Abelian insights on Kinematic Algebras

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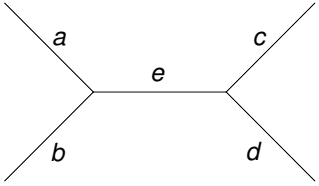
Overview

What are kinematic algebras, and what can we learn about them
by studying simple (abelian) gauge theories?

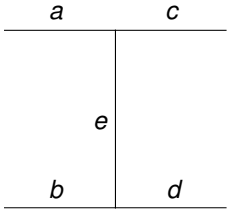
- Introduction to BCJ duality.
- Kinematic algebras and known special cases.
- The diffeomorphism algebra of (restricted) abelian gauge theories.
- Interacting theories from abelian building blocks.
- Open questions.

Amplitudes and colour factors

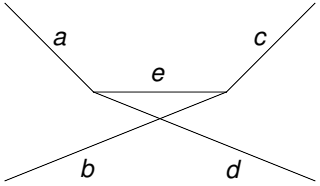
- Scattering amplitudes in non-abelian gauge theories can be obtained by considering certain cubic graphs, each of which has a colour factor c_i.
- Each graph also has a kinematic numerator n_i, which is a function of momenta, polarisation etc.
- Colour factors of different graphs are related by **Jacobi identities**.
- Remarkably, the kinematic numerators can be chosen to obey similar identities.
- This is called **BCJ Duality** (Bern, Carrasco, Johansson), and is conjectured at all loop orders.



$c_s = f^{aeb} f^{ecd}$



$c_t = f^{aec} f^{ebd}$

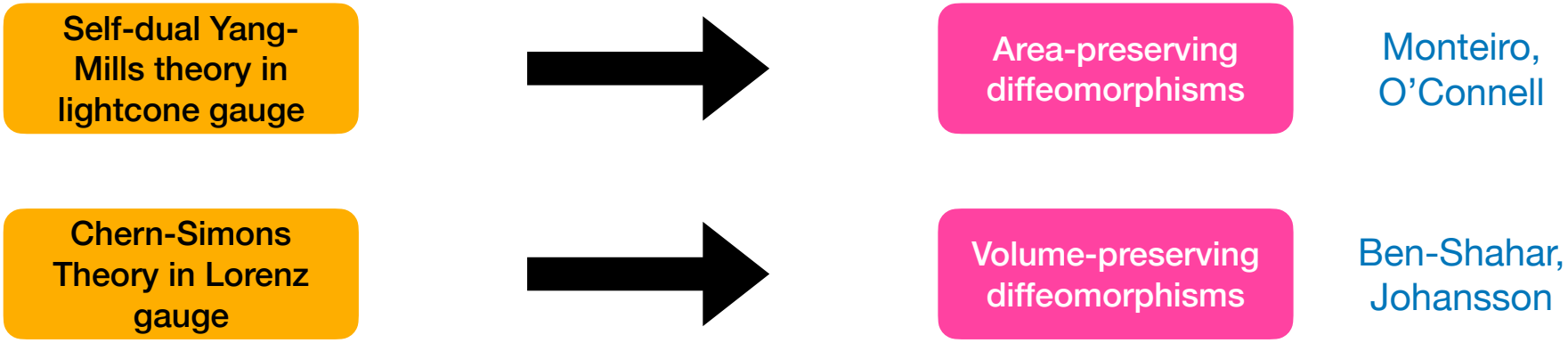


$c_u = f^{aed} f^{ecb}$

$$c_s = c_t + c_u$$

Kinematic algebras

- The Jacobi identity for colour factors is a consequence of the colour Lie algebra.
- Thus, BCJ duality suggests there is a **kinematic algebra**, that somehow mirrors the colour algebra.
- For some very special cases, this is known to be a Lie algebra:

