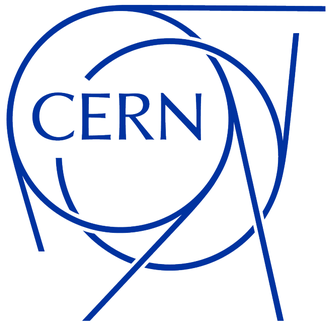
Multi-leg amplitudes in QCD

Simone Zoia

QCD meets Gravity 2023, CERN, 15th Dec 2023



Precision physics at the Large Hadron Collider

Run 3 of the LHC has just started

We have seen just a few percent of the total data the LHC is expected to deliver

Improvements in systematic uncertainties

⇒ Measurements at the **percent-level precision** will soon be available

We **must** keep the theoretical and experimental uncertainties in line

➔ A. Huss' talk

Precision demands at least NNLO QCD

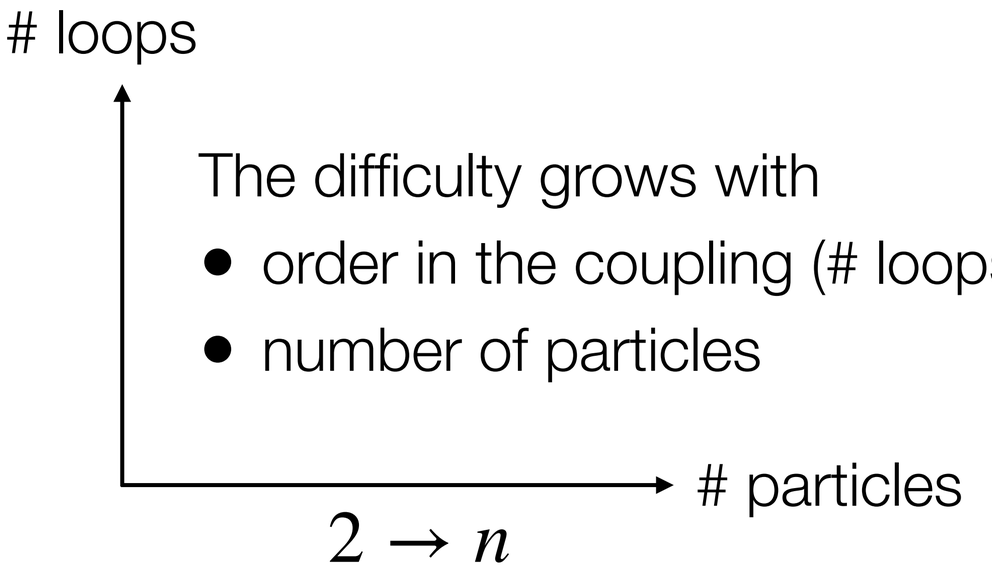
$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{LO}} + \alpha_{\mathrm{S}}\,\mathrm{d}\sigma^{\mathrm{NLO}} + \alpha_{\mathrm{S}}^{2}\,\mathrm{d}\sigma^{\mathrm{NNLO}} + \ldots$$

$\approx 10 - 30\,\%$ $\approx 1 - 10\,\%$

Precision demands at least NNLO QCD

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{LO}} + \alpha_{\mathrm{S}}\,\mathrm{d}\sigma^{\mathrm{NLO}} + \alpha_{\mathrm{S}}^{2}\,\mathrm{d}\sigma^{\mathrm{NNLO}} + \ldots$$

$\approx 10 - 30\,\%$ $\approx 1 - 10\,\%$



The difficulty grows with

- order in the coupling (# loops)
- number of particles

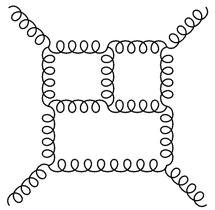
Precision demands at least NNLO QCD

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{LO}} + \alpha_S\,\mathrm{d}\sigma^{\mathrm{NLO}} + \alpha_S^2\,\mathrm{d}\sigma^{\mathrm{NNLO}} + \ldots$$

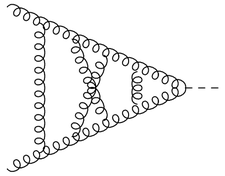
$\approx 10 - 30\,\%$ $\approx 1 - 10\,\%$

