Multipole Moments, Time-derivative Operators, and the Kerr Compton Amplitude

QCD Meets Gravity 2023

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Based on:

2311.18421 J. Vines, TS

2308.14176 Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, F. Teng, J. Vines, TS

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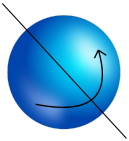
Outline

- How many #'s necessary in GR/EM at each order in spin?
- **Spin magnitude change in QFT & WL** – Discuss different number of #'s in QFT used by Z. Bern, A. Luna, R. Roiban, C. Shen, M. Zeng 2005.03071 and the worldline, describe a way to add dofs and consequently more #'s to allow spin magnitude change.



- **Generic spinning body Comptons in GR & EM** – All-orders in spin unparameterized amplitude, parameterized expression in terms of all possible #'s through $\mathcal{O}(S^5)/\mathcal{O}(S^3)$ in GR & EM.
- **Orthogonality relations for generic spinning body DMF's from Dixon's multipoles** – Consistency constraints on the WL action necessary to match the MPD equations.

- Mathisson–Papapetrou–Dixon (**MPD**) gives exact WL dynamics (but not closed) W. G. Dixon (1974)
- **Minimal** Poincare dofs $\implies$ **closed** minimal MPD eoms.
- Conservative minimal MPD has various WL descriptions (such as DMF). Various high-spin QFTs give equivalent observables. Alessio, Arkani-Hamed, Aoude, Bautista, Ben-Shahar, Bern, Bini, Buonanno, Cachazo, Cangemi, Chiodaroli, Chen, Cho, Chung, Cordero, Damour, Damgaard, Ehlers, Gatica, Goldberger, Guevara, Haddad, Helset, Holstein, Huang, Huang, Jakobsen, Johansson, Kim, Kraus, Kosmopolous, Lee, Levi, Lin, Liu, Luna, Maybee, Menezes, Mogull, O'Connell, Ochirov, Pichini, Plefka, Porto, Riva, Roiban, Ross, Rothstein, Rudolph, Ruf, Saketh, Sauer, Sergola, Shen, Siemonsen, Skvortsov, Steinhoff, Teng, Vaydia, di Vecchia, Vernizzi, Vines, Wong, Xu, Yang, Yin, Zeng,...
- Different number of Wilson coefficients in the high-spin QFT used by Z. Bern, A. Luna, R. Roiban, C. Shen, M. Zeng `2005.03071` and worldline

- Conservative & minimal worldline theory interaction with GR and EM is described by an action of the form:

$$\mathcal{S}[\alpha,\beta,z,p,\Lambda,S]=\int_{-\infty}^{\infty}\left((p_\mu+qA_\mu)\dot{z}^\mu+\frac{1}{2}\Omega^{\mu\nu}S_{\mu\nu}-\frac{\alpha}{2}(p^2+\mathcal{M}^2(z,p,S))+\beta_\mu S^{\mu\nu}p_\nu\right)d\lambda.$$

- The spin tensor and vector are related using the SSC:

$$S^2=S^\mu S_\mu=\frac{1}{2}S^{\mu\nu}S_{\mu\nu}\qquad S^\mu=-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu S_{\rho\sigma}$$
$$S^{\mu\nu}p_\nu=0\implies S_{\mu\nu}=\epsilon_{\mu\nu\rho\sigma}u^\rho S^\sigma$$

- Automatically conserves spin magnitude (it is a Casimir operator):

$$\frac{d}{d\lambda}(S^2)=0.$$

- To have spin magnitude change, need a dof which does not Poisson commute with S^2.
- Drop the β term from the action gives new extended theory with more #'s and more dof.
- Special values of new #'s $\implies$ reduce the extended theory to minimal theory at the level of eoms.
- In the extended theory, $S^{\mu\nu}S_{\mu\nu}$ is conserved, but $S^2 = S^\mu S_\mu$ is not conserved. (One model of spin magnitude change.)
- Extra dofs captured by K^μ:

$$K^\mu = S^{\mu\nu}u_\nu \implies S_{\mu\nu} = u_\mu K_\nu - u_\nu K_\mu + \epsilon_{\mu\nu\rho\sigma}u^\rho S^\sigma$$

- Still consistent with the full MPD equations, simply contains nonminimal number of dofs.
- K^μ can just be thought of as extra dofs, does not have to be packaged with S^μ.

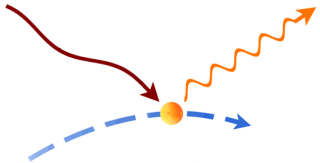
- K-extended WL in EM has same number #'s as high-spin QFT of Z. Bern, A. Luna, R. Roiban, C. Shen, M. Zeng `2005.03071`
- In that QFT, spin s fields reducible (allow for spin magnitude change).
- Irreduciblizing projectors in propagators $\implies$ QFT observables = minimal WL observables
- No projectors $\implies$ QFT observables = K-extended WL observables
- J.P. Gatica `2312.04680` showed that QFT observables (impulse and spin kick) without projectors can be computed using KMOC and described by an eikonal formula, also confirmed matching between QFT and extended WL.
- Extended formalism gives way to add dofs with spin magnitude change in EM, studying match in GR.

