Hot dark matter in N-body simulations

Giovanni Pierobon, UNSW Sydney

With Markus Mosbech, Amol Upadhye, Yvonne Wong

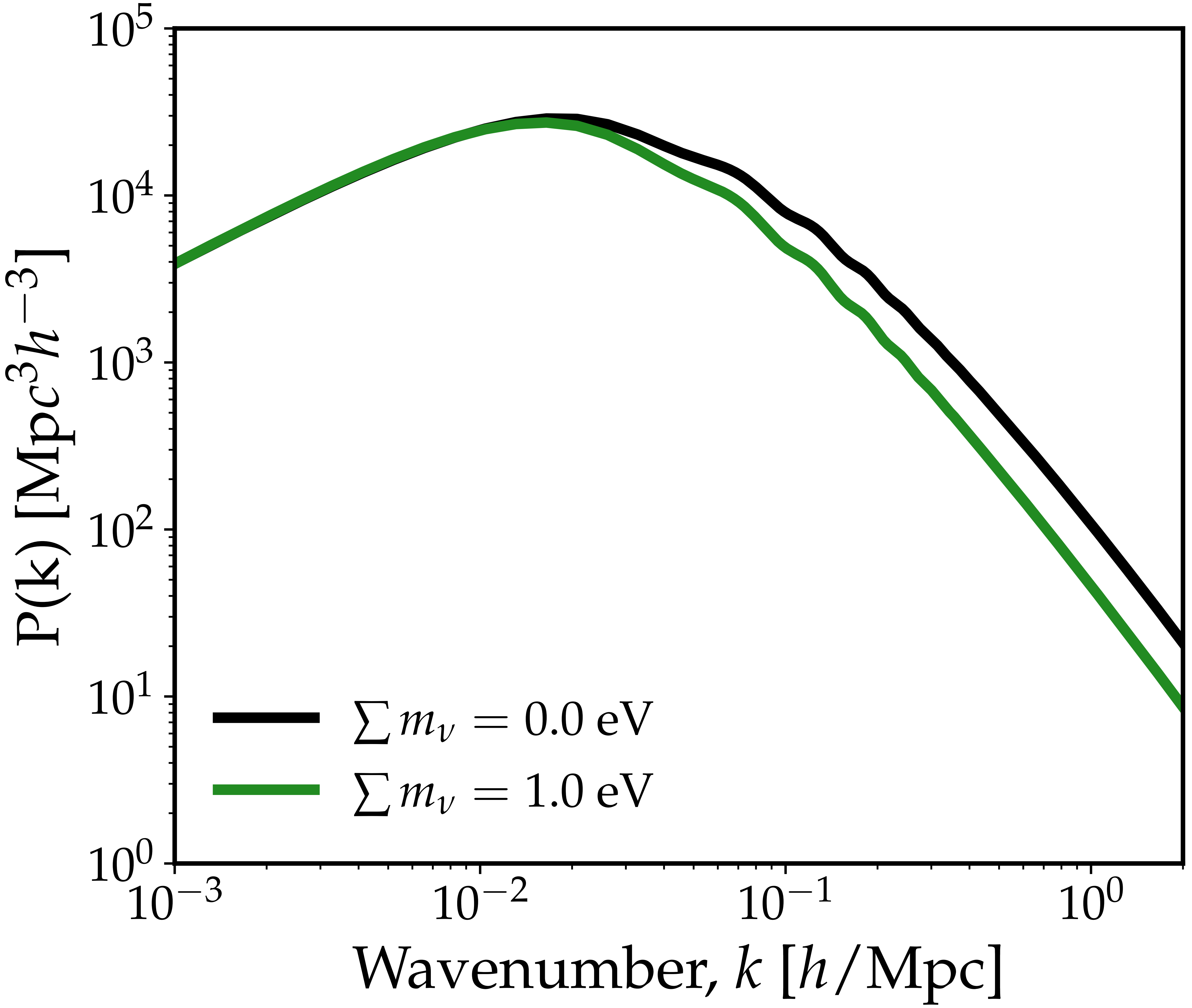
arXiv:2410.05815

Gordon Godfrey Workshop, Sydney, December 9-13, 2024

Overview