Loop frontier

N^{3}LO $2 \to 2$



N^{4}LO $2 \to 1$



loops

The difficulty grows with

- order in the coupling (# loops)
- number of particles

particles

$2 \to n$

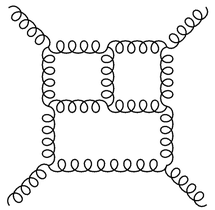
Precision demands at least NNLO QCD

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{LO}} + \alpha_S\,\mathrm{d}\sigma^{\mathrm{NLO}} + \alpha_S^2\,\mathrm{d}\sigma^{\mathrm{NNLO}} + \ldots$$

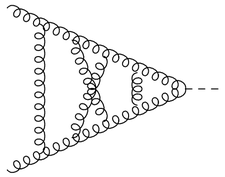
$\approx 10 - 30\,\%$ $\approx 1 - 10\,\%$

Loop frontier

N^{3}LO $2 \to 2$



N^{4}LO $2 \to 1$



loops

The difficulty grows with

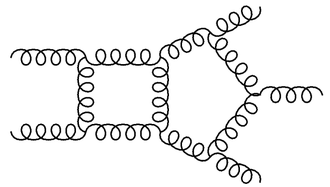
- order in the coupling (# loops)
- number of particles

particles

$2 \to n$

Multiplicity frontier

NNLO $2 \to 3$



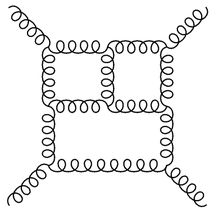
Precision demands at least NNLO QCD

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{\mathrm{LO}} + \alpha_S\,\mathrm{d}\sigma^{\mathrm{NLO}} + \alpha_S^2\,\mathrm{d}\sigma^{\mathrm{NNLO}} + \ldots$$

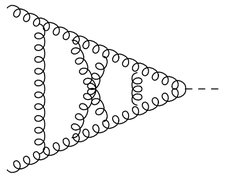
$\approx 10 - 30\,\%$ $\approx 1 - 10\,\%$

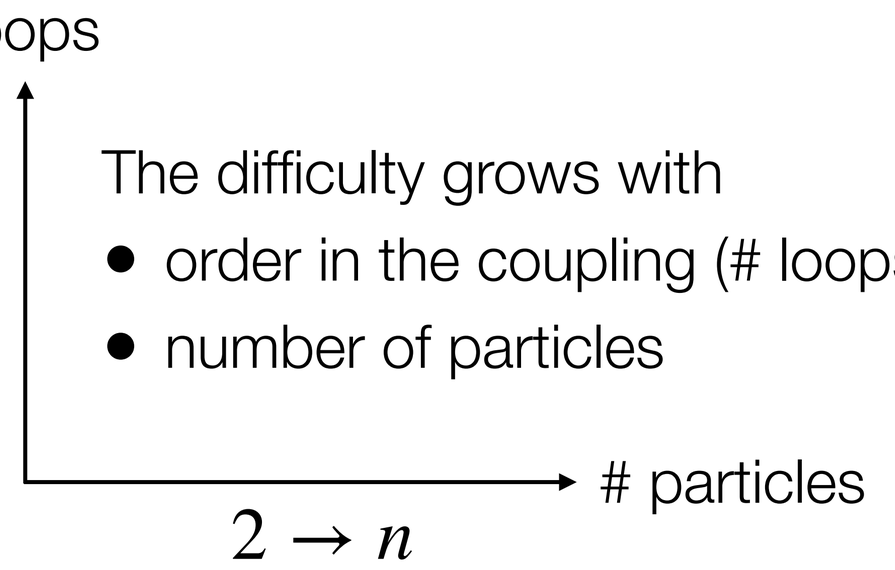
Loop frontier

N^{3}LO $2 \to 2$



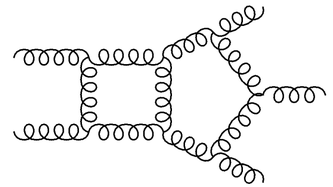
N^{4}LO $2 \to 1$





Multiplicity frontier

NNLO $2 \to 3$



Bottleneck: 2-loop 5-pt amplitudes

Explosion of results in the last couple of years

Massless

- 3γ *[Abreu, Page, Pascual, Sotnikov 2020; Chawdhry, Czakon, Mitov, Poncelet 2021; Abreu, De Laurentis, Ita, Klinkert, Page, Sotnikov, 2023]*
- $2\gamma + \mathrm{j}$ *[Agarwal, Buccioni, von Manteuffel, Tancredi 2021; Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Brönnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]*
- $\gamma + 2\mathrm{j}$ *[Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, **SZ** 2023]*
- $3\mathrm{j}$ *[Abreu, Febres-Cordero, Ita, Page, Sotnikov 2021; De Laurentis, Ita, Klinkert, Sotnikov 2023; Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi 2023; De Laurentis, Ita, Sotnikov 2023]*

