Understanding the NLO Scattering Waveform

Soft Limit and Post-Newtonian Expansion

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December 14, 2023

Herderschee, Roiban, **FT**, arxiv:2303.06112
Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**, in progress

Focus:

NLO scattering waveform from KMOC formalism

- Contribution from virtual five-point amplitude and cut [cf. Giroux's talk]
- Soft limit of the waveform up to the first non-universal term [cf. Russo's talk]
- In progress: full comparison up to 2.5PN with the MPM waveform
  [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**]

KMOC formalism

Kosower, Maybee, O'Connell, 1811.10950
[Jones' talk, Novichkov's talk, Russo's talk]

We can compute observables by dressing amplitudes with the corresponding operators,

$$\begin{aligned}\Delta O &= \langle \psi_{\rm out}|\mathbb{O}|\psi_{\rm out}\rangle - \langle \psi_{\rm in}|\mathbb{O}|\psi_{\rm in}\rangle \\ &= \langle \psi_{\rm in}|\hat{S}^\dagger \mathbb{O}\hat{S}|\psi_{\rm in}\rangle - \langle \psi_{\rm in}|\mathbb{O}|\psi_{\rm in}\rangle \\ &= i\langle \psi_{\rm in}|[\mathbb{O},\hat{T}]|\psi_{\rm in}\rangle + \langle \psi_{\rm in}|\hat{T}^\dagger[\mathbb{O},\hat{T}]|\psi_{\rm in}\rangle\end{aligned}$$

$$\hat{S} = 1 + i\hat{T}$$
$$\hat{T} - \hat{T}^\dagger = i\hat{T}^\dagger\hat{T}$$

For classical two-body scattering problems ($\lambda_{\rm Compton} \ll r_s \ll b$):

$$\begin{aligned}\Delta O = \int & {\rm d}\Phi[p_1]{\rm d}\Phi[p_2]{\rm d}\Phi[p_1']{\rm d}\Phi[p_2']\phi(p_1)\phi(p_2)\phi(p_1')\phi(p_2')e^{i(p_1-p_1')\cdot b_1 + i(p_2-p_2')\cdot b_2} \\ & \times\Big[i\langle p_1'p_2'|[\mathbb{O},\hat{T}]|p_1p_2\rangle + \langle p_1'p_2'|\hat{T}^\dagger[\mathbb{O},\hat{T}]|p_1p_2\rangle\Big] \\ \simeq {\rm FT}&\Big[i\langle p_1-q_1,p_2-q_2|[\mathbb{O},\hat{T}]|p_1p_2\rangle + \langle p_1-q_1,p_2-q_2|\hat{T}^\dagger[\mathbb{O},\hat{T}]|p_1p_2\rangle\Big]\end{aligned}$$

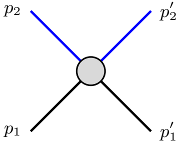
$${\rm FT}[f(q_1,q_2)] \equiv \int \frac{{\rm d}^d q_1}{(2\pi)^{d-2}}\frac{{\rm d}^d q_2}{(2\pi)^{d-2}}\delta(2p_1\cdot q_1)\delta(2p_2\cdot q_2)e^{iq_1\cdot b_1 + iq_2\cdot b_2}f(q_1,q_2)$$

Impulse

Kosower, Maybee, O'Connell, 1811.10950
Through 3PM: Hermann, Parra-Martinez, Ruf, Zeng, 2104.03957

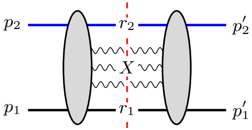
- Operator: $\mathbb{O} = \mathbb{P}_1^\mu \,, \quad \mathbb{P}_1^\mu |p_1\rangle = p_1^\mu |p_1\rangle$
- Virtual amplitude contribution (starting from LO)

$$i\langle p_1' p_2' | [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2 \rangle = i(p_1' - p_1)^\mu$$



- Cut contribution (starting from NLO)

$$\langle p_1' p_2' | \hat{T}^\dagger [\mathbb{P}_1^\mu, \hat{T}] | p_1 p_2 \rangle = \sum_X \int \mathrm{d}\Phi[r_1] \mathrm{d}\Phi[r_2] (r_1 - p_1)^\mu$$



$$\Delta p^\mu_{(0)} = -\frac{m_1 m_2}{\sqrt{-b^2}} \frac{2(2\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \tilde{b}^\mu \qquad \sigma = u_1 \cdot u_2$$

$$\Delta p^\mu_{(1,\perp)} = -\frac{m_1 m_2 (m_1 + m_2)}{(-b^2)} \frac{3\pi}{4} \frac{5\sigma^2 - 1}{\sqrt{\sigma^2 - 1}} \tilde{b}^\mu$$

$$\Delta p^\mu_{(1,\parallel)} = \frac{m_1^2 m_2^2}{-b^2} \frac{2(2\sigma^2 - 1)^2}{(\sigma^2 - 1)^2} \left[\frac{(\sigma u_2 - u_1)^\mu}{m_1} - \frac{(\sigma u_1 - u_2)^\mu}{m_2} \right]$$