- Minimal WL action gives eom:

$$\dot{z}^\mu = u^\mu + \dots \qquad \dot{p}_\mu = q\mathcal{F}_{\mu\nu}\dot{z}^\nu - \frac{\partial \mathcal{M}}{\partial z^\mu} \qquad \dot{S}^\mu = u^\mu \dot{u}\cdot S + \epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho \frac{\partial \mathcal{M}}{\partial S^\sigma}$$

- Using minimal WL eoms we compute the response of the body to an incoming plane wave:

$$A_\mu = \epsilon \mathcal{E}_{1\mu} e^{ik_1\cdot x} + qA_\mu^{\text{stat}} + q^2 \epsilon \delta A_\mu + \mathcal{O}(q^3)$$
$$\mathcal{E}_2^{*\mu}\delta A_\mu = \frac{e^{i\omega(r-t)}}{4\pi r}\frac{\mathcal{A}}{2m} + \mathcal{O}\left(\frac{1}{r^2}\right).$$



- The amplitude only depends on DMF through the combinations:

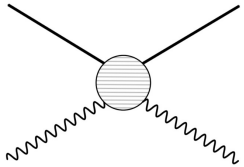
$$\mathcal{N}(u,S,k,\mathcal{E}) = \sum_{n=0}^{\infty} \frac{i^n}{2m}\frac{\partial \delta \mathcal{M}_1^2}{\partial \partial^n_{\rho_1\dots\rho_n}\mathcal{F}_{\mu\nu}} k_{\rho_1}\dots k_{\rho_n} f_{\mu\nu}$$

$$\mathcal{P}(u,S,k,\mathcal{E},k',\mathcal{E}') = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{i^{l-n}}{2m}\frac{\partial \delta \mathcal{M}_2^2}{\partial \partial^n_{\rho_1\dots\rho_n}\mathcal{F}_{\mu\nu}\partial \partial^l_{\kappa_1\dots\kappa_l}\mathcal{F}_{\tau\omega}} k'_{\rho_1}\dots k'_{\rho_n} f'^*_{\mu\nu} k_{\kappa_1}\dots k_{\kappa_l} f_{\tau\omega}.$$

- WL eoms give compact manifestly gauge-invariant **all-orders in spin** Compton for general bodies:

$$\mathcal{A} = 2m\left(f^*_{2\mu\nu}\delta\widetilde{z}^\mu v^\nu + ik_2\cdot\delta\widetilde{z}\mathcal{N}_2^* - \delta\widetilde{u}^\sigma\frac{\partial\mathcal{N}_2^*}{\partial v^\sigma} - \delta\widetilde{S}^\sigma\frac{\partial\mathcal{N}_2^*}{\partial s^\sigma} + \frac{\mathcal{N}_1\mathcal{N}_2^*}{m} - \mathcal{P}_{12}\right)$$

$$\begin{aligned}
\delta\widetilde{u}^\mu =& \frac{-if_1^{\mu\nu}v_\nu}{mk_1\cdot v} - \frac{\mathcal{N}_1}{m}\left(v^\mu + \frac{k_1^\mu}{k_1\cdot v}\right)\\
\delta\widetilde{S}^\mu =& v^\mu\delta\widetilde{u}\cdot s - \frac{i}{k_1\cdot v}\epsilon^{\mu\nu\rho\sigma}v_\nu s_\rho\frac{\partial\mathcal{N}_1}{\partial s^\sigma}\\
\delta\widetilde{z}^\mu =& \frac{-i}{k_1\cdot v}\left(\delta\widetilde{u}^\mu + \frac{\delta^\mu_\alpha + v^\mu v_\alpha}{m^2}\star f_1^{\alpha\beta}s_\beta + \frac{\eta^{\mu\nu}+v^\mu v^\nu}{m}\frac{\partial\mathcal{N}_1}{\partial v^\nu}\right.\\
&\left. + \frac{s^\mu}{m}v^\nu\frac{\partial\mathcal{N}_1}{\partial s^\nu} + \frac{i}{m^2}\mathcal{N}_1\epsilon^{\mu\nu\rho\sigma}v_\nu s_\rho k_{1\sigma}\right).
\end{aligned}$$



- Most general DMF for Compton through $\mathcal{O}(S^3)$ is:

$$\delta\mathcal{M}_1^2 = 2C_1 {}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu + \frac{C_2}{m}(S\cdot\partial)\mathcal{F}_{\mu\nu}S^\mu u^\nu - \frac{C_3}{3m^2}(S\cdot\partial)^2{}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu + \frac{E_3}{3m^2}S^2(u\cdot\partial)^2{}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu$$

$$\begin{aligned}\delta\mathcal{M}_2^2 = {} & \frac{D_{2a}}{m^2}(\mathcal{F}_{\mu\nu}S^\mu u^\nu)^2 + \frac{D_{2b}}{m^2}({}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu)^2 + \frac{D_{2c}}{m^2}S^2\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{D_{2d}}{m^2}S^2u^\mu\mathcal{F}_{\mu\nu}\mathcal{F}^{\nu\rho}u_\rho \longrightarrow 4\ D\text{'s at } \mathcal{O}(S^2)\\ & + \frac{D_{3a}}{m^3}{}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu(S\cdot\partial)\mathcal{F}_{\rho\sigma}S^\rho u^\sigma + \frac{D_{3b}}{m^3}S^2{}^\star\mathcal{F}^{\mu\nu}u_\mu\partial_\nu\mathcal{F}_{\rho\sigma}S^\rho u^\sigma \longrightarrow 6\ D\text{'s at } \mathcal{O}(S^3)\\ & + \frac{D_{3c}}{m^3}S^2{}^\star\mathcal{F}^{\mu\nu}u_\mu(S\cdot\partial)\mathcal{F}_{\nu\rho}u^\rho + \frac{D_{3d}}{m^3}S^2{}^\star\mathcal{F}^{\mu\nu}S_\mu(u\cdot\partial)\mathcal{F}_{\nu\rho}u^\rho\\ & + \frac{D_{3e}}{m^3}{}^\star\mathcal{F}^{\mu\nu}S_\mu(S\cdot\partial)\mathcal{F}_{\nu\rho}S^\rho + \frac{D_{3f}}{m^3}S^2{}^\star\mathcal{F}^{\mu\nu}(S\cdot\partial)\mathcal{F}_{\mu\nu}\end{aligned}$$

- Formula on previous slide gives full amplitude directly
- Amplitude is independent of 1 linear combination $\implies$ E_3 is removable by field-definition.