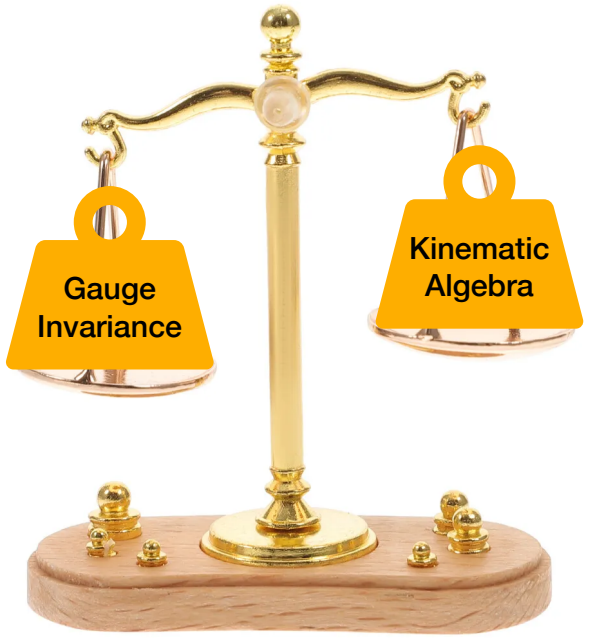


General kinematic algebras

- More generally, kinematic algebras are not expected to be Lie algebras…
- …but more general mathematical structures.
- A particular candidate is a so-called $\mathrm{BV}^{\square}_{\infty}$ algebra, or **homotopy algebra** (Reiterer; then Borsten, Jurco, Kim, Macrelli, Saemann, Wolf; Bonezzi, Chiaffrino, Diaz-Jaramilo, Hohm, Plefka).
- There are higher-order brackets that generalise the Lie bracket of gauge fields.
- Jacobi identities are satisfied only up to terms involving higher-order brackets, leading to an intricate structure of constraints.
- Somehow, this must all reduce to a straightforward Lie algebra in those cases where this is possible.

Why do we care?

- Kinematic algebras remain mysterious.
- But they are relatively **new**, **exciting** and certainly **fun** 😀!
- New way to think about gauge theories?
- If so, we should explore all ways of gaining intuition about the many open questions.



Which theories have kinematic algebras?

Are previous known cases related?

How do kinematic algebras depend upon the gauge?

When do we get a Lie algebra?

How do we visualise kinematic algebras geometrically?

Abelian insights

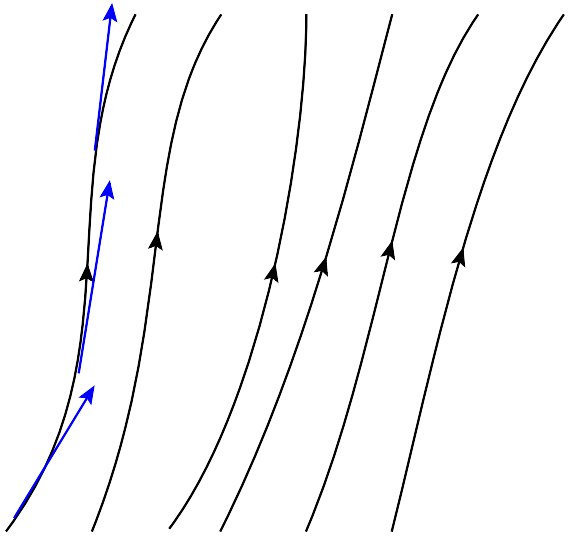
- It is common in physics to use simple cases to gain intuition.
- To this end, let us look at **abelian** gauge theories (or **linearised non-abelian**).
- BCJ duality is conventionally a purely non-linear phenomenon: structure constants colour / kinematic algebras occur together.
- But there can also be “colour-kinematics” duality at linearised level.
- A non-abelian gauge field is Lie-algebra valued in two Lie algebras:

$$\mathbf{A} = A^{\mu a}\,\mathbf{T}^{a}\,\partial_{\mu}$$

Infinitesimal gauge transformation

Infinitesimal diffeomorphism

Diffeomorphisms



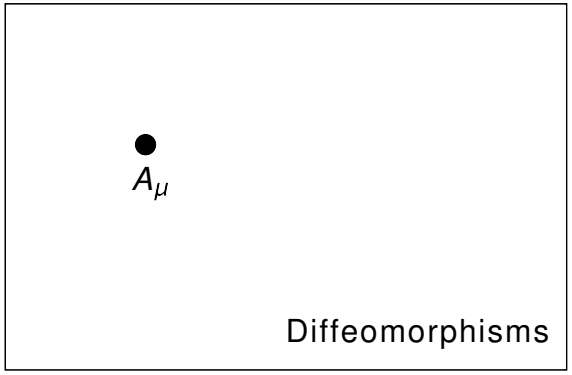
- Diffeomorphism = simultaneous translation along all integral curves (field lines) of the vector field.
- In known cases where linear solutions double / zeroth copy, this amounts to replacing generators of colour / kinematic algebras appropriately.
- This is colour / kinematics duality involving **generators**, rather than **structure constants**.
- We can associate the “kinematic algebra” of a linear theory with these diffeomorphisms… (see also Fu, Krasnov)
- …and will eventually see how these ideas are relevant for exploring non-linear theories.