Waveform

Cristofoli, Gonzo, Kosower, O'Connell, 2107.10193
[Novichkov's talk]

Operator: linearized Riemann tensor

$$\mathbb{O} = \mathbb{R}_{\mu\nu\rho\sigma}(x) = \frac{\kappa}{2} \sum_{h=\pm} \int \mathrm{d}\Phi[k] \left[k_{[\mu}\varepsilon^*_{\nu],h}\, k_{[\rho}\varepsilon^*_{\sigma],h}\, e^{-ik\cdot x}\, \hat{a}_{hh}(k) + \mathrm{c.c} \right]$$

Since there is no radiation at $t \to -\infty$,

$$\Delta R_{\mu\nu\rho\sigma} = \langle\psi_{\rm in}|\hat{S}^\dagger \mathbb{R}_{\mu\nu\rho\sigma}\hat{S}|\psi_{\rm in}\rangle = i \int \mathrm{d}\Phi[k] \left[\tilde{J}_{\mu\nu\rho\sigma}(k) e^{-ik\cdot x} - \mathrm{c.c}\right]$$

$$\tilde{J}_{\mu\nu\rho\sigma}(k) = \frac{\kappa}{2} \sum_{h=\pm} k_{[\mu}\varepsilon^*_{\nu],h}\, k_{[\rho}\varepsilon^*_{\sigma],h} (-i) \langle\psi_{\rm in}|\hat{S}^\dagger \hat{a}_{hh}(k)\hat{S}|\psi_{\rm in}\rangle$$

Integrating over k gives both retarded and advanced Green's function. We discarded the advanced one due to the boundary condition,

$$\Delta R_{\mu\nu\rho\sigma} = \int \mathrm{d}^4 y J_{\mu\nu\rho\sigma}(y) G_{\rm ret}(x-y)$$

Assuming the spatial current $J(y)$ is localized around $y = 0$,

$$\Delta R_{\mu\nu\rho\sigma}\Big|_{|\boldsymbol{x}|\to\infty} = \frac{1}{4\pi|\boldsymbol{x}|} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{J}_{\mu\nu\rho\sigma}(\omega, \omega\boldsymbol{n}) e^{-i\omega\tau}$$

Newman-Penrose scalar and asymptotic metric

The slowest decaying component of the Riemann tensor [Newman, Penrose, 1962]

$$\Psi_4(x) = -N^\mu M^{*\nu} N^\rho M^{*\rho} R_{\mu\nu\rho\sigma} = \frac{1}{|\boldsymbol{x}|}\Psi_4^\infty + \ldots = \frac{\kappa}{8\pi|\boldsymbol{x}|}(\ddot{h}_+^\infty + i\ddot{h}_\times^\infty)$$

where $h_{+,\times}^\infty$ are the two polarizations of the linearized metric perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\kappa^2(m_1+m_2)}{8\pi|\boldsymbol{x}|}h_{\mu\nu}$$

Asymptotic metric (waveform) in the frequency domain

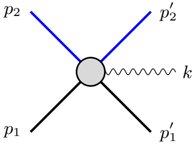
$$\tilde{h}_+^\infty + i\tilde{h}_\times^\infty = -i\Big[\Theta(\omega)\langle\psi_{\rm in}|\hat{S}^\dagger\hat{a}_{--}(\omega,\omega\boldsymbol{n})\hat{S}|\psi_{\rm in}\rangle - \Theta(-\omega)\langle\psi_{\rm in}|\hat{S}^\dagger\hat{a}_{++}^\dagger(|\omega|,|\omega|\boldsymbol{n})\hat{S}|\psi_{\rm in}\rangle\Big]$$

Amplitudes and cuts

$$\langle p_1' p_2' | \hat{S}^\dagger \hat{a}_{hh}(k) \hat{S} | p_1 p_2 \rangle = \langle p_1' p_2' k^h | i\hat{T} | p_1 p_2 \rangle + \langle p_1' p_2' | \hat{T}^\dagger \hat{a}_{hh}(k) \hat{T} | p_1 p_2 \rangle$$