Role of neutrinos in modern cosmology

$$\sum m_\nu \lesssim 0.1 \text{ eV}$$



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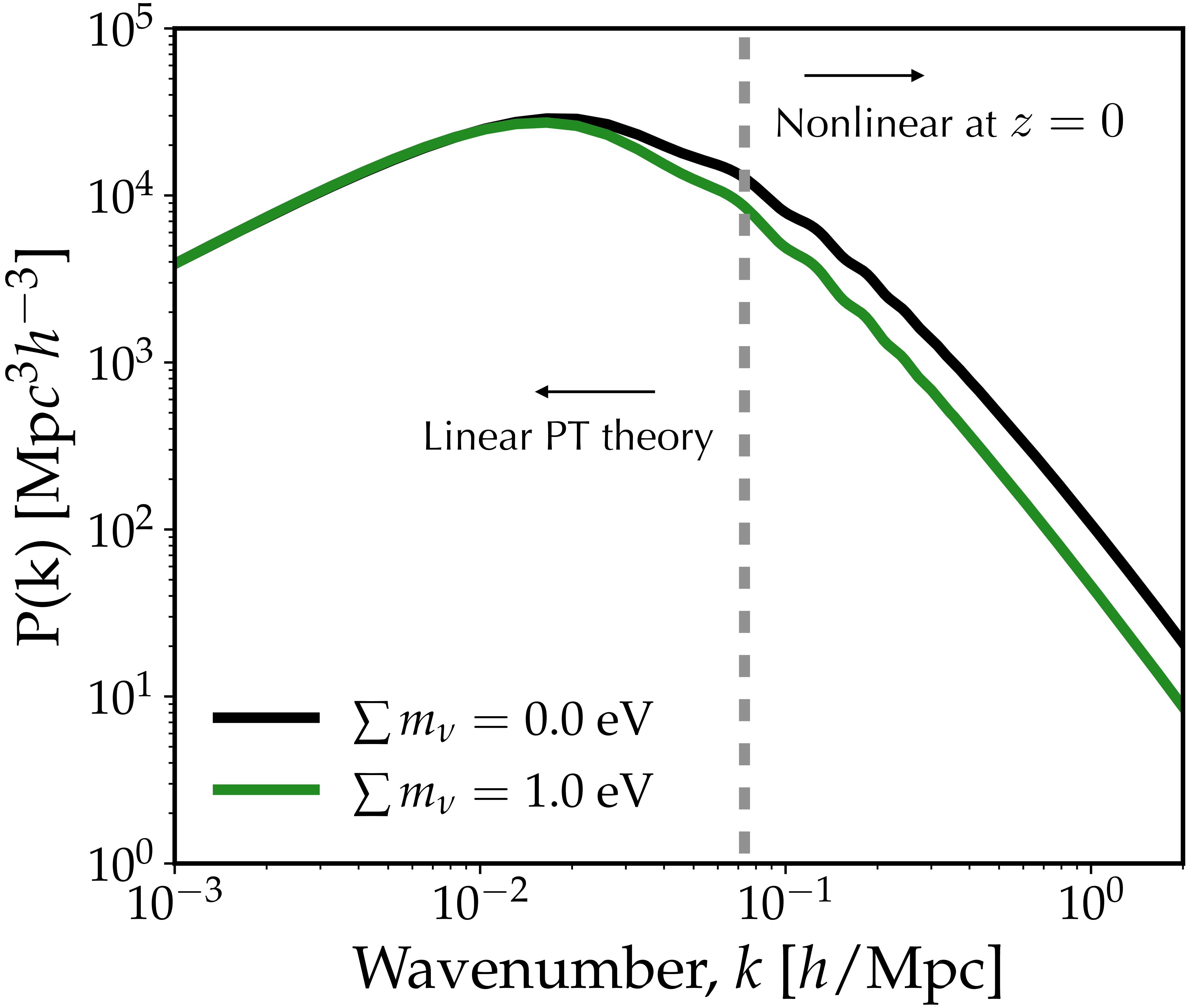
Predicting neutrino/HDM signatures on non-linear scales through gravitational clustering