1 external mass

- $\mathrm{W} + \mathrm{b\bar{b}}$ (planar) *[Badger, Bayu Hartanto, **SZ** 2021; Bayu Hartanto, Poncelet, Popescu, **SZ** 2022]*
- $\mathrm{W} + 2\mathrm{j}$ (planar) *[Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov 2022]*
- $\mathrm{H} + \mathrm{b\bar{b}}$ (planar) *[Badger, Bayu Hartanto, Kryś, **SZ** 2021]*
- $\mathrm{W} + \gamma + \mathrm{j}$ (planar) *[Badger, Bayu Hartanto, Kryś, **SZ** 2022]*

Ready for deployment in NNLO QCD phenomenology

Leading colour @ 2 loops

- $pp \to 3\gamma$ *[Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]*
- $pp \to 2\gamma + j$ *[Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]*
- $pp \to 3j$ *[Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]*
- $pp \to W + b\bar{b}$ *[Bayu Hartanto, Poncelet, Popescu, **SZ** 2022; Buonocore, Devoto, Kallweit, Mazzitelli, Rottoli, Savoini 2023]*
- $pp \to \gamma + 2j$ *[Badger, Czakon, Bayu Hartanto, Moodie, Peraro, Poncelet, **SZ** 2023]*

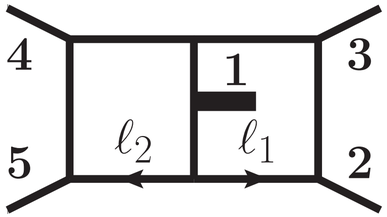
Important milestones:

All massless 2→3 LHC processes now analysed at NNLO QCD ✓

All their two-loop amplitudes known analytically in full colour ✓

Algebraic and analytic complexity

amplitude $= \sum$ **algebraic coeffs** $\times$ **special funcs**



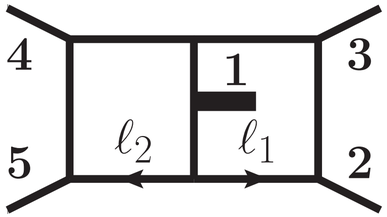
6 scalar invariants
10 square roots

Many variables, complicated algebraic expressions, terabytes of data ⚠️

Feynman integrals
Intricate branch cuts
Functional identities
Which functions?! ⚠️

Algebraic and analytic complexity

amplitude $= \sum$ **algebraic coeffs** $\times$ **special funcs**



6 scalar invariants
10 square roots

Many variables, complicated algebraic expressions, terabytes of data ⚠️

Finite-field arithmetic & functional reconstruction

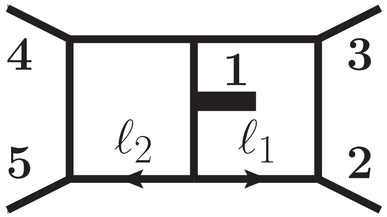
Partial fraction decompositions

➔ **this talk**

Feynman integrals
Intricate branch cuts
Functional identities
Which functions?! ⚠️

Algebraic and analytic complexity

amplitude $= \sum$ **algebraic coeffs** $\times$ **special funcs**



6 scalar invariants
10 square roots

Many variables, complicated algebraic expressions, terabytes of data ⚠️

Finite-field arithmetic & functional reconstruction

Partial fraction decompositions

➔ **this talk**

Feynman integrals
Intricate branch cuts
Functional identities
Which functions?! ⚠️

Canonical DEs &
Chen iterated integrals

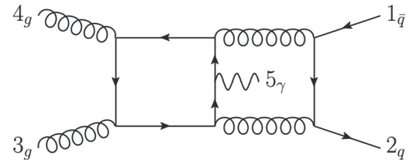
➔ **Vasily Sotnikov's talk**

Amplitude workflow

Amplitude
integrand

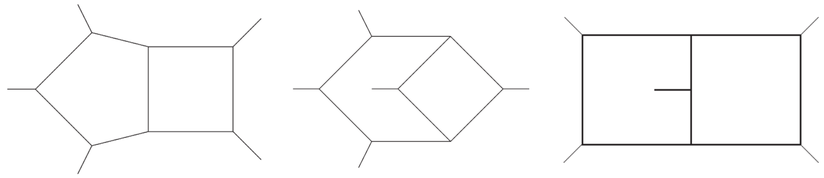
Helicity amplitude, projector method…

Feynman diagrams, unitarity methods…



$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i c_i\left(\{p\},\epsilon\right)\ \mathrm{I}_i\left(\{p\},\epsilon\right)$$