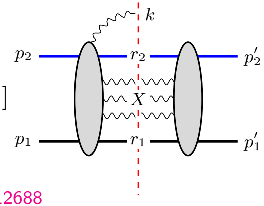
- Virtual amplitude contribution (starting from LO)

$$\langle p_1' p_2' k^h | i\hat{T} | p_1 p_2 \rangle = i$$



- Cut contribution (starting from NLO)

$$\langle p_1' p_2' | \hat{T}^\dagger \hat{a}_{hh}(k) \hat{T} | p_1 p_2 \rangle = \sum_X \int \mathrm{d}\Phi[r_1] \mathrm{d}\Phi[r_2]$$



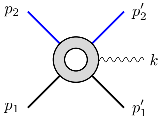
LO waveform $\mathcal{W}^{(0)}$: [spinning case: Novichkov's talk]
From GR: Kovacs, Thorne, Astrophys. J. 224 62, 1978
From world line QFT: Jakobsen, Mogull, Plefka, Steinhoff, 2101.12688
From KMOC: Cristofoli, Gonzo, Kosower, O'Connell, 2107.10193

One-loop matrix elements

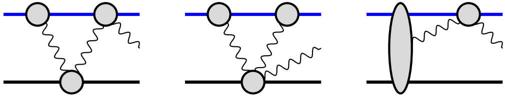
Herderschee, Roiban, **FT**, 2303.06112

$$\langle p_1' p_2' | \hat{S}^\dagger \hat{a}_{hh}(k) \hat{S} | p_1 p_2 \rangle = \langle p_1' p_2' k^h | i\hat{T} | p_1 p_2 \rangle + \langle p_1' p_2' | \hat{T}^\dagger \hat{a}_{hh}(k) \hat{T} | p_1 p_2 \rangle$$

$$\langle p_1' p_2' k^h | i\hat{T} | p_1 p_2 \rangle \Big|_{\text{1-loop}} = i$$



- We use generalized unitarity to construct the GR integrand



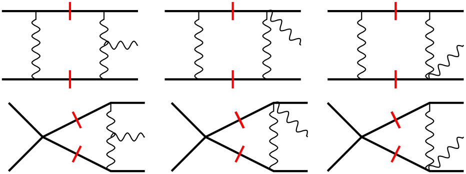
- We then expand the integrand in the classical limit:
  - Matter propagator: $\dfrac{1}{(p+\ell)^2 - m^2 + i0} = \dfrac{1}{2p\cdot\ell + \ell^2 + i0} = \dfrac{1}{2p\cdot\ell + i0}\sum_{n=0}^{\infty}\left(-\dfrac{\ell^2}{2p\cdot\ell + i0}\right)^n$
  - One matter line per loop
  - Matter lines do not touch before IBP

Implementing method of region
Beneke, Smirnov, hep-ph/9711391

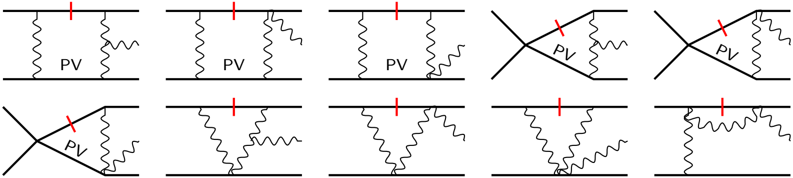
Master integrals: amplitude

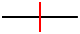
Herderschee, Roiban, **FT**, 2303.06112

Super-classical: $$\frac{m_1^3 m_2^3}{\hbar^2}\sum_i \alpha_i \mathcal{I}_i^{\text{cut,cut}}$$



Classical: $$\frac{1}{\hbar}\sum_i (m_1^3 m_2^2 \beta_i \mathcal{I}_i^{\text{cut,pv}} + m_1^2 m_2^3 \gamma_i \mathcal{I}_i^{\text{pv,cut}}) + \frac{1}{\hbar}(\text{triangles and bubbles})$$

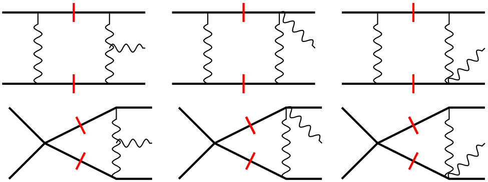


 $= \hat{\delta}(2u_2 \cdot \ell)$ 

$= \frac{1}{2u_1 \cdot \ell + i0} + \frac{1}{2u_1 \cdot \ell - i0}$

Master integrals: cut

$$\langle p_1' p_2' | \hat{S}^\dagger \hat{a}_{hh}(k) \hat{S} | p_1 p_2 \rangle = \langle p_1' p_2' k^h | i\hat{T} | p_1 p_2 \rangle + \langle p_1' p_2' | \hat{T}^\dagger \hat{a}_{hh}(k) \hat{T} | p_1 p_2 \rangle$$



$$\sum_i \left(-\frac{1}{\hbar^2} m_1^3 m_2^3 \alpha_i + \frac{1}{\hbar} m_1^3 m_2^2 \beta_i - \frac{1}{\hbar} m_1^2 m_2^3 \gamma_i \right) \mathcal{I}_i^{\text{cut,cut}}$$