Neutrinos have large thermal velocities which makes their modelling difficult



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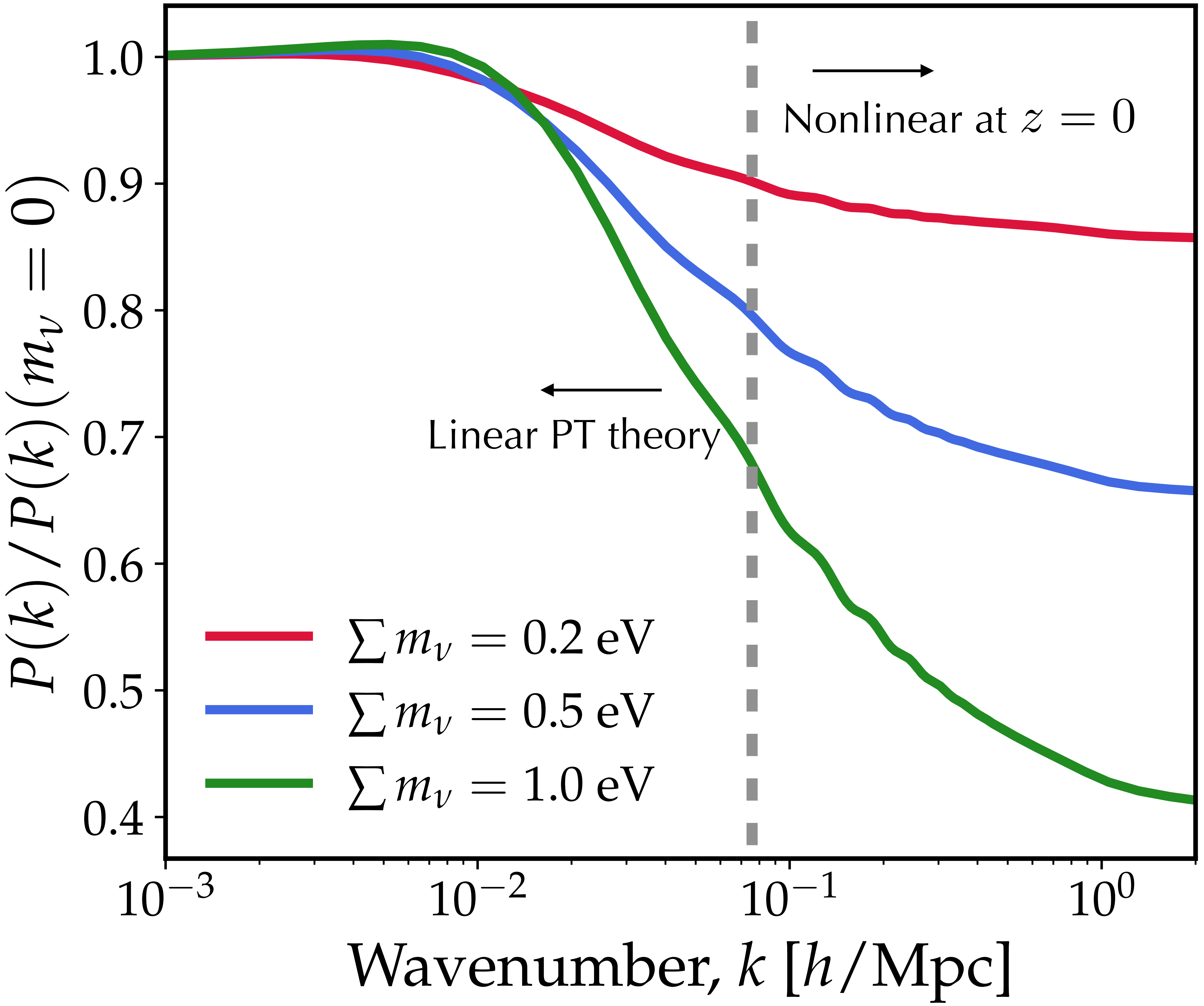

Outline

Part I: methods

Easily include the effects of neutrinos in N-body simulations: *SuperEasy*

Extension to any hot dark matter (HDM): *Generalised SuperEasy*

Other methods: multi-fluid theory and particle simulations

Part II: applications

Mixed hot dark matter models: SM neutrinos and axions

Can we distinguish effects of BSM relics from neutrinos on nonlinear scales?

Part I: methods

Linear response for HDM

Relic neutrinos as a nonrelativistic gas of collisionless particles
in an expanding background

Vlasov-Poisson system

$$\frac{\partial f}{\partial s} + \frac{\vec{p}}{m} \cdot \nabla_{\vec{x}} \, f - a^2 m \nabla_{\vec{x}} \, \Phi \cdot \nabla_{\vec{p}} \, f = 0$$

$$\nabla^2_{\vec{x}} \, \Phi(\vec{x}, s) = \frac{3}{2} \mathcal{H}^2(s) \, \Omega_{\rm m}(s) \, \delta_{\rm m}(\vec{x}, s)$$

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Neutrino phase space density $f(\vec{x}, \vec{p}, s)$

Superconformal time $s = \int a^{-2}\mathrm{d}t$

Gravitational potential

Total matter density contrast $\delta_{\mathrm{m}} = f_{\mathrm{cb}}\delta_{\mathrm{cb}} + f_{\nu}\delta_{\nu}$

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Total matter density contrast
$\delta_{\rm m} = f_{\rm cb}\delta_{\rm cb} + f_\nu \delta_\nu$

Bertschinger, 1993

$$\delta f = \boxed{\delta f(s_{\rm i})\, e^{-\frac{i\vec{k}\cdot\vec{p}}{m}(s - s_{\rm i})}} + \boxed{im\vec{k} \cdot \nabla_{\vec{p}} \bar{f} \int_{s_{\rm i}}^{s} \mathrm{d}s'\, a^2\, \Phi\, e^{-\frac{i\vec{k}\cdot\vec{p}}{m}(s - s')}}$$

- Linearisation $\nabla_{\vec{p}}|f - \bar{f}| \ll |\nabla_{\vec{p}}|\bar{f}|$
- External potential
- Solving for $\delta f = f - \bar{f}$

Homegeneous part
Free-streaming of initial conditions

Inhomogeneous part
neutrino **response** to the external potential

HDM: neutrinos

Integrate over momentum: $$\delta_\nu \simeq k^2 \int_{s_{\mathrm{i}}}^{s} \mathrm{d}s' a^2 \Phi(s-s') F\left[\frac{T_{\nu,0}k(s-s')}{m_\nu}\right]$$

Integral linear response

$$F(q) = \frac{m_\nu}{\bar{\rho}_\nu(s)} \int \mathrm{d}^3 p \bar{f}(p) e^{-i\vec{q}\cdot\vec{p}/T_{\nu,0}}$$

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SuperEasy linear response: find analytical solutions in large and small k limits

Ringwald & Wong, 2004
Chen, Upadhye & Wong, 2020

Free-streaming limit $k \gg k_\mathrm{FS}$

$$\delta_\nu(\vec{k}, s) \simeq \frac{k_\mathrm{FS}^2}{k^2} \delta_\mathrm{m}(\vec{k}, s)$$

Clustering limit $k \ll k_\mathrm{FS}$

$$\delta_\nu(\vec{k}, s) \simeq \delta_\mathrm{cb}(\vec{k}, s) \simeq \delta_\mathrm{m}(\vec{k}, s)$$

Free-streaming scale is ‘integrated’ over Fermi-Dirac

only depends on mass and time

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Interpolation

Clustering limit $k \ll k_\mathrm{FS}$

$$\delta_\nu(\vec{k},s) \simeq \delta_\mathrm{cb}(\vec{k},s) \simeq \delta_\mathrm{m}(\vec{k},s)$$

$$\delta_\nu(\vec{k},s) = \frac{k_\mathrm{FS}^2}{[k+k_\mathrm{FS}(s)]^2}\delta_\mathrm{m}(\vec{k},s)$$

Free-streaming scale is
'integrated' over Fermi-Dirac

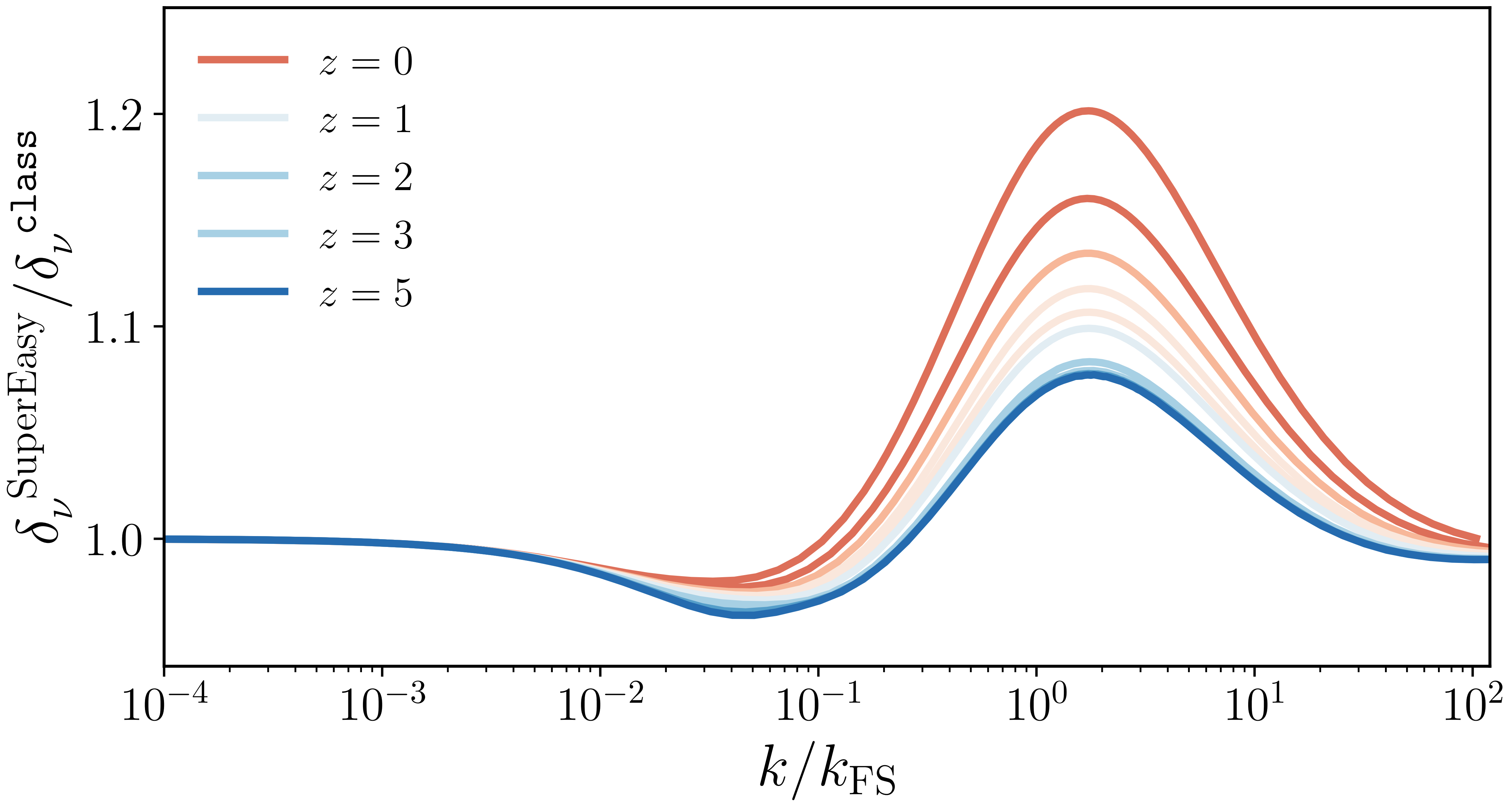
only depends on mass and
time

Note: momentum integration
comes before interpolation!

SuperEasy for neutrinos

Neutrino density contrast responding to CDM density

$$\delta_\nu(\vec{k}, s) = \frac{k_{\rm FS}^2(s)(1-f_\nu)}{[k + k_{\rm FS}(s)]^2 - k_{\rm FS}^2(s) f_\nu} \delta_{\rm cb}(\vec{k}, s)$$



SuperEasy for neutrinos

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