$$Z = -6E_3 + D_{3f} + 2C_1D_{3b} - 2(1+C_1)D_{3c} + 4D_{3d}$$

Comptons - $\sqrt{\text{Kerr}}$ values through S^3

- C coefficients are fixed by stationary $\sqrt{\text{Kerr}}$ / Newman-Janis shift / 3pt amplitude. Levi, Steinhoff
- Through $\mathcal{O}(S^2)$ for Compton, Guevara-Ochirov-Vines (GOV) spin-exponentiation / shift-symmetry of R. Aoude, K. Haddad, A. Helset `2203.06197` and Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, F. Teng `2203.06202` are consistent with each other and determine all contact terms.
- Beginning at $\mathcal{O}(S^3)$, GOV can only be maintained for one helicity combination. Maintaining GOV + shift-symmetry leaves 1 free coefficient in the Compton.

$$\mathcal{A}_{++} = \mathcal{A}^{(0)}_{++} \exp\left(a \cdot (k_1 - k_2)\right) + \mathcal{O}(S^4)$$

$$\mathcal{A}_{+-} = \mathcal{A}^{(0)}_{+-} \exp(a \cdot (k_1 + k_2)) \sum_{n=0}^{3} \frac{\bar{J}_n}{n!} + \mathcal{O}(S^4)$$

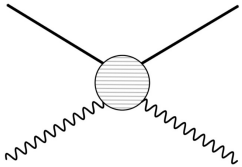
$$\bar{J}_1 = -2a \cdot w, \quad \bar{J}_2 = (2a \cdot w)^2, \quad \bar{J}_3 = -(3D_{3e} + 1)(a \cdot w)^2 a \cdot (k_1 + k_2).$$

- Exponential requires D's vanish at $\mathcal{O}(S^2)$ but not $\mathcal{O}(S^3)$

- Minimal WL action gives eoms:

$$\dot{z}^\mu = u^\mu + \dots \qquad \frac{Dp_\mu}{D\lambda} = -R^\star_{\mu\nu\rho\sigma}\dot{z}^\nu u^\rho S^\sigma - \nabla_\mu \mathcal{M} \qquad \frac{DS^\mu}{D\lambda} = u^\mu S_\nu \frac{Du^\nu}{D\lambda} + \epsilon^{\mu\nu\rho\sigma} u_\nu S_\rho \frac{\partial \mathcal{M}}{\partial S^\sigma}$$

- Following the same steps as EM, we derived an all-orders in spin GR Compton formula for a generic body



$$\mathcal{A} = \mathcal{A}_{\text{EM-like}} + \mathcal{A}_{\text{3g}} + \sum_{n=0}^{\infty} \delta\mathcal{A}_n$$

$$\begin{aligned}\mathcal{A}_{\text{EM-like}} =& 2m \Bigg(f^*_{2\mu\nu} k_2^\mu \Delta\widetilde{z}^{\nu\star} f^*_{2\rho\sigma} v^\rho s^\sigma - \frac{i}{4} \frac{v_\alpha f_2^{*\alpha\beta} f_{1\beta\gamma} v^\gamma}{(k_1\cdot v)^3} v_\mu f_1^{\mu\nu} \epsilon_{\nu\rho v s} f_2^{*\rho\sigma} v_\sigma \\ &+ i(k_2\cdot\Delta\widetilde{z})\mathcal{N}_2^* - \Delta\widetilde{u}^\alpha \frac{\partial \mathcal{N}_2^*}{\partial v^\alpha} - \Delta\widetilde{S}^\alpha \frac{\partial \mathcal{N}_2^*}{\partial s^\alpha} + \frac{\mathcal{N}_1\mathcal{N}_2^*}{m} - \mathcal{P}_{12} \Bigg).\end{aligned}$$

(Remaining pieces in supplementary slides)

- General R^1 piece of the DMF:

$$\delta\mathcal{M}^2_{1S^2} = -C_2 R_{uSuS} - \frac{E_{2a}}{3} R_{SS} + E_{2b} S^2 R_{uu} + \frac{E_{2c}}{6} S^2 R,$$

$$\delta\mathcal{M}^2_{1S^3} = \frac{C_3}{3m}(S\cdot\nabla) R^\star_{uSuS} + \frac{E_3}{3m} S^2 \nabla^\rho R^\star_{\rho uuS},$$

$$\begin{aligned}
\delta\mathcal{M}^2_{1S^4} = {} & \frac{C_4}{12m^2}(S\cdot\nabla)^2 R_{uSuS} - \frac{E_{4a}}{20m^2} S^2 (u\cdot\nabla)^2 R_{uSuS} - \frac{E_{4b}}{12m^2} S^2\nabla^2 R_{uSuS} \\
& + \frac{E_{4c}}{30m^2}(S\cdot\nabla)^2 R_{SS} - \frac{E_{4d}}{30m^2} S^2 (u\cdot\nabla)^2 R_{SS} - \frac{E_{4e}}{30m^2} S^2\nabla^2 R_{SS} \\
& - \frac{E_{4f}}{12m^2} S^2 (S\cdot\nabla)^2 R_{uu} + \frac{E_{4g}}{20m^2} S^4 (u\cdot\nabla)^2 R_{uu} + \frac{E_{4h}}{12m^2} S^4\nabla^2 R_{uu} \\
& - \frac{E_{4i}}{60m^2} S^2 (S\cdot\nabla)^2 R - \frac{E_{4j}}{60m^2} S^4 (u\cdot\nabla)^2 R + \frac{E_{4k}}{60m^2} S^4\nabla^2 R,
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{M}^2_{1S^5} = {} & -\frac{C_5}{60m^3}(S\cdot\nabla)^3 R^\star_{uSuS} + \frac{E_{5a}}{60m^3} S^2 (u\cdot\nabla)^2 (S\cdot\nabla) R^\star_{uSuS} + \frac{E_{5b}}{60m^3}(S\cdot\nabla)\nabla^2 R^\star_{uSuS} \\
& - \frac{E_{5c}}{60m^3} S^2 (S\cdot\nabla)^2 \nabla^\rho R^\star_{\rho uuS} + \frac{E_{5d}}{60m^3} S^4 (u\cdot\nabla)^2 \nabla^\rho R^\star_{\rho uuS} + \frac{E_{5e}}{60m^3} S^4\nabla^2\nabla^\rho R^\star_{\rho uuS}
\end{aligned}$$

- Red coefficients don't contribute to Compton, Blue coefficients do.