[Giroux's talk]

- They exactly cancel the super-classical terms in the virtual amplitude
- They convert the PV matter propagators to retarded

Caron-Huot, Giroux, Hannesdottir, Mizera, 2308.02125

$$\text{PV}\frac{1}{2u_1 \cdot \ell} \longrightarrow \frac{1}{2u_1 \cdot \ell - i0} \qquad \text{PV}\frac{1}{-2u_2 \cdot \ell} \longrightarrow \frac{1}{-2u_2 \cdot \ell - i0}$$

- Loop integrals only give imaginary IR divergences

Amplitude contribution

Herderschee, Roiban, **FT**, 2303.06112
GR result confirmed by: Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini, 2303.06111
Georgoudis, Heissenberg, Vazquez-Holm, 2303.07006

$$
\begin{aligned}
\mathcal{W}^{(1)}_{\text{amp}} = &-\frac{i\,\kappa^5}{32\pi}(k\cdot p_1 + k\cdot p_2)\left[\frac{1}{\epsilon} - \log\frac{(k\cdot u_1)(k\cdot u_2)}{\mu^2}\right] M^{\text{cl.}}_{5,\text{tree}} \\
&+ \kappa^5\left[A^R_{\text{rat}} + \frac{A^R_1}{\sqrt{(k\cdot u_2)^2 - q_1^2}} + \frac{A^R_2}{\sqrt{(k\cdot u_1)^2 - q_2^2}} + \frac{A^R_3}{\sqrt{-q_1^2}} + \frac{A^R_4}{\sqrt{-q_2^2}}\right] \\
&+ i\,\kappa^5\left[A^I_{\text{rat}} + A^I_1\frac{\operatorname{arcsinh}\frac{k\cdot u_2}{\sqrt{-q_1^2}}}{\sqrt{(k\cdot u_2)^2 - q_1^2}} + A^I_2\frac{\operatorname{arcsinh}\frac{k\cdot u_1}{\sqrt{-q_2^2}}}{\sqrt{(k\cdot u_1)^2 - q_2^2}} + A^I_3\log\frac{q_2^2}{q_1^2}\right. \\
&\qquad\left. + A^I_4\log\frac{k\cdot u_1}{k\cdot u_2} + A^I_5\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{3/2}}\right]
\end{aligned}
$$

The coefficients A_i contain complicated spurious poles originated from IBP

Cut contribution

$$\mathcal{W}_{\rm cut}^{(1)} = \frac{i\,\kappa^5}{32\pi}\frac{\sigma(\sigma^2-3/2)}{(\sigma^2-1)^{3/2}}(k\cdot p_1 + k\cdot p_2)\left[\frac{1}{\epsilon} - \log\frac{(k\cdot u_1)(k\cdot u_2)}{(\sigma^2-1)\mu^2}\right] M_{5,\rm tree}^{\rm cl.}$$
$$- \frac{i\,\kappa^5}{256\pi}\frac{(2\sigma^2-1)^2}{(\sigma^2-1)^{3/2}}(k\cdot p_1 + k\cdot p_2)\left[\frac{1}{\epsilon} - \log\frac{(k\cdot u_1)(k\cdot u_2)}{(\sigma^2-1)\mu^2}\right] A_{\rm UV}^{\rm cut}$$
$$- i\,\kappa^5\left[A_{\rm rat}^{\rm cut} + A_1^{\rm cut}\log\frac{k\cdot u_1}{k\cdot u_2} + A_2^{\rm cut}\log\frac{(k\cdot u_1)(k\cdot u_2)}{-q_1^2}\right.$$
$$\left. + A_3^{\rm cut}\log\frac{(k\cdot u_1)(k\cdot u_2)}{-q_2^2} + A_4^{\rm cut}\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{3/2}}\right]$$

The UV divergence is local and contributes only a contact interaction, which can be ignored for large impact parameters,

$$A_{\rm UV}^{\rm cut} = \frac{m_1^2 m_2^2 (u_1\cdot f\cdot u_2)^2[(k\cdot u_1)^2 + \sigma(k\cdot u_1)(k\cdot u_2) + (k\cdot u_2)^2]}{(k\cdot u_1)^3(k\cdot u_2)^3}$$