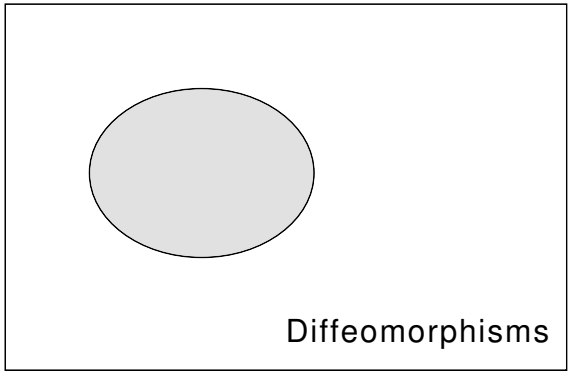
Abelian theories

- A particular A_μ is a point in the space of all possible diffeomorphisms:
- There is no non-trivial kinematic algebra: diffeomorphisms along the same field lines commute.
- We can instead associate a non-trivial algebra if we can find a set of solutions closed under the Lie algebra of diffeomorphisms:

$$[A^{(1)\mu}\,\partial_\mu, A^{(2)\nu}\,\partial_\nu] = A^{(3)\mu}\,\partial_\mu$$

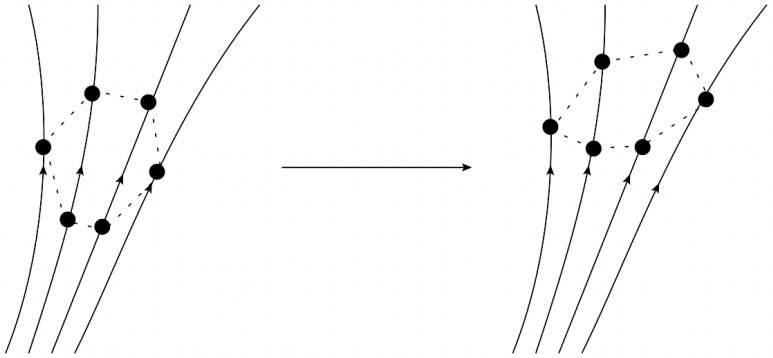
$$A^{(3)}_\mu = A^{(1)} \cdot \partial A^{(2)}_\mu - A^{(2)} \cdot \partial A^{(1)}_\mu$$





Closed subgroups of diffeomorphisms

- There are indeed subgroups of the diffeomorphism group.
- A notable one is **volume-preserving diffeomorphisms**.
- The “volume” being preserved is that of a shape which is dragged along the field lines.



- Turns out to imply $\partial \cdot A = 0$, whose physical interpretation is that we are in **Lorenz gauge**.
- Further subgroups exist, corresponding to preserving volume in a lower-dimensional subspace.

Symplectomorphisms

- Another interesting subgroup is that of **symplectomorphisms** (a.k.a. **canonical transformations**).
- The gauge field is a so-called **Hamiltonian vector field**, with the special form

$$A^\mu = \hat{k}^\mu \phi, \quad \hat{k}_\mu = \Omega_{\mu\nu}\partial^\nu, \quad \Omega_{\mu\nu} = -\Omega_{\nu\mu}$$

Scalar field ("Hamiltonian")

"Symplectic form"

- This is a subset of solutions of the theory i.e. more than just a gauge choice.
- **Self-dual** electromagnetism in **lightcone gauge** is a special case.

Poisson brackets

- If we have a symplectic form, we can form a **Poisson bracket**:

$$\{\phi_1, \phi_2\} = \Omega^{\mu\nu}(\partial_\mu \phi_1)(\partial_\nu \phi_2)$$

- The Lie bracket of two Hamiltonian vector fields is also Hamiltonian, where the relevant scalar function is given by the Poisson bracket of the two original scalars:

$$[(\hat{k}^\mu \phi_1)\partial_\mu, (\hat{k}^\nu \phi_2)\partial_\nu] = (\hat{k}^\mu \phi_3)\partial_\mu, \quad \phi_3 = -\{\phi_1, \phi_2\}$$

- Thus, the Poisson bracket encodes the Lie algebra of diffeomorphisms generated by Hamiltonian vector fields A_μ.

Lightcone gauge electromagnetism

- As an example of these ideas, consider the coordinate system

$$u = \frac{t - z}{\sqrt{2}}, \quad v = \frac{t + z}{\sqrt{2}}, \quad X = \frac{x + iy}{\sqrt{2}}, \quad Y = \frac{x - iy}{\sqrt{2}}$$

- A gauge field in the lightcone gauge $A_u = 0$ can be shown to have the form

$$A_\mu = \hat{k}_\mu \phi + \hat{k}_\mu^\dagger \phi^\dagger, \quad \hat{k}_\mu \equiv (0, \partial_Y, \partial_u, 0)$$

- The two terms can be shown to generate diffeomorphisms in families of (u,X) and (u,Y) planes respectively.
- Furthermore, they are area-preserving in these planes.
- Thus, we can directly visualise the “kinematic algebra” of lightcone gauge EM.

