Abelian insights on Kinematic Algebras

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Overview

What are kinematic algebras, and what can we learn about them by studying simple (abelian) gauge theories?

- Introduction to BCJ duality.
- Kinematic algebras and known special cases.
- The diffeomorphism algebra of (restricted) abelian gauge theories.
- Interacting theories from abelian building blocks.
- Open questions.
Amplitudes and colour factors

• Scattering amplitudes in non-abelian gauge theories can be obtained by considering certain cubic graphs, each of which has a colour factor $C_i$.

• Each graph also has a kinematic numerator $n_i$, which is a function of momenta, polarisation etc.

• Colour factors of different graphs are related by Jacobi identities.

• Remarkably, the kinematic numerators can be chosen to obey similar identities.

• This is called BCJ Duality (Bern, Carrasco, Johansson), and is conjectured at all loop orders.
Kinematic algebras

• The Jacobi identity for colour factors is a consequence of the colour Lie algebra.

• Thus, BCJ duality suggests there is a **kinematic algebra**, that somehow mirrors the colour algebra.

• For some very special cases, this is known to be a Lie algebra:

  - Self-dual Yang-Mills theory in lightcone gauge
  - Area-preserving diffeomorphisms
  - Monteiro, O’Connell
  - Chern-Simons Theory in Lorenz gauge
  - Volume-preserving diffeomorphisms
  - Ben-Shahar, Johansson
General kinematic algebras

• More generally, kinematic algebras are not expected to be Lie algebras…

• …but more general mathematical structures.

• A particular candidate is a so-called BV∞ algebra, or homotopy algebra (Reiterer; then Borsten, Jurco, Kim, Macrelli, Saemann, Wolf; Bonezzi, Chiaffrino, Diaz-Jaramilo, Hohm, Plefka).

• There are higher-order brackets that generalise the Lie bracket of gauge fields.

• Jacobi identities are satisfied only up to terms involving higher-order brackets, leading to an intricate structure of constraints.

• Somehow, this must all reduce to a straightforward Lie algebra in those cases where this is possible.
Why do we care?

- Kinematic algebras remain mysterious.
- But they are relatively **new, exciting** and certainly **fun 😊**!
- New way to think about gauge theories?
- If so, we should explore all ways of gaining intuition about the many open questions.

- Which theories have kinematic algebras?
- Are previous known cases related?
- How do kinematic algebras depend upon the gauge?
- When do we get a Lie algebra?
- How do we visualise kinematic algebras geometrically?
Abelian insights

• It is common in physics to use simple cases to gain intuition.

• To this end, let us look at abelian gauge theories (or linearised non-abelian).

• BCJ duality is conventionally a purely non-linear phenomenon: structure constants colour / kinematic algebras occur together.

• But there can also be “colour-kinematics” duality at linearised level.

• A non-abelian gauge field is Lie-algebra valued in two Lie algebras:

\[ A = A^{\mu a} \mathbf{T}^a \partial_\mu \]
Diffeomorphisms

• Diffeomorphism = simultaneous translation along all integral curves (field lines) of the vector field.

• In known cases where linear solutions double / zeroth copy, this amounts to replacing generators of colour / kinematic algebras appropriately.

• This is colour / kinematics duality involving generators, rather than structure constants.

• We can associate the “kinematic algebra” of a linear theory with these diffeomorphisms… (see also Fu, Krasnov)

• …and will eventually see how these ideas are relevant for exploring non-linear theories.
Abelian theories

• A particular $A_\mu$ is a point in the space of all possible diffeomorphisms:

• There is no non-trivial kinematic algebra: diffeomorphisms along the same field lines commute.

• We can instead associate a non-trivial algebra if we can find a set of solutions closed under the Lie algebra of diffeomorphisms:

\[
[A^{(1)}_{\mu} \partial_\mu, A^{(2)}_{\nu} \partial_\nu] = A^{(3)}_{\mu} \partial_\mu
\]

\[
A^{(3)}_{\mu} = A^{(1)} \cdot \partial A^{(2)}_{\mu} - A^{(2)} \cdot \partial A^{(1)}_{\mu}
\]
Closed subgroups of diffeomorphisms

- There are indeed subgroups of the diffeomorphism group.
- A notable one is **volume-preserving diffeomorphisms**.
- The “volume” being preserved is that of a shape which is dragged along the field lines.
- Turns out to imply $\partial \cdot A = 0$, whose physical interpretation is that we are in **Lorenz gauge**.
- Further subgroups exist, corresponding to preserving volume in a lower-dimensional subspace.
Symplectomorphisms

- Another interesting subgroup is that of symplectomorphisms (a.k.a. canonical transformations).