- All R^2 terms that can contribute to Compton can be expressed in terms of the tidal tensors $E_{\mu\nu}$, $B_{\mu\nu}$:

$$
\begin{aligned}
E_{\mu\nu} \mp iB_{\mu\nu} &= (R_{\rho\mu\sigma\nu} \mp i\,{}^\star R_{\rho\mu\sigma\nu})u^\rho u^\sigma = 2\,{}^{\pm}R_{\rho\mu\sigma\nu}u^\rho u^\sigma, \\
E_{\lambda\mu\nu} \mp iB_{\lambda\mu\nu} &= 2\nabla_\kappa\,{}^{\pm}R^\rho{}_{(\lambda}{}^\sigma{}_\mu\left(\delta_{\nu)}{}^\kappa + u_{\nu)}u^\kappa\right)u_\rho u_\sigma, \\
\dot{E}_{\mu\nu} \mp i\dot{B}_{\mu\nu} &= 2u^\lambda\nabla_\lambda\,{}^{\pm}R_{\rho\mu\sigma\nu}u^\rho u^\sigma.
\end{aligned}
$$

- General R^2 DMF for Compton:

$$
\begin{aligned}
\delta\mathcal{M}^2_{2S^4} = {} & \frac{D_{4a}}{m^2}(E_{SS})^2 + \frac{D_{4b}}{m^2}S^2E^\mu{}_S E_{\mu S} + \frac{D_{4c}}{m^2}S^4E^{\mu\nu}E_{\mu\nu} && \longrightarrow 6\ D\text{'s at } \mathcal{O}(S^4) \\
& + \frac{D_{4d}}{m^2}(B_{SS})^2 + \frac{D_{4e}}{m^2}S^2B^\mu{}_S B_{\mu S} + \frac{D_{4f}}{m^2}S^4B^{\mu\nu}B_{\mu\nu} \\
\delta\mathcal{M}^2_{2S^5} = {} & \frac{D_{5a}}{m^3}B_{SS}E_{SSS} + \frac{D_{5b}}{m^3}S^2B^\mu{}_S E_{\mu SS} + \frac{D_{5c}}{m^3}S^4B^{\mu\nu}E_{\mu\nu S} && \longrightarrow 10\ D\text{'s at } \mathcal{O}(S^5) \\
& + \frac{D_{5d}}{m^3}E_{SS}B_{SSS} + \frac{D_{5e}}{m^3}S^2E^\mu{}_S B_{\mu SS} + \frac{D_{5f}}{m^3}S^4E^{\mu\nu}B_{\mu\nu S} \\
& + \left(\frac{D_{5g}}{m^3}E_{S\mu}\dot{E}_{S\nu} + \frac{D_{5h}}{m^3}S^2E^\lambda{}_\mu\dot{E}_{\lambda\nu}\right)\epsilon^{\mu\nu}{}_{\rho\sigma}u^\rho S^\sigma S^2 \\
& + \left(\frac{D_{5i}}{m^3}B_{S\mu}\dot{B}_{S\nu} + \frac{D_{5j}}{m^3}S^2B^\lambda{}_\mu\dot{B}_{\lambda\nu}\right)\epsilon^{\mu\nu}{}_{\rho\sigma}u^\rho S^\sigma S^2.
\end{aligned}
$$

- Returning the DMF to our expression for the amplitude gives that the amplitude is independent of precisely 7 linear combinations at $\mathcal{O}(S^5)$:

$$
\begin{aligned}
Z_4 &= E_{4b} + \frac{D_{4b}}{2} - \frac{D_{4c}}{6} - \frac{D_{4e}}{2} + \frac{D_{4f}}{6} \\
Z_{5a} &= E_{5a} + \frac{D_{5e}}{180} + \frac{D_{5g}}{60} - \frac{D_{5i}}{90} \\
Z_{5b} &= E_{5b} - \frac{4}{15} D_{5b} + \frac{D_{5c}}{10} - \frac{4}{15} D_{5e} + \frac{D_{5f}}{10} + \frac{D_{5h}}{30} - \frac{D_{5j}}{30} \\
Z_{5c} &= E_{5c} - \frac{D_{5b}}{12} + \frac{D_{5c}}{36} - \frac{17}{180} D_{5e} + \frac{D_{5f}}{30} - \frac{D_{5g}}{180} + \frac{D_{5h}}{180} - \frac{D_{5i}}{180} \\
Z_{5d} &= E_{5d} - \frac{D_{5f}}{180} - \frac{D_{5h}}{90} + \frac{D_{5j}}{180} \\
Z_{5e} &= E_{5e} + \frac{D_{5c}}{60} + \frac{D_{5f}}{60} + \frac{D_{5h}}{20} - \frac{D_{5j}}{20} \\
Z_{5f} &= E_{4a} - \frac{D_{5b}}{20} - \frac{C_2}{10} D_{5g} + \frac{3}{20} D_{5i}
\end{aligned}
$$

$\implies$ E's killed by field definitions.

#'s	R^1	R^2
S^1	0	0
S^2	1	0
S^3	1	0
S^4	1	6
S^5	1	10

- No obstruction to pushing to higher orders in spin.

Comptons - Kerr values through S^5

- C coefficients are fixed by stationary Kerr / Newman-Janis shift / 3pt amplitude. Levi, Steinhoff

- Through $\mathcal{O}(S^4)$ for Compton, Guevara-Ochirov-Vines (GOV) spin-exponentiation / shift-symmetry of R. Aoude, K. Haddad, A. Helset `2203.06197` and Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, F. Teng `2203.06202` / Teukolsky equation results are all consistent with each other and determine quadratic in Riemann coefficients.