Scalar Feynman integrals



$$\mathrm{I} = \int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2}{k_1^2 (k_1+p_1)^2 (k_1+p_1+p_2)^2 \ldots}$$

Integration-by-parts (IBPs)

➔ **J. Usovitsch's talk**

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\ \mathrm{MI}_i\left(\{p\},\epsilon\right)$$

"Master integrals"

Simplification from the ϵ-expansion

Amplitudes needed only up to a certain order in $\epsilon = (4 - D)/2$

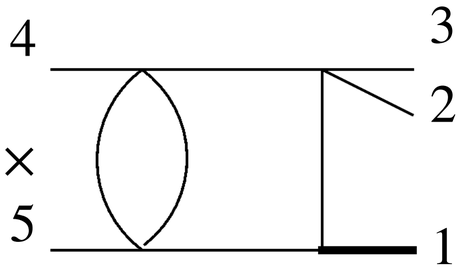
- Significantly simpler expressions
- Check IR/UV poles (IR factorisation + UV renormalisation)

$$A^{(2)}\left(\{p\}, \epsilon\right) = \sum_i d_i\left(\{p\}, \epsilon\right) \, \mathrm{MI}_i\left(\{p\}, \epsilon\right)$$

We need the ϵ-expansion of the master integrals ⇒ **Special functions!**

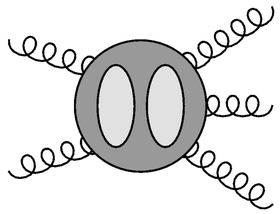
Pentagon functions

Choose **pure MIs** and map them onto a **basis** of special functions

$$\epsilon^3(1-2\epsilon)\sqrt{\Delta_3^{(1)}} \times$$ 

$$= \epsilon^2 f_{23}^{(2)} + \epsilon^3 \left[\frac{1}{4}\left(f_1^{(1)} - f_6^{(1)}\right) f_{23}^{(2)} + \frac{1}{2} f_3^{(3)} - \frac{1}{2} f_{29}^{(3)}\right] + \epsilon^4 f_{47}^{(4)} + \mathcal{O}\left(\epsilon^5\right)$$

Efficient numerical evaluation based on 1-fold integrations

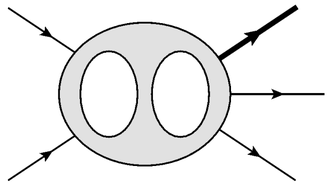


Massless 2-loop 5-point

[Gehrmann, Henn, Lo Presti 2018]
[Chicherin, Sotnikov 2020]