- The gauge field is a so-called Hamiltonian vector field, with the special form

\[ A^\mu = \hat{k}^\mu \phi, \quad \hat{k}_\mu = \Omega_{\mu\nu} \partial^{\nu}, \quad \Omega_{\mu\nu} = - \Omega_{\nu\mu} \]

- This is a subset of solutions of the theory i.e. more than just a gauge choice.

- **Self-dual** electromagnetism in lightcone gauge is a special case.
Poisson brackets

- If we have a symplectic form, we can form a **Poisson bracket**:
  \[
  \{ \phi_1, \phi_2 \} = \Omega^{\mu\nu} (\partial_\mu \phi_1)(\partial_\nu \phi_2)
  \]

- The Lie bracket of two Hamiltonian vector fields is also Hamiltonian, where the relevant scalar function is given by the Poisson bracket of the two original scalars:
  \[
  [(\hat{k}^\mu \phi_1)\partial_\mu, (\hat{k}^\nu \phi_2)\partial_\nu] = (\hat{k}^\mu \phi_3)\partial_\mu, \quad \phi_3 = - \{ \phi_1, \phi_2 \}
  \]

- Thus, the Poisson bracket encodes the Lie algebra of diffeomorphisms generated by Hamiltonian vector fields $A_\mu$. 
Lightcone gauge electromagnetism

• As an example of these ideas, consider the coordinate system

\[ u = \frac{t-z}{\sqrt{2}}, \quad v = \frac{t+z}{\sqrt{2}}, \quad X = \frac{x+iy}{\sqrt{2}}, \quad Y = \frac{x-iy}{\sqrt{2}} \]

• A gauge field in the lightcone gauge \( A_u = 0 \) can be shown to have the form

\[ A_\mu = \hat{k}_\mu \phi + \hat{k}_\mu^\dagger \phi^\dagger, \quad \hat{k}_\mu \equiv (0, \partial_Y, \partial_u, 0) \]

• The two terms can be shown to generate diffeomorphisms in families of \((u,X)\) and \((u,Y)\) planes respectively.

• Furthermore, they are area-preserving in these planes.

• Thus, we can directly visualise the “kinematic algebra” of lightcone gauge EM.
**Lightcone gauge electromagnetism**

- The “kinematic algebra” of electromagnetism in lightcone gauge will be a subgroup of
  \[
  \text{Diff}_{(u,X)} \times \text{Diff}_{(u,Y)},
  \]
picked out by the gauge field being real in Lorentzian signature.

- One may show in general that this comprises 3d-volume preserving diffeomorphisms in the \((u,x,y)\) subspace...

- …that become area-preserving diffeomorphisms in the \((u,x)\) or \((u,y)\) planes if \(\phi\) is pure real or imaginary.
Gauge dependence of kinematic algebras

- The gauge dependence of kinematic algebras can also be thought about at linearised level.

- A **gauge orbit** (set of gauge fields related by gauge transformations) shows up as a line in the space of all possible diffeomorphisms.

- If we start with a Hamiltonian vector field $A_\mu$, gauge transformations act as

  $$A_\mu \rightarrow A_\mu + \partial_\mu \alpha$$

- If $\alpha$ is harmonic, gauge transformations will move $A_\mu$ to the set of volume-preserving diffeomorphisms.

- No special set will be obtained for general $\alpha$. 
Interacting theories

• Of course, usually by “kinematic algebras”, we mean structures that act on interacting theories.

• The canonical example is self-dual Yang-Mills theory, whose lightcone gauge field equation may be written as

\[ \partial^2 \Phi + g\{[\Phi, \Phi]\} = 0 \]

Scalar field \( \Phi = \Phi^a T^a \)

Coupling constant

Double bracket

\[ \{[\Phi, \Phi]\} = \Omega^{\mu
\nu} f^{abc}(\partial_\mu \Phi^b)(\partial_\nu \Phi^c)T^a \]

• The gauge field is Hamiltonian: \( A^a_\mu = \hat{k}_\mu \Phi^a \), and the double bracket combines the colour Lie bracket with the Poisson bracket for the scalar components.

• The kinematic algebra is one of area-preserving diffeomorphisms.
There is an interesting way to reinterpret this.

We can begin with linearised Yang-Mills theory, and require the gauge field to be Hamiltonian.

Then we can make a Poisson bracket, and postulate an interaction term that combines this with the colour Lie algebra.

Self-dual Yang-Mills theory is one possible case, but other choice of $\Omega_{\mu\nu}$ give alternative theories, that have been briefly explored before (Bjerrum-Bohr, Damgaard, Monteiro, O’Connell; Chacón, García-Compeán, Luna, Monteiro, White).

The kinematic algebra of the interacting theory is made out of building blocks already present at linear level, which have a clear geometric interpretation.

Furthermore, this recipe can be used to generate new examples of theories with simple kinematic algebras!
Electromagnetism coupled to scalar matter

• Let us start with pure electromagnetism, and require the gauge field to be Hamiltonian: $A_\mu = \Omega_{\mu\nu} \partial^\nu \phi \implies \partial^2 \phi = 0$.

• To find an interacting version with a kinematic algebra, we can again make a Poisson bracket out of $\Omega_{\mu\nu}$.

• For this to be non-zero, we need a second scalar field $\psi$, and can postulate the field equation

$$\partial^2 \psi + c_1 \{ \phi, \psi \} = 0.$$  

• Requiring invariance under gauge transformations that preserve $\partial \cdot A = 0$ gives

$$c_1 = -2ie, \quad \Omega_{\mu\nu} \Omega^\mu_\alpha = 0.$$  

• There are indeed many solutions, such that (a closed subsector of) EM coupled to scalar matter has a well-defined kinematic Lie algebra.
Chern-Simons theory

• As well as using the Poisson bracket to make interaction terms, one can also work with $A_\mu$ directly, and make a double bracket using the Lie bracket of diffeomorphisms.

• As an example, we may consider abelian Chern-Simons theory, whose field equation is

$$
\epsilon^{\mu\nu\rho} \left[ \partial_\nu A_\rho - \partial_\rho A_\nu \right] = 0.
$$

• After contracting with $\epsilon_{\sigma\mu\alpha}\partial^{\alpha}$ to get a second-order equation, we can add an interaction proportional to the double Lie bracket

$$
\left[ [A, A] \right] = T^a f^{abc} \left[ A_\mu^b \partial_\mu, A_\nu^c \partial_\nu \right], \quad A = A^{\mu a} T^a \partial_\mu
$$
Chern-Simons theory

• Requiring the form of the equation to remain invariant under (partial) gauge transformations, one finds

\[ \partial \cdot A^{\mu a} = 0. \]

• Thus, a Lie kinematic algebra in the equation of motion is only obtained in Lorenz gauge, verifying Ben-Shahar & Johansson.

• In other gauges, the interaction terms do not have a direct Lie algebra interpretation…

• …although a Lie algebra for this theory (without gauge fixing) turns out to be possible in the BV\(_\infty\) approach (Bonezzi, Chiaffrino, Díaz-Jaramillo, Hohm).

• Would be nice to relate these different approaches 😊
Fluid mechanics

• A further novel example of a theory with a Lie kinematic algebra is a non-abelian Navier-Stokes equation considered by Cheung & Mangan.

• In terms of a non-abelian velocity field $\mathbf{u} \equiv u_i^a T^a \partial_i$, its equation of motion is

$$ (\partial_0 - \nu \nabla^2) \mathbf{u} + \frac{1}{2} [[\mathbf{u}, \mathbf{u}]] = \mathbf{J} $$

- Viscosity
- Double Lie Bracket
- Source current

• Again the interaction term contains building blocks already present at linear level.

• In this case, there is a very physical origin for the double Lie bracket, in terms of convection.

• Might further useful physical insights be gained from this, or from a recent gauge theory for shallow water waves (Tong; Sheikh-Jabbari, Taghiloo, Vahidinia)?
Conclusions

• Kinematic algebras are an intriguing phenomenon of field theories.

• General mathematical frameworks exist in principle for characterising them...

• ...but many questions about these algebras remain.

• One can gain useful intuition about kinematic algebras by considering linearised theories...

• ...and using their (subgroups of) diffeomorphisms to construct interacting theories.

• This allows us to obtain cases of theories with Lie kinematic algebras, and also to probe when the Lie algebras may break down.
Open questions

• How are these ideas related to the study of homotopy algebras?

• Can we get useful physical insights about kinematic algebras by looking at fluid mechanics?

• Can we make interesting new interacting theories out of abelian building blocks, that have geometrically visualisable kinematic algebras?

• Might some of these theories be useful for condensed matter physics?

• What is the proper geometric interpretation of the double Lie bracket?

• If a kinematic algebra is not Lie, is there some alternative description of gauge theories (not fibre-bundle based) that gives a geometric meaning to the kinematic algebra?