- Beginning at $\mathcal{O}(S^5)$, GOV can only be maintained for one helicity combination, and shift-symmetry disagrees with Teukolsky. Y.F. Bautista, A. Guevara, C. Kavanagh, J. Vines `2212.07965`. The combination GOV+Teukolsky match

$$\mathcal{A}_{++} = \mathcal{A}^{(0)}_{++} \exp\left(a \cdot (k_1 - k_2)\right) + \mathcal{O}(S^6)$$

$$\mathcal{A}_{+-} = \mathcal{A}^{(0)}_{+-} \exp(a \cdot (k_1 + k_2)) \sum_{n=0}^{5} \frac{\bar{J}_n}{n!} + \mathcal{O}(S^6)$$

- Match to GOV+Teukolsky leaves 0 free #'s in Compton.

- Any body with a physically reasonable stress tensor $T^{\mu\nu}$: $\nabla_\nu T^{\mu\nu} = 0$ has a unique center of mass worldline $z^\mu(\lambda)$, total body momentum $p_\mu(\lambda)$, and total spin vector $S^\mu(\lambda)$:

$$u^\mu = \frac{p^\mu}{\sqrt{|p^2|}}, \qquad S_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} u^\rho S^\sigma$$

$$p^\kappa = \int_\Sigma (-\sigma^{-1}_{\alpha\lambda}) \sigma^{\lambda\kappa} T^{\alpha\beta} d\Sigma_\beta, \qquad S^{\kappa\lambda} = 2 \int_\Sigma \sigma^{[\kappa} (\sigma^{-1})^{\lambda]}{}_\alpha T^{\alpha\beta} d\Sigma_\beta$$

  where σ is Synge's worldfunction and $\Sigma(\lambda) = \{X : \sigma_\mu u^\mu(\lambda) = 0\}$.

- p, S satisfy the full (covariant SSC) MPD eoms:

$$\frac{Dp_\mu}{D\lambda} = -\frac{1}{2} R_{\mu\nu\rho\sigma} \dot{z}^\nu S^{\rho\sigma} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n!} I_n^{\nu_1 \ldots \nu_n \rho\sigma} \nabla_\mu g_{\rho\sigma,\nu_1 \ldots \nu_n}$$

$$\frac{DS^{\mu\nu}}{D\lambda} = 2p^{[\mu} \dot{z}^{\nu]} + \sum_{n=1}^{\infty} \frac{1}{n!} g^{\sigma[\mu} I_{n+1}^{\nu]\rho_1 \ldots \rho_n \tau\omega} g_{\{\sigma\omega,\tau\}\rho_1 \ldots \rho_n}$$

- The evolution of the I moments is completely unconstrained by $\nabla_\nu T^{\mu\nu} = 0$.

- Reduced multipole moments (I moments) $\neq$ stationary multipole moments.
- Naive multipole moments of a tensor field:

$$F_N^{\mu_1\cdots\mu_N\rho_1\cdots\rho_m}{}_{\nu_1\ldots\nu_n}[\phi](\lambda) = \int_\Sigma (-\sigma^{\mu_1})....(-\sigma^{\mu_N})\phi'^{\rho_1\cdots\rho_m}{}_{\nu_1\ldots\nu_n} w^\alpha d\Sigma_\alpha$$

$$\phi'^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}(Z,v) = (-\sigma^{\mu_1}{}_{\alpha_1})...(-\sigma^{\mu_m}{}_{\alpha_m})(-\sigma^{-\beta_1}{}_{\nu_1})...(-\sigma^{-\beta_n}{}_{\nu_n})\phi^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}(\exp_Z(v)).$$

- For a conserved tensor field, these multiple moment contain infinitely many very complicated interdependencies.

$$\int \psi^* \nabla_\alpha \phi^\alpha D^4x = -\int_{-\infty}^{\infty} \int \widetilde{\psi}^*(z,k) \sum_{n=1}^{\infty} \frac{(-i)^n}{(n-1)!} k_{\mu_1}...k_{\mu_n} F_{n-1}^{(\mu_1\cdots\mu_n)} \frac{D^4k}{(2\pi)^2} D\lambda.$$

- Even if $\nabla_\alpha \phi^\alpha = 0$, cannot make any useful deductions about the naive moments.

- Dixon defined the reduced multipole moments precisely to remove interdependencies for conserved tensor fields (formula in supplementary slides). The only conditions they are restricted by are:

$$I_n^{\lambda_1\ldots\lambda_n\mu\nu} = I_n^{(\lambda_1\ldots\lambda_n)(\mu\nu)}, \qquad I_n^{(\lambda_1\ldots\lambda_n\mu)\nu} = 0,$$
$$u_{\lambda_1} I_n^{\lambda_1\ldots\lambda_{n-2}[\lambda_{n-1}['\lambda_n\mu]\nu]'} = 0, \qquad (n \geq 3)$$

- Integration against test functions determines stress tensor:

$$\int_\Sigma h^*_{\alpha\beta}(X) T^{\alpha\beta}(X) w^\gamma d\Sigma_\gamma = \int \widetilde{h}^*_{\mu\nu}(Z,k) I^{\mu\nu}(\lambda,k) \frac{D^4k}{(2\pi)^2}$$
$$I^{\mu\nu}(\lambda,k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} I_n^{\lambda_1\ldots\lambda_n\mu\nu} k_{\lambda_1}\ldots k_{\lambda_n}.$$

- Unique choice of multipole moments that remove interdependencies for general motion.