1-mass 2-loop 5-point

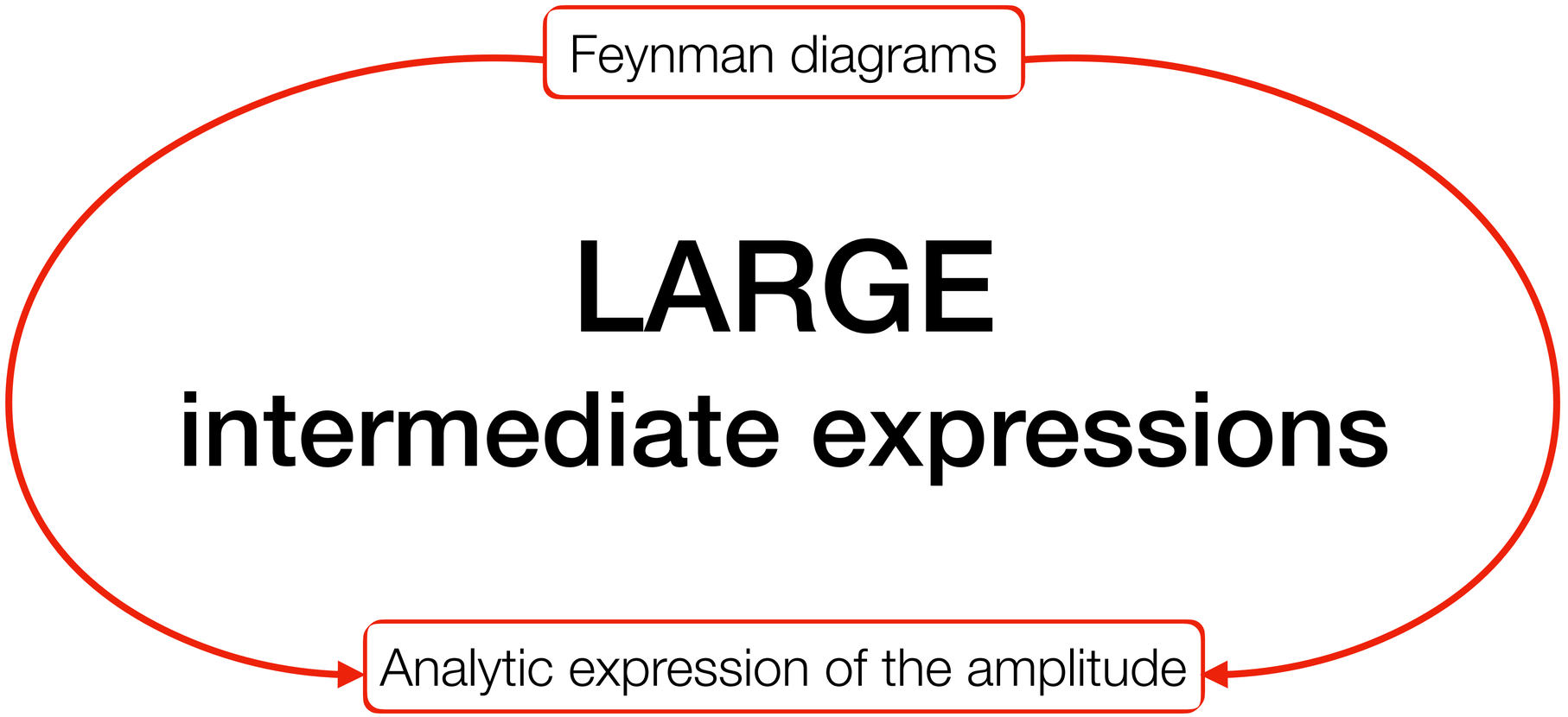
*[Chicherin, Sotnikov, **SZ** 2022; Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, **SZ** 2023]*



⇒ Unique, simpler expression of the amplitudes 👍

⇒ Efficient numerical evaluation 👍

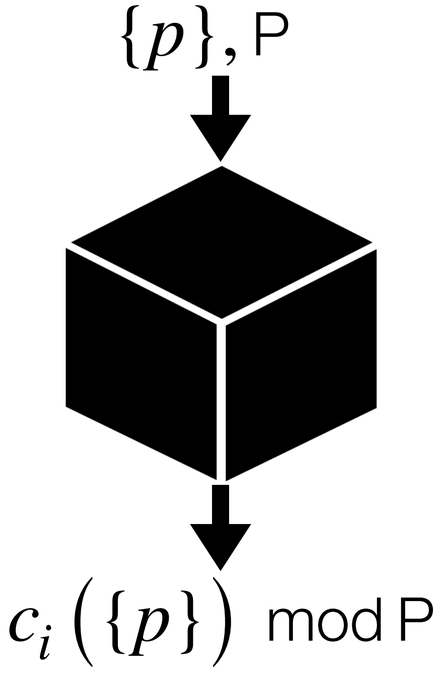
➔ **V. Sotnikov's talk**


Feynman diagrams
LARGE
intermediate expressions
Analytic expression of the amplitude

Finite field revolution

[von Manteuffel, Schabinger 2015; Peraro 2016]

- Evaluate rational functions at numerical integer points $(\{p\}, \epsilon)$ modulo prime number $\longrightarrow$ finite field
- Perform intermediate rational operations numerically
- Reconstruct the analytic expression of the final result from multiple numerical evaluations



Sidestep intermediate expression swell

What to reconstruct?

