytical results for SF at large $r$: useful for both TD and FD approaches.

- Improved TD methods also under development, including spectral methods with hyperboloidal slicing and compactification.

- Routes to gravity?
  - Direct Lorenz-gauge calculation \cite{Ackay, Warburton, Barack 2013}
  - Radiation-gauge reconstruction \cite{Pound, Merlin, Barack 2013}
  - Lorenz-gauge reconstruction \cite{Dolan, Durkan, Kavanagh, Wardell 2023}

- Second order?
  - Easier than bound? No disparate timescales.
  - Would give conservative dynamics to 6PM.
  - Some way off.