Orthogonality - MPD compared to WL eom

- Minimal WL has evolution:

$$\frac{Dp_\mu}{D\lambda} = -R^\star_{\mu\nu\rho\sigma}\dot{z}^\nu u^\rho S^\sigma - \nabla_\mu \mathcal{M}$$
$$\frac{DS^\mu}{D\lambda} = u^\mu S_\nu \frac{Du^\nu}{D\lambda} + \epsilon^{\mu\nu\rho\sigma} u_\nu S_\rho \frac{\partial \mathcal{M}}{\partial S^\sigma}.$$

- Requiring match MPD is not automatic due to orthogonality relations!

$$\frac{Dp_\mu}{D\lambda} = -R^\star_{\mu\nu\rho\sigma}\dot{z}^\nu u^\rho S^\sigma + \frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n!} I_n^{\nu_1\ldots\nu_n\rho\sigma}\nabla_\mu g_{\rho\sigma,\nu_1\ldots\nu_n}$$
$$\frac{DS^\mu}{D\lambda} = u^\mu S_\nu \frac{Du^\nu}{D\lambda} - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\nu \sum_{n=1}^{\infty}\frac{1}{n!} g^{\lambda\rho} I_{n+1}^{\sigma\kappa_1\ldots\kappa_n\tau\omega} g_{\{\lambda\omega,\tau\}\kappa_1\ldots\kappa_n}$$

$$u_{\lambda_1} I_n^{\lambda_1\ldots\lambda_{n-2}[\lambda_{n-1}['\lambda_n\mu]\nu]'} = 0$$

- The stress tensor computed from the minimal MPD action is:

$$T_{\alpha\beta} = \int_{-\infty}^{\infty} \left(\dot{z}_{(\alpha} p_{\beta)} \delta(X,Z) + \nabla^{\gamma} (\dot{z}_{(\alpha} \epsilon_{\beta)\gamma\rho\sigma} u^{\rho} S^{\sigma} \delta(X,Z)) \right) d\lambda + \sum_{n=0}^{\infty} \Psi^{(n)}_{\alpha\beta}$$

$$\begin{aligned}
\Psi^{(n)}_{\alpha\beta} &= \sum_{k=0}^{n-1} (-1)^k \nabla^k_{\lambda_{n-k+1}\ldots\lambda_n} \Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma} g_{\lambda_{n-k}(\alpha} \nabla_{\beta)} \nabla^{n-k-1}_{\lambda_1\ldots\lambda_{n-k-1}} R_{\mu\nu\rho\sigma} \\
&\quad + \sum_{k=0}^{n-1} 2(-1)^k \nabla_{\gamma} \left(\nabla^k_{\lambda_{n-k+1}\ldots\lambda_n} \Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma} g_{\lambda_{n-k}(\alpha} g_{\beta)\delta} \hat{G}^{[\gamma\delta]} (\nabla^{n-k-1}_{\lambda_1\ldots\lambda_{n-k-1}} R_{\mu\nu\rho\sigma}) \right) \\
&\quad + 2(-1)^n \nabla^n_{\lambda_1\ldots\lambda_n} \Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma} g_{\mu(\alpha} R_{\beta)\nu\rho\sigma} + 4(-1)^n \nabla^2_{\mu\nu} \nabla^n_{\lambda_1\ldots\lambda_n} \Phi_n^{\lambda_1\ldots\lambda_n\mu}{}_{(\alpha}{}^{\nu}{}_{\beta)}.
\end{aligned}$$

$$\Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma} = \int_{-\infty}^{\infty} \frac{\partial \mathcal{M}}{\partial \nabla^n_{(\lambda_1\ldots\lambda_n)} R_{\mu\nu\rho\sigma}} \delta(X,Z) D\lambda.$$

- We can use this stress tensor to relate I moments and DMF couplings using Dixon's definitions.

Orthogonality - Multipole Constraints on the DMF 2

- Returning the action's stress tensor to Dixon's moments implies:

$$u_{\lambda_1} \frac{\partial \mathcal{M}}{\partial g_{[\mu['\nu,|\lambda_1 \ldots \lambda_{n-2}|\lambda_{n-1}]\lambda_n]'}} = 0 \implies u_{\lambda_1} \frac{\partial \mathcal{M}}{\partial \nabla^n_{(\lambda_1 \ldots \lambda_n)} R_{\rho\mu\sigma\nu}} = 0 + \mathcal{O}(R)$$

- Higher order version corrections determined by a recurrence relation converting tensor extensions of the metric to derivatives of the Riemann tensor.
- The analogous calculation in EM gives precisely: $u_{\lambda_1} \frac{\partial \mathcal{M}}{\partial \partial^n_{\lambda_1 \ldots \lambda_n} \mathcal{F}_{\mu\nu}} = 0.$
- Does these constraints affect asymptotic observables?
- Through order $\mathcal{O}(R^1 S^5)/\mathcal{O}(\mathcal{F}^1 S^3)$ these constraints eliminate the same #'s that can be removed by field redefinition.
- If this correspondence continues at higher order (which we are working to understand), they will provide a clean way to eliminate redundant operators early (would make calculations easier!)

Summary

- Interacting tower of different spin s fields gives observables with spin magnitude change, can be matched to a WL with nonminimal dof.
- Minimal WL Compton scattering in EM/GR has a compact all-orders in spin expression.
- We have explicit results at $\mathcal{O}(S^5)$. Only effort in pushing spin-orders in the Compton is enumerating operators
- Requiring exact match of WL eom to MPD places an infinite tower of constraints on the WL action.
- These restrictions through the order computed exactly remove all operators that can be removed by field definition
- If this persists in general, it provides an efficient way to remove degenerate operators before calculation.

Thank you!

(Additional) Full WL eoms

- EM

$$\left(1-\frac{q}{\mathcal{M}^2}{}^\star\mathcal{F}^{\alpha\beta}u_\alpha S_\beta\right)\dot{z}^\mu = u^\mu + \frac{1}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u_\mu} + u^\mu\frac{u^\nu}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u^\nu} + S^\mu\frac{u^\nu}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial S^\nu} + \frac{1}{\mathcal{M}^2}\epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\frac{\partial\mathcal{M}}{\partial z^\sigma}$$
$$+\frac{q}{\mathcal{M}^2}{}^\star\mathcal{F}^{\mu\nu}S_\nu + \frac{q}{\mathcal{M}^3}\left(S^\rho\frac{\partial\mathcal{M}}{\partial u^\rho} + S^2u^\rho\frac{\partial\mathcal{M}}{\partial S^\rho}\right){}^\star\mathcal{F}^{\mu\nu}u_\nu$$
$$\dot{p}_\mu = q\mathcal{F}_{\mu\nu}\dot{z}^\nu - \frac{\partial\mathcal{M}}{\partial z^\mu}$$
$$\dot{S}^\mu = u^\mu\dot{u}\cdot S + \epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\frac{\partial\mathcal{M}}{\partial S^\sigma}.$$