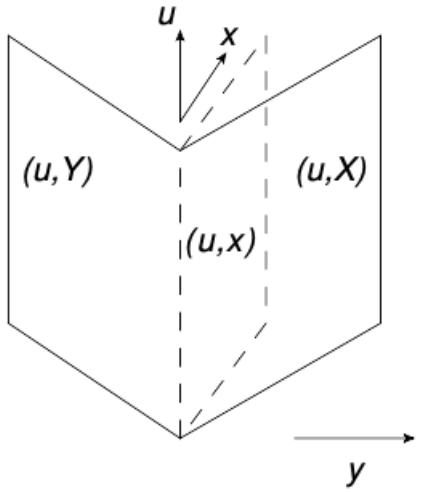
Lightcone gauge electromagnetism

- The “kinematic algebra” of electromagnetism in lightcone gauge will be a subgroup of

$$\mathrm{Diff}_{(u,X)} \times \mathrm{Diff}_{(u,Y)} ,$$

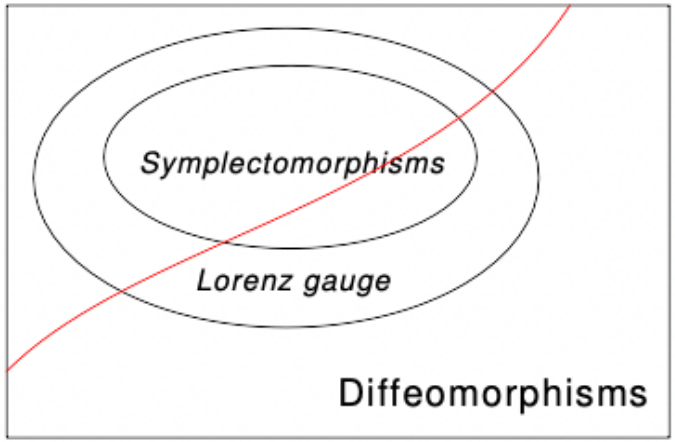
picked out by the gauge field being real in Lorentzian signature.

- One may show in general that this comprises 3d-volume preserving diffeomorphisms in the (u,x,y) subspace…
- …that become area-preserving diffeomorphisms in the (u,x) or (u,y) planes if ϕ is pure real or imaginary.



Gauge dependence of kinematic algebras

- The gauge dependence of kinematic algebras can also be thought about at linearised level.
- A **gauge orbit** (set of gauge fields related by gauge transformations) shows up as a line in the space of all possible diffeomorphisms.
- If we start with a Hamiltonian vector field A_μ , gauge transformations act as



$$A_\mu \to A_\mu + \partial_\mu \alpha$$

- If α is harmonic, gauge transformations will move A_μ to the set of volume-preserving diffeomorphisms.
- No special set will be obtained for general α.

Interacting theories

- Of course, usually by “kinematic algebras”, we mean structures that act on interacting theories.
- The canonical example is self-dual Yang-Mills theory, whose lightcone gauge field equation may be written as

$$\partial^2 \mathbf{\Phi} + g\{[\mathbf{\Phi}, \mathbf{\Phi}]\} = 0$$

Scalar field $\mathbf{\Phi} = \Phi^a \mathbf{T}^a$

Coupling constant

Double bracket $\{[\mathbf{\Phi}, \mathbf{\Phi}]\} = \Omega^{\mu\nu} f^{abc} (\partial_\mu \Phi^b)(\partial_\nu \Phi^c)\mathbf{T}^a$

- The gauge field is Hamiltonian: $A_\mu^a = \hat{k}_\mu \Phi^a$, and the double bracket combines the colour Lie bracket with the Poisson bracket for the scalar components.
- The kinematic algebra is one of area-preserving diffeomorphisms.

Kinematic algebras from the ground up

- There is an interesting way to reinterpret this.
- We can begin with linearised Yang-Mills theory, and require the gauge field to be Hamiltonian.
- Then we can make a Poisson bracket, and postulate an interaction term that combines this with the colour Lie algebra.
- Self-dual Yang-Mills theory is one possible case, but other choice of $\Omega_{\mu\nu}$ give alternative theories, that have been briefly explored before (Bjerrum-Bohr, Damgaard, Monteiro, O'Connell; Chacón, García-Compeán, Luna, Monteiro, White).
- The kinematic algebra of the interacting theory is made out of building blocks already present at linear level, which have a clear geometric interpretation.
- Furthermore, this recipe can be used to generate new examples of theories with simple kinematic algebras!