Momentum space and frequency domain waveform:

$$(-i)\langle p_1 - q_1, p_2 - q_2|\hat{S}^\dagger \hat{a}_{hh}(k)\hat{S}|p_1 p_2\rangle_{1\ \rm loop}^{\rm cl.} = \mathcal{W}_{\rm amp}^{(1)} - \mathcal{W}_{\rm cut}^{(1)} \equiv \mathcal{W}_{\rm KMOC}^{(1)}$$

The fate of IR divergences

Herderschee, Roiban, **FT**, 2303.06112

Imaginary IR divergence leads to a divergent phase [Giroux's talk]

Weinberg, 1965

peculiar to in-in correlator
Caron-Huot, etc, 2308.02125

$$\frac{\mathcal{W}^{(1)}_{\rm KMOC}(p_1 p_2 \to p_1' p_2' k)}{\mathcal{W}^{(0)}(p_1 p_2 \to p_1' p_2' k)} = \exp\left[-iG(p_1\cdot k + p_2 \cdot k)\left(\frac{1}{\epsilon} - \log\frac{\Lambda_{\rm IR}^2}{\mu^2}\right)\left(1 + \frac{\sigma(\sigma^2-3/2)}{(\sigma^2-1)^{3/2}}\right)\right]$$

It can be absorbed in the definition of the retarded time

Goldberger, Ross, 0912.4254
Porto, Ross, Rothstein, 1203.2962

$$\Delta R_{\mu\nu\rho\sigma}\Big|_{|\boldsymbol{x}|\to\infty} = \frac{1}{|\boldsymbol{x}|}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi} f_{\mu\nu\rho\sigma}(\omega,\boldsymbol{n})e^{-i\omega\tau}$$

$$\tau \to \tau - G(p_1\cdot k + p_2\cdot k)\left(\frac{1}{\epsilon} - \log\frac{\Lambda_{\rm IR}^2}{\mu^2}\right)\left(1 + \frac{\sigma(\sigma^2-3/2)}{(\sigma^2-1)^{3/2}}\right)$$

The time shift replaces the arbitrary scale μ by the physical IR cutoff $\Lambda_{\rm IR}$

How to justify our result?

- Comparison with GR calculation need to be carried out in the position space
- Fourier transform to the position space can only be done numerically for generic kinematic setup
- Analytic computation under better control in the soft graviton limit and/or small relative velocity (PN expansion)

Soft graviton limit

We consider the soft expansion of the position space waveform:

$$\mathcal{W}(\omega, n) \sim \frac{\mathcal{A}}{\omega} + \mathcal{B}\,\log\omega + \mathcal{C}\,\omega(\log\omega)^2 + \mathcal{D}\,\omega\log\omega + \ldots$$

- The coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are universal
- The $\mathcal{D}$ coefficient is the first non-universal case in the expansion

Saha, Sahoo, Sen, 1912.06413
Sahoo, Sen, 2105.08739

LO soft and memory

$$\mathcal{X} = \frac{m_1 m_2 (u_1 \cdot f \cdot u_2)}{(u_1 \cdot n)(u_2 \cdot n)} \left[\frac{u_2 \cdot f \cdot \hat{b}}{u_2 \cdot n} + \frac{u_1 \cdot f \cdot \hat{b}}{u_1 \cdot n} \right]$$

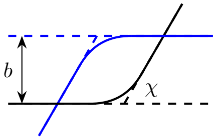
$$\hat{b}^\mu = b^\mu / \sqrt{-b^2} \qquad n^\mu = k^\mu / \omega \qquad f^{\mu\nu} = n^\mu \varepsilon^\nu - n^\nu \varepsilon^\mu$$

Tree-level:

$$\mathcal{A}^{(0)} = \frac{i}{16\pi\sqrt{-b^2}} \frac{(2\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \mathcal{X}$$

One-loop:

$$\begin{aligned}
\mathcal{A}^{(1)} &= \mathcal{A}^{(1)}_{\text{amp}} + \mathcal{A}^{(1)}_{\text{cut}} \\
&= \frac{3i\,(m_1 + m_2)}{4096\pi(-b^2)} \frac{(5\sigma^2 - 1)}{\sqrt{\sigma^2 - 1}} \mathcal{X} + \frac{i\, m_1 m_2}{256\pi^2(-b^2)} \frac{(2\sigma^2 - 1)^2}{(\sigma^2 - 1)} \left[\frac{m_2 (u_1 \cdot f \cdot \hat{b})^2}{(u_1 \cdot n)^3} + \frac{m_1 (u_2 \cdot f \cdot \hat{b})^2}{(u_2 \cdot n)^3} \right. \\
&\quad \left. - \frac{(u_1 \cdot f \cdot u_2)^2}{2(\sigma^2 - 1)(u_1 \cdot n)(u_2 \cdot n)} \left(\frac{\sigma m_1 + m_2}{u_2 \cdot n} + \frac{\sigma m_2 + m_1}{u_1 \cdot n} \right) \right]
\end{aligned}$$