- GR

$$\left(1+\frac{{}^\star R^\star_{uSuS}}{\mathcal{M}^2}\right)\dot{z}^\mu = u^\mu + \frac{1}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u_\mu} + u^\mu\frac{u^\nu}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u^\nu} + S^\mu\frac{u^\nu}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial S^\nu} + \frac{1}{\mathcal{M}^2}\epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\nabla_\sigma\mathcal{M}$$
$$-\frac{{}^\star R^{\star\mu SuS}}{\mathcal{M}^2} - \frac{1}{\mathcal{M}^3}\left(S^\nu\frac{\partial\mathcal{M}}{\partial u^\nu} + S^2u^\nu\frac{\partial\mathcal{M}}{\partial S^\nu}\right){}^\star R^{\star\mu uuS}$$
$$\frac{Dp_\mu}{D\lambda} = -R^\star_{\mu\nu\rho\sigma}\dot{z}^\nu u^\rho S^\sigma - \nabla_\mu\mathcal{M}$$
$$\frac{DS^\mu}{D\lambda} = u^\mu S_\nu\frac{Du^\nu}{D\lambda} + \epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\frac{\partial\mathcal{M}}{\partial S^\sigma}$$

(Additional) GR eom solutions

$$\delta\widetilde{u}^\mu = \frac{i}{2m}\mathcal{E}_1^\mu \epsilon^{vsk_1\mathcal{E}_1} - \frac{\mathcal{N}_1}{m}\left(v^\mu + \frac{k_1^\mu}{k_1\cdot v}\right)$$

$$\delta\widetilde{S}^\mu = \frac{1}{2}(\mathcal{E}_1\cdot s)\mathcal{E}_1^\mu + v^\mu s\cdot\delta\widetilde{u} - \frac{i}{k_1\cdot v}\epsilon^{\mu\nu\rho\sigma}v_\nu s_\rho\frac{\partial\mathcal{N}_1}{\partial s^\sigma}$$

$$\delta\widetilde{z}^\mu = \frac{-i}{k_1\cdot v}\left(\delta\widetilde{u}^\mu + \frac{\epsilon^{vsk_1\mathcal{E}_1}}{2m^2}\left(\delta_\nu^\mu + v^\mu v_\nu\right)\epsilon^{\nu sk_1\mathcal{E}_1} + \frac{\eta^{\mu\nu}+v^\mu v^\nu}{m}\frac{\partial\mathcal{N}_1}{\partial v^\nu} + \frac{s^\mu v^\nu}{m}\frac{\partial\mathcal{N}_1}{\partial s^\nu} + \frac{i}{m^2}\mathcal{N}_1\epsilon^{\mu\nu\rho\sigma}v_\nu s_\rho k_{1\sigma}\right).$$

(Additional) Synge's Worldfunction

- Let $\zeta(s, Z, v)$ be geodesic with affine parameter s, $\zeta^\mu(0, Z, v) = z^\mu$, $\frac{d\zeta^\mu}{ds}(0, Z, v) = v^\mu$. Then:

$$\exp_Z(v) = \zeta(1, Z, v).$$

- Consider geodesic $\zeta(s)$ with affine parameter s, $\zeta^\mu(0) = z^\mu$, $\zeta^\alpha(1) = x^\alpha$ (for Z and X sufficiently close for one only such geodesic to exist). Then:

$$\sigma(Z, X) = \frac{1}{2}\int_0^1 g_{\alpha\beta}(\zeta)\frac{d\zeta^\alpha}{ds}\frac{d\zeta^\beta}{ds}ds.$$

- Indices on σ to indicate covariant derivatives with $\alpha, \beta, ...$ indices for x and $\mu, \nu, ...$ indices for z:

$$\sigma_\alpha = \nabla_\alpha\sigma = \frac{\partial\sigma}{\partial x^\alpha} = \frac{d\zeta_\alpha}{ds}(1), \qquad \sigma_\mu = \nabla_\mu\sigma = \frac{\partial\sigma}{\partial z^\mu} = -\frac{d\zeta_\mu}{ds}(0).$$

- We introduce the inverse matrix σ^{-1}:

$$\sigma^{-\alpha}{}_\mu\sigma^\mu{}_\beta = \delta^\alpha_\beta.$$

(Additional) Explicit formula for reduced moments

$$\Theta_n^{\kappa\lambda\mu\nu}(Z,X) = (n-1)\int_0^1 \sigma^{\kappa\alpha}\sigma^{(\mu}{}_{\alpha}\sigma^{\nu)}{}_{\beta}\sigma^{\lambda\beta}t^{n-2}dt, \quad (n\geq 2)$$

$$\mathfrak{p}_n^{\kappa_1\ldots\kappa_n\lambda\mu\nu} = 2(-1)^n\int_\Sigma \sigma^{\kappa_1}\ldots\sigma^{\kappa_n}\Theta_n^{\rho\nu\lambda\mu}(-\sigma^{-1}_{\alpha\rho})T^{\alpha\beta}d\Sigma_\beta, \quad (n\geq 2)$$

$$\mathfrak{t}_n^{\kappa_1\ldots\kappa_n\lambda\mu} = (-1)^n\int_\Sigma \sigma^{\kappa_1}\ldots\sigma^{\kappa_n}\sigma^{\lambda}{}_{\alpha}\sigma^{\mu}{}_{\beta}T^{\alpha\beta}w^{\gamma}d\Sigma_\gamma, \quad (n\geq 2)$$

$$\mathcal{J}_n^{\kappa_1\ldots\kappa_n\lambda\mu\nu\rho} = \mathfrak{t}_{n+2}^{\kappa_1\ldots\kappa_n[\lambda['\nu\mu]\rho]'} + \frac{1}{n+1}\mathfrak{p}_{n+2}^{\kappa_1\ldots\kappa_n[\lambda['\nu\mu]\rho]'\tau}\dot{z}_\tau$$

$$I_n^{\lambda_1\ldots\lambda_n\mu\nu} = \frac{4(n-1)}{n+1}\mathcal{J}_{n-2}^{(\lambda_1\ldots\lambda_{n-1}|\mu|\lambda_n)\nu}, \quad (n\geq 2).$$

(Additional) Tensor Extensions

$$N_{\alpha_1 \ldots \alpha_n \beta \gamma}{}^{\delta} \to \partial^n_{\alpha_1 \ldots \alpha_n} \Gamma^{\delta}{}_{\beta \gamma}$$

$$N_{\alpha\beta\gamma\delta} = \frac{2}{3} R_{\alpha(\beta\gamma)\delta}$$

$$E_{\alpha_1 \ldots \alpha_n} v_\beta = \nabla_{(\alpha_1 \ldots \alpha_n)} v_\beta + \sum_{r=2}^{n} \sum_{s=r}^{n} \binom{s-1}{r-1} N_{(\alpha_1 \ldots \alpha_r |\beta|}{}^{\gamma} E_{\alpha_{r+1} \ldots \alpha_s} \nabla_{\alpha_{s+1} \ldots \alpha_n)} v_\gamma$$

$$\begin{aligned}\mathcal{A}_{3\mathrm{g}} = & -\frac{m(\mathcal{E}_1\cdot\mathcal{E}_2^*)^2}{2(k_1\cdot k_2)}M_{k_1k_1} - \frac{m(\mathcal{E}_1\cdot k_2)^2}{2(k_1\cdot k_2)}M_{\mathcal{E}_2^*\mathcal{E}_2^*} - \frac{m(\mathcal{E}_2^*\cdot k_1)^2}{2(k_1\cdot k_2)}M_{\mathcal{E}_1\mathcal{E}_1} \\ & + \frac{m(\mathcal{E}_1\cdot k_2)(\mathcal{E}_1\cdot\mathcal{E}_2^*)}{(k_1\cdot k_2)}M_{k_1\mathcal{E}_2^*} + m(\mathcal{E}_1\cdot\mathcal{E}_2^*)M_{\mathcal{E}_1\mathcal{E}_2^*} \\ & + \frac{m(\mathcal{E}_1\cdot\mathcal{E}_2^*)(k_1\cdot\mathcal{E}_2^*)}{(k_1\cdot k_2)}M_{k_1\mathcal{E}_1} - \frac{m(\mathcal{E}_1\cdot k_2)(\mathcal{E}_2^*\cdot k_1)}{(k_1\cdot k_2)}M_{\mathcal{E}_1\mathcal{E}_2^*}\end{aligned}$$

$$M^{\alpha\beta}(k) = mv^\alpha v^\beta + iv^{(\alpha}\epsilon^{\beta)\gamma\rho\sigma}k_\gamma v_\rho s_\sigma - \frac{2}{m}\sum_{n=0}^{\infty}(-i)^n T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}(\eta,v,s)k_{\lambda_1}\ldots k_{\lambda_n}k_\mu k_\nu.$$

$$
\begin{aligned}
\delta\mathcal{A}_n = & -\frac{1}{2}\sum_{l=0}^{n-1} i^n(-1)^l(k_1\cdot\mathcal{E}_2^*)T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}k_{1\lambda_1}\ldots k_{1\lambda_{n-l-1}}\mathcal{E}^*_{2\lambda_{n-l}}q_{\lambda_{n-l+1}}\ldots q_{\lambda_n}\delta\widetilde{R}_{\rho\mu\sigma\nu} \\
& + \sum_{l=0}^{n-1} i^n(-1)^l T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}\mathcal{E}_2^{*[\tau}k_2^{\omega]}\hat{G}_{\tau\omega}(k_{1\lambda_1}\ldots k_{1\lambda_{n-l-1}}\delta\widetilde{R}_{\rho\mu\sigma\nu})\mathcal{E}^*_{2\lambda_{n-l}}q_{\lambda_{n-l+1}}\ldots q_{\lambda_n} \\
& - (-i)^n T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}q_{\lambda_1}\ldots q_{\lambda_n}\mathcal{E}^*_{2\rho}\mathcal{E}_2^{*\alpha}\delta\widetilde{R}_{\alpha\mu\sigma\nu} \\
& + 2(-1)^n\sum_{l=0}^{n-1} i^{n-1}\mathcal{E}^*_{2\alpha}\mathcal{E}^*_{2\beta}k_{2\mu}k_{2\nu}k_{2\lambda_{n-l+1}}\ldots k_{2\lambda_n}\delta\widetilde{\Gamma}^{\sigma}{}_{\rho\lambda_{n-l}}\hat{G}^{\rho}{}_{\sigma}(T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}q_{\lambda_1}\ldots q_{\lambda_{n-l-1}}) \\
& - 2i(-i)^n\mathcal{E}^*_{2\alpha}\mathcal{E}^*_{2\beta}k_{2\mu}\delta\widetilde{\Gamma}^{\sigma}{}_{\rho\nu}\hat{G}^{\rho}{}_{\sigma}(T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}q_{\lambda_1}\ldots q_{\lambda_n}) \\
& - 2i(-i)^n\mathcal{E}^*_{2\alpha}\mathcal{E}^*_{2\beta}\delta\widetilde{\Gamma}^{\sigma}{}_{\rho\mu}\hat{G}^{\rho}{}_{\sigma}(T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}q_{\nu}q_{\lambda_1}\ldots q_{\lambda_n}) \\
& + (-i)^n k_{2\lambda_1}\ldots k_{2\lambda_n}k_{2\rho}\mathcal{E}^*_{2\mu}k_{2\sigma}\mathcal{E}^*_{2\nu}\mathcal{E}_1^{\alpha}\mathcal{E}_1^{\beta}\hat{G}_{\alpha\beta}(T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu})
\end{aligned}
$$