1. Reconstruct the IBP tables

$$\mathrm{I}_i\left(\{p\},\epsilon\right) = \sum_j C_{ij}\left(\{p\},\epsilon\right)\ \mathrm{MI}_j\left(\{p\},\epsilon\right)$$

Exceedingly bulky (terabytes of data)

Spectacular simplification through **multivariate partial fraction decomposition**

[Leinartas '78; Raichev 2012; Meyer 2016; Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 2019; Heller, von Manteuffel 2021; Bendle, Boehm, Heymann, Ma, Rahn, Ristau, Wittmann, Wu, Xu, Zhang 2021]

`MultivariateApart` *[Heller, von Manteuffel 2021]*

`pfd` *[Bendle, Boehm, Heymann, Ma, Rahn, Ristau, Wittmann, Wu, Xu, Zhang 2021]*

```
MultivariateApart[(2 y - x) / (y (x + y) (y - x))]
```

$$-\frac{1}{2\,(x-y)\,y}+\frac{3}{2\,y\,(x+y)}$$

Use simplified IBP reductions in amplitude computation

E.g. 2-loop 5-parton amplitudes for 3-jet production *[Agarwal, Buccioni, Devoto, Gambuti, von Manteuffel, Tancredi 2023]*

2. Reconstruct the amplitude coefficients

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \text{Feynman diagram}_i$$

IBP reduction

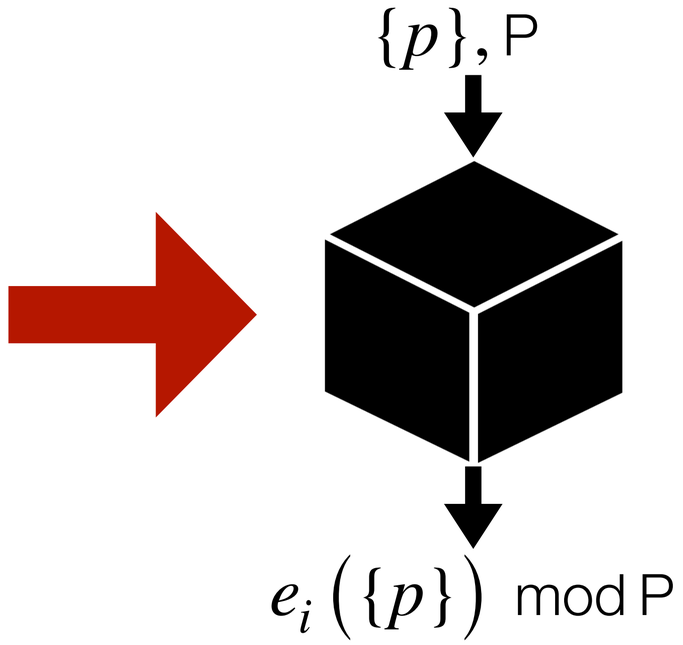
$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\ \text{MI}_i\left(\{p\},\epsilon\right)$$

ϵ expansion

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_{w=-4}^{\infty} \epsilon^w \sum_i c_i^{(w)}\left(\{p\}\right)\ \text{mon}_{w,i}\left[f\right]$$

IR/UV subtraction

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \text{mon}_i\left[f\right]$$



FiniteFlow *[Peraro 2019]*

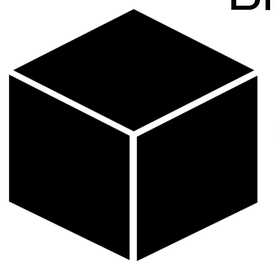
FireFly *[Klappert, Lange 2019]*

Caravel *[Abreu, Dormans, Febres Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov 2020]*

Rational reconstruction

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \mathrm{mon}_i\left[f\right]$$

Black box interpolation



$$\left(\{p\}, \mathrm{P}\right) \rightarrow \qquad \rightarrow e_i\left(\{p\}\right) \bmod \mathrm{P}$$

$e_i(\{p\})$ reconstructed in $\dfrac{\#\ \text{points} \times \text{eval. time}}{\#\ \text{CPUs}}$

Rational reconstruction

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \mathrm{mon}_i\left[f\right]$$

Black box interpolation

$$\left(\{p\}, \mathrm{P}\right) \rightarrow \blacksquare \rightarrow e_i\left(\{p\}\right) \bmod \mathrm{P}$$

$e_i(\{p\})$ reconstructed in $\dfrac{\#\ \text{points} \times \text{eval. time}}{\#\ \text{CPUs}}$ ⏳

Make a good ansatz

💰

Improve the IBP system

➔ J. Usovitsch's talk

Guessing the denominators

$$F^{(2)}\left(\{p\}\right) = \sum_i e_i\left(\{p\}\right)\ \mathrm{mon}_i\left[f\right] \qquad e_i(s) = \frac{N_i(s)}{D_i(s)} \qquad s := \{p\}$$

Ansatz for the denominators informed by singularities of Feynman integrals

$$D_i = \prod_j w_j^{a_j}$$

Determined by reconstructing the coefficients on univariate phase-space slices

[Abreu, Dormans, Febres Cordero, Ita, Page 2018]

$$e_i(s) = \frac{N_i(s)}{D_i(s)}$$

Degrees known

Entirely known ✓

Can we use this information to construct an ansatz for the numerators?