LO soft and memory

The LO soft agrees exactly with [Sahoo, Sen, 2105.08739]

$$\mathcal{A} = i\left[\frac{(\varepsilon\cdot p_1)^2}{p_1\cdot n} + \frac{(\varepsilon\cdot p_2)^2}{p_2\cdot n} - \frac{(\varepsilon\cdot p_3)^2}{p_3\cdot n} - \frac{(\varepsilon\cdot p_4)^2}{p_4\cdot n}\right]$$

$$p_4 = p_1 + \Delta p \qquad p_3 = p_2 - \Delta p \qquad \Delta p = G\Delta p_{(0)} + G^2(\Delta p_{(1,\perp)} + \Delta p_{(1,\parallel)})$$

The LO and NLO impulse are given by [Hermann, Parra-Martinez, Ruf, Zeng, 2104.03957]

$$\Delta p^\mu_{(0)} = -\frac{m_1 m_2}{\sqrt{-b^2}}\frac{2(2\sigma^2-1)}{\sqrt{\sigma^2-1}}\hat{b}^\mu$$

$$\Delta p^\mu_{(1,\perp)} = -\frac{m_1 m_2(m_1+m_2)}{-b^2}\frac{3\pi}{4}\frac{5\sigma^2-1}{\sqrt{\sigma^2-1}}\hat{b}^\mu$$

$$\Delta p^\mu_{(1,\parallel)} = \frac{m_1^2 m_2^2}{-b^2}\frac{2(2\sigma^2-1)^2}{(\sigma^2-1)^2}\left[\frac{(\sigma u_2 - u_1)^\mu}{m_1} - \frac{(\sigma u_1 - u_2)^\mu}{m_2}\right]$$

More specifically,

- $\mathcal{A}^{(0)}$ and $\mathcal{A}^{(1)}_{\mathrm{amp}}$ are contributed respectively by $\Delta p_{(0)}$ and $\Delta p_{(1,\perp)}$
- $\mathcal{A}^{(1)}_{\mathrm{cut}}$ is contributed by $\Delta p_{1,\parallel}$ and the iteration of $\Delta p_{(0)}$

Sub-leading and sub-sub-leading soft

$\mathcal{B}$ and $\mathcal{C}$ coefficients also agree with [Sahoo, Sen, 2105.08739]

$$\mathcal{B}^{(0)} = -\frac{\Gamma\,\mathcal{Y}}{16\pi}$$

$$\mathcal{B}^{(1)} = \mathcal{B}^{(1)}_{\text{amp}} + \mathcal{B}^{(1)}_{\text{cut}} = -\frac{(2+\Gamma)(2\sigma^2-1)\mathcal{M}\,\mathcal{X}}{512\pi^2\sqrt{\sigma^2-1}\sqrt{-b^2}}$$

$$\mathcal{C}^{(1)} = \mathcal{C}^{(1)}_{\text{amp}} = \frac{i\mathcal{M}}{16\pi}\mathcal{B}^{(0)}$$

The cut contribution has physical consequences on $\mathcal{W}_{\text{KMOC}}$

[Giroux's talk]

$$\mathcal{X} = \frac{m_1 m_2 (u_1\cdot f\cdot u_2)}{(u_1\cdot n)(u_2\cdot n)}\left[\frac{u_2\cdot f\cdot\hat{b}}{u_2\cdot n} + \frac{u_1\cdot f\cdot\hat{b}}{u_1\cdot n}\right] \qquad \mathcal{Y} = \frac{m_1 m_2 (u_1\cdot f\cdot u_2)^2}{(u_1\cdot n)(u_2\cdot n)}$$

$$\mathcal{M} = m_1(u_1\cdot n) + m_2(u_2\cdot n) \qquad \Gamma = \frac{3\sigma - 2\sigma^3}{(\sigma^2-1)^{3/2}}$$

The non-universal term

There are two regions that contributes to $\mathcal{D}$ in the Fourier transform

$$\begin{aligned} &\text{region 1:} \qquad q_i \sim q_i \quad k \sim \omega k \\ &\text{region 2:} \qquad q_i \sim \omega q_i \quad k \sim \omega k \end{aligned}$$

while only region 1 contributes to $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$

Examples of the two-region Fourier transforms:

$$\mathrm{FT}\left[\frac{1}{q_1^2 q_2^2}\right] = \mathrm{FT}\left[\frac{1}{q_1^2 (k-q_1)^2}\right] = \frac{1}{\omega^2}\frac{\operatorname{arccosh}(\sigma)}{16\pi m_1 m_2 (u_1\cdot n)(u_2\cdot n)} + \dots$$

$$\mathrm{FT}\left[\frac{\log(-q_1^2)}{q_1^2 q_2^2}\right] = \frac{\log\omega}{\omega^2}\frac{\operatorname{arccosh}(\sigma)}{8\pi m_1 m_2 (u_1\cdot n)(u_2\cdot n)} + \dots$$

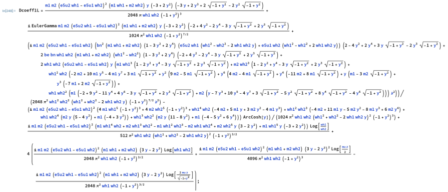
The non-universal term

$$\mathcal{D}^{(0)} = \frac{i\,\Gamma m_1 m_2}{32\pi}(u_1\cdot f\cdot u_2)\left[\frac{u_1\cdot f\cdot\hat{b}}{u_1\cdot n}+\frac{u_2\cdot f\cdot\hat{b}}{u_2\cdot n}\right]$$

[Ghosh, Sahoo, 2106.10741]

$$\mathcal{D}^{(1)} = \frac{(2+\Gamma)}{64\pi}\pi\mathcal{M}\,\mathcal{B}^{(0)} + \frac{i\mathcal{M}\,\mathcal{B}^{(0)}}{8\pi}\gamma_{\rm E} + \frac{i\mathcal{M}\,\mathcal{B}^{(0)}}{16\pi}\log(u_1\cdot n)(u_2\cdot n) + \frac{i\left(\mathcal{Z}_1+\mathcal{Z}_2\,\mathrm{arccosh}(\sigma)+\mathcal{Z}_3\log\frac{u_1\cdot n}{u_2\cdot n}\right)}{(u_1\cdot n)^2+(u_2\cdot n)^2-2\sigma(u_1\cdot n)(u_2\cdot n)}$$

- The spurious pole in $\mathcal{D}^{(1)}$ indeed cancels
- Time shift $\delta\tau \propto G\mathcal{M}$ leads to $\delta\mathcal{D}^{(1)} \propto G\mathcal{M}\,\mathcal{B}^{(0)}$



MPM waveform [Bini, Damour, Geralico, 2309.14925]

(Multipolar-Post-Minkowskian)

How to compare $\mathcal{W}_{\text{KMOC}}$ with the GR result $\mathcal{W}_{\text{MPM}}$

- $\mathcal{W}_{\text{MPM}}$ is given in terms of kinematic data in the eikonal COM frame, which is related to the incoming COM frame through a $(\chi/2)$-rotation

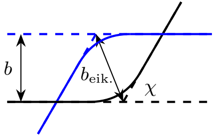
$$\frac{\chi_{\text{LO}}}{2} = \frac{M\sqrt{1+2\nu(\sigma-1)}}{32\pi\sqrt{-b^2}}\frac{2\sigma^2-1}{\sigma^2-1}$$

$M = m_1 + m_2$

$\nu = m_1 m_2 / M^2$

- The difference between $|v|$, $|b|$ and $|v_{\text{eik}}|$, $|b_{\text{eik}}|$ is $\mathcal{O}(G^2)$ and thus irrelevant at one-loop
- The KMOC waveform in this new frame is given by

$$\mathcal{W}^{(1)}_{\text{KMOC eik.}}(\theta,\phi) = \mathcal{W}^{(1)}_{\text{KMOC}}(\theta,\phi) + \frac{\chi_{\text{LO}}}{2}\frac{\partial}{\partial\phi}\mathcal{W}^{(0)}(\theta,\phi)$$



[Russo's talk]

MPM waveform [Bini, Damour, Geralico, 2309.14925]

With only partial result ($\mathcal{W}^{(1)}_{\rm amp}$) available, [BDG] observed that if we redefine $\mathcal{W}^{(1)}_{\rm amp}$ as a function in the eikonal COM frame, better agreement is found with $\mathcal{W}_{\rm MPM}$:

- The two waveform agree at Newtonian and 1PN order
- They disagree at 2.5PN

Remark:

- The comparison is carried out in the scattering plane ($\theta = \pi/2$) and for the terms even under $\phi \to \phi + \pi$
- The match requires a finite time shift starting at $\delta t \sim \frac{G\mathcal{M}}{(\sigma^2-1)^{1/2}}$.
- The comparison (a) between 1PN and 2.5PN (b) for parity odd contribution was not carried out due to technical difficulties in the Fourier transform of the KMOC waveform (current in progress)

Current status Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**

In the soft regime, the rotation to the eikonal COM frame completely removes the cut contribution to $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A}^{(1)}_{\text{amp}}(\theta,\phi) = \mathcal{A}^{(1)}(\theta,\phi) + \frac{\chi_{\text{LO}}}{2}\frac{\partial}{\partial\phi}\mathcal{A}^{(0)}(\theta,\phi)$$

$$\mathcal{B}^{(1)}_{\text{amp}}(\theta,\phi) = \mathcal{B}^{(1)}(\theta,\phi) + \frac{\chi_{\text{LO}}}{2}\frac{\partial}{\partial\phi}\mathcal{B}^{(0)}(\theta,\phi)$$

The cut contribution to $\mathcal{D}$ can be removed by a further time shift:

$$\mathcal{D}^{(1)}_{\text{amp}}(\theta,\phi) = \mathcal{D}^{(1)}(\theta,\phi) + \frac{\chi_{\text{LO}}}{2}\frac{\partial}{\partial\phi}\mathcal{D}^{(0)}(\theta,\phi) + \frac{i\mathcal{M}\,\mathcal{B}^{(0)}(\theta,\phi)}{16\pi\sqrt{\sigma^2-1}}\left[\frac{(2\sigma^2-1)\nu}{2+4\nu(\sigma-1)} - \sigma\right]$$

Does it suggest that the frame rotation and time shift may remove the cut contribution?

$$\mathcal{W}^{(1)}_{\text{amp or MPM}}(\theta,\phi) \stackrel{?}{=} \mathcal{W}^{(1)}_{\text{KMOC}}(\theta,\phi) + \frac{\chi_{\text{LO}}}{2}\frac{\partial}{\partial\phi}\mathcal{W}^{(0)}(\theta,\phi) + \frac{i\mathcal{M}\delta\tau}{32\pi}\mathcal{W}^{(0)}(\theta,\phi)$$

Currently we are checking this relation at the 1PN order. Preliminary results suggest that additional terms might be necessary for this relation to hold

Current status

Including the cut contribution does resolve the disagreement at 2.5PN in $\mathcal{D}^{(1)}$ of [BDG]

$$\begin{aligned}
\mathcal{D}^{G^2(\eta^3+\eta^5)} &= \frac{\eta^3 G^2 M^2 \nu}{p_\infty}\frac{1}{12}[(24\gamma_E i + 6\pi - 35i)\cos(2\phi) + (24\gamma_E i + 6\pi - 11i)] \\
&+ \frac{\eta^5 G^2 M^2 p_\infty \nu}{40}\left[\frac{1}{3}(360 i\nu\gamma_E - 480 i\gamma_E + 90\pi\nu - 873 i\nu - 120\pi + 398 i)\cos(2\phi)\right. \\
&+ \frac{1}{2}(120 i\nu\gamma_E - 40\gamma_E i + 30\pi\nu - 421 i\nu - 10\pi + 136 i)\cos(4\phi) \\
&+ \left.\frac{1}{6}(360 i\nu\gamma_E - 840\gamma_E i + 90\pi\nu - 243 i\nu - 210\pi + 1348 i)\right].
\end{aligned}$$

The difference

$$\delta\mathcal{D} = -\frac{iM^3\nu(\cos\phi)^2(4 - 3\nu + \nu\cos 2\phi)}{4096\pi^2}$$

can be removed by a time shift and a BMS supertranslation
[see Georgoudis, Heissenberg, Russo, 2312.07452; Russo's talk]

Outlook Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, **FT**

Goal: Full comparison of the KMOC and MPM waveform up to 2.5PN in the eikonal COM frame

- Analytic PN expansion of $\mathcal{W}_{\mathrm{KMOC}}$ requires a significant simplification in its spurious pole structures: rewrite $\mathcal{W}_{\mathrm{KMOC}}$ in terms of spurious-pole-free log combinations
- Will time shift and BMS transform resolve the difference between the KMOC and MPM waveforms? Is there something else we have not considered yet?

Future:

- Full analytic NLO waveform $\rightarrow$ momentum and angular momentum loss at 3-loop
- NNLO waveform and frame dependency?
- Bound state waveform and possible B2B map?

in the spirit of Kälin, Porto, 1910.03008, 1911.09130
Cho, Kälin, Porto, 2112.03976

Thanks for listening!