Electromagnetism coupled to scalar matter

- Let us start with pure electromagnetism, and require the gauge field to be Hamiltonian: $A_\mu = \Omega_{\mu\nu}\partial^\nu\phi \quad \Rightarrow \quad \partial^2\phi = 0.$
- To find an interacting version with a kinematic algebra, we can again make a Poisson bracket out of $\Omega_{\mu\nu}$.
- For this to be non-zero, we need a second scalar field ψ, and can postulate the field equation

$$\partial^2\psi + c_1\{\phi,\psi\} = 0.$$

- Requiring invariance under gauge transformations that preserve $\partial \cdot A = 0$ gives

$$c_1 = -2ie, \quad \Omega_{\mu\nu}\Omega^\mu{}_\alpha = 0.$$

- There are indeed many solutions, such that (a closed subsector of) EM coupled to scalar matter has a well-defined kinematic Lie algebra.

Chern-Simons theory

- As well as using the Poisson bracket to make interaction terms, one can also work with A_μ directly, and make a double bracket using the Lie bracket of diffeomorphisms.
- As an example, we may consider abelian Chern-Simons theory, whose field equation is

$$\epsilon^{\mu\nu\rho}\left[\partial_\nu A_\rho - \partial_\rho A_\nu\right] = 0.$$

- After contracting with $\epsilon_{\sigma\mu\alpha}\partial^\alpha$ to get a second-order equation, we can add an interaction proportional to the double Lie bracket

$$\Big[\big[\mathbf{A},\mathbf{A}\big]\Big] = \mathbf{T}^a f^{abc}\left[A^{\mu\,b}\partial_\mu, A^{\nu\,c}\partial_\nu\right], \quad \mathbf{A} = A^{\mu a}\mathbf{T}^a\partial_\mu$$

Chern-Simons theory

- Requiring the form of the equation to remain invariant under (partial) gauge transformations, one finds

$$\partial \cdot A^{\mu a} = 0.$$

- Thus, a Lie kinematic algebra **in the equation of motion** is only obtained in Lorenz gauge,verifying Ben-Shahar & Johansson.
- In other gauges, the interaction terms do not have a direct Lie algebra interpretation…
- …although a Lie algebra for this theory (without gauge fixing) turns out to be possible in the $\mathrm{BV}^{\square}_{\infty}$ approach (Bonezzi, Chiaffrino, Díaz-Jaramillo, Hohm).
- Would be nice to relate these different approaches 😀

Fluid mechanics

- A further novel example of a theory with a Lie kinematic algebra is a non-abelian Navier-Stokes equation considered by Cheung & Mangan.

- In terms of a non-abelian velocity field $\mathbf{u} \equiv u_i^a \mathbf{T}^a \partial_i$, its equation of motion is

$$(\partial_0 - \nu\nabla^2)\mathbf{u} + \frac{1}{2}[[\mathbf{u}, \mathbf{u}]] = \mathbf{J}$$

Viscosity

Double Lie Bracket

Source current

- Again the interaction term contains building blocks already present at linear level.

- In this case, there is a very physical origin for the double Lie bracket, in terms of **convection**.

- Might further useful physical insights be gained from this, or from a recent gauge theory for shallow water waves (Tong; Sheikh-Jabbari, Taghiloo, Vahidinia)?

Conclusions

- Kinematic algebras are an intriguing phenomenon of field theories.
- General mathematical frameworks exist in principle for characterising them…
- …but many questions about these algebras remain.
- One can gain useful intuition about kinematic algebras by considering linearised theories…
- …and using their (subgroups of) diffeomorphisms to construct interacting theories.
- This allows us to obtain cases of theories with Lie kinematic algebras, and also to probe when the Lie algebras may break down.

Open questions

- How are these ideas related to the study of **homotopy algebras**?
- Can we get useful physical insights about kinematic algebras by looking at fluid mechanics?
- Can we make interesting new interacting theories out of abelian building blocks, that have geometrically visualisable kinematic algebras?
- Might some of these theories be useful for condensed matter physics?
- What is the proper geometric interpretation of the double Lie bracket?
- If a kinematic algebra is not Lie, is there some alternative description of gauge theories (not fibre-bundle based) that gives a geometric meaning to the kinematic algebra?