Univariate partial fraction decomposition

*[Badger, Bayu Hartanto, **SZ** 2021; Badger, Brönnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Peraro, **SZ** 2021]*

$$e(x,y) = \frac{-2x^4 - 4x^3y + 5x^2y^2 - xy^3 + 4y^4}{(x-y)y^2(x^2+y^2)} = -\frac{2x}{y^2} - \frac{6}{y} + \frac{1}{x-y} + \frac{3y}{x^2+y^2}$$

Construct ansatz based on the knowledge of degrees and denominators

$$e(x,y) = \frac{q_1(x)}{y^2} + \frac{q_2(x)}{y} + \frac{q_3(x)}{x-y} + \frac{q_4(x) + q_5(x)\,y}{x^2+y^2}$$

Linear fit to reconstruct the coefficients $q_i(x)$

- One fewer variables
- Lower degrees

Ansätze from singularity constraints

Characterize the **divergences** of the coefficients, and construct ansatz matching observed behaviour in terms of **spinor helicity variables**

[De Laurentis, Maître 2020; De Laurentis, Page 2022]

Combine coefficients to decorrelate poles and reduce order of **unphysical poles** systematically

All rational coefficients of 2-loop 5-gluon amplitudes (full colour!) spelled out explicitly in the paper!

[De Laurentis, Ita, Klinkert, Sotnikov 2023; De Laurentis, Ita, Sotnikov 2023]

$$\tilde{r}_1^{--} = \frac{\langle 45\rangle^2}{\langle 12\rangle\langle 13\rangle\langle 23\rangle}$$

$$\tilde{r}_2^{--} = \frac{\langle 45\rangle^3}{\langle 12\rangle^2\langle 34\rangle\langle 35\rangle}$$

$$\tilde{r}_3^{--} = \frac{\langle 45\rangle^3}{\langle 12\rangle\langle 15\rangle\langle 23\rangle\langle 34\rangle}$$

$$\tilde{r}_4^{--} = \frac{[14][12][35]}{\langle 23\rangle[45]^3}$$

$$\tilde{r}_5^{--} = \frac{\langle 45\rangle^2\langle 24\rangle}{\langle 12\rangle^2\langle 23\rangle\langle 34\rangle}$$

Conclusions

Efficient methodology for multi-particle computations based on

- **finite-field arithmetic** to keep algebraic complexity under control
- **functional reconstruction** enhanced by physical ideas / algebraic geometry

All massless 2→3 LHC processes analysed at NNLO QCD (leading colour)

All massless 2-loop 5-particle amplitudes known analytically, full colour

Conclusions

Efficient methodology for multi-particle computations based on

- **finite-field arithmetic** to keep algebraic complexity under control
- **functional reconstruction** enhanced by physical ideas / algebraic geometry

All massless 2→3 LHC processes analysed at NNLO QCD (leading colour)

All massless 2-loop 5-particle amplitudes known analytically, full colour

Thank you!

Substantial drop in the polynomial degrees

Set $s_{12} = 1$

Most complicated 2-loop partial amplitudes for $0 \to \bar{q}qgg\gamma$

Most complicated @ leading colour

		4 variables			3 variables	
amplitude	helicity	original	stage 1	stage 2	stage 3	stage 4
$A^{(2),1}_{34;q}$	$-++-+$	94/91	74/71	74/0	22/18	22/0
$A^{(2),1}_{34;q}$	$-+-++$	93/89	90/86	90/0	24/14	18/0
$A^{(2),1/N_c^2}_{34;q}$	$-++-+$	90/88	73/71	73/0	23/18	22/0
$A^{(2),1/N_c^2}_{34;q}$	$-+-++$	90/86	86/82	86/0	24/14	19/0
$A^{(2),1/N_c}_{34;l}$	$-+-++$	89/82	74/67	73/0	27/14	20/0
$A^{(2),1/N_c}_{34;l}$	$-++-+$	85/81	61/58	60/0	27/18	20/0
$A^{(2),N_c^2}_{34;q}$	$-+-++$	58/55	54/51	53/0	20/16	20/0

$$e_i(s) = \frac{N_i(s)}{D_i(s)}$$

Linear relations

Partial fraction

Denominator guessing

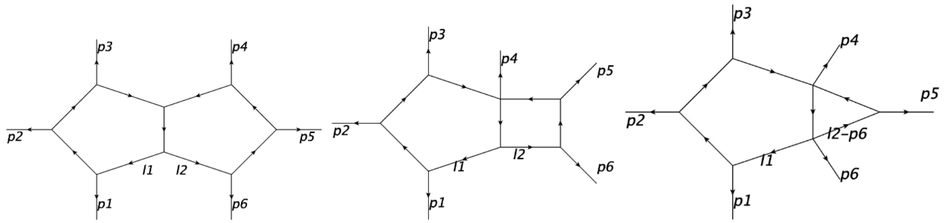
Example:

$$e(x,y) = \frac{-2x^4 - 4x^3y + 5x^2y^2 - xy^3 + 4y^4}{(x-y)y^2(x^2+y^2)} \to 4/5$$

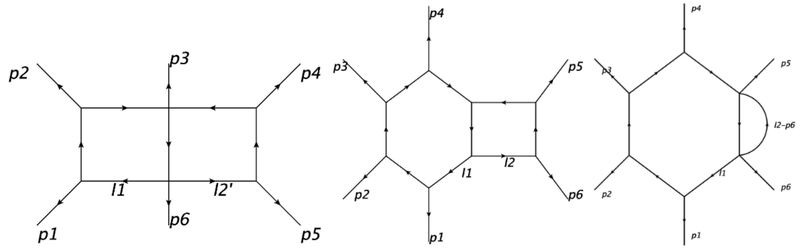
Going beyond 5 particles/massless propagators

Planar 2-loop 6-point Feynman integrals on the maximal cuts

[Henn, Peraro, Xu, Zhang 2022]



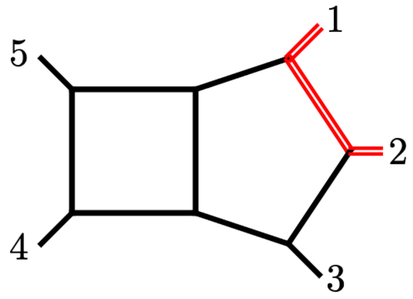
(a) Double-pentagon (b) Pentagon-box (c) Pentagon-triangle



(d) Double-box (e) Hexagon-box (f) Hexagon-bubble

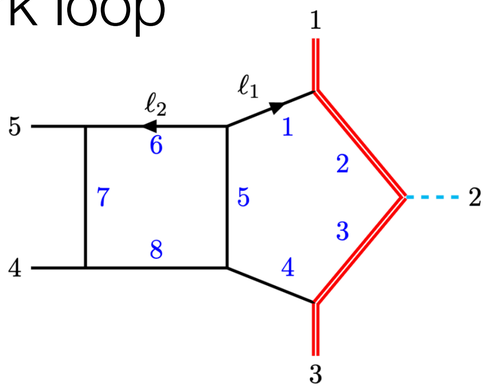
Planar 2-loop 5-point Feynman integrals for $\mathrm{t\bar{t}}$+jet production (only 1 family)

[Badger, Becchetti, Chaubey, Marzucca 2022]



Planar 2-loop 5-point Feynman integrals for $\mathrm{t\bar{t}H}$ with a light-quark loop

*[Febres Cordero, Figueiredo, Kraus, Page, Reina, **yesterday!**]*



Improved IBPs through syzygy equations

[Gluza, Kajda, Kosower 2011; Ita 2016; Larsen, Zhang 2016]

Standard way to generate IBP relations: $$\int \mathrm{d}^D k_1 \, \mathrm{d}^D k_2 \, \frac{\partial}{\partial k_i^\mu} \frac{q_j^\mu}{D_1^{\nu_1} \ldots D_m^{\nu_m}} = 0$$

The resulting IBPs contain integrals which are not needed for the amplitudes

- Integrals with doubled propagators
- Dimensionally-shifted integrals

We generated optimised IBP systems by solving polynomial “syzygy” equations for all two-loop integral families

[Boehm, Georgoudis, Larsen, Schulze, Zhang 2017; von Manteuffel, Schabinger 2019]

⇒ speed-up in IBP solution + substantial reduction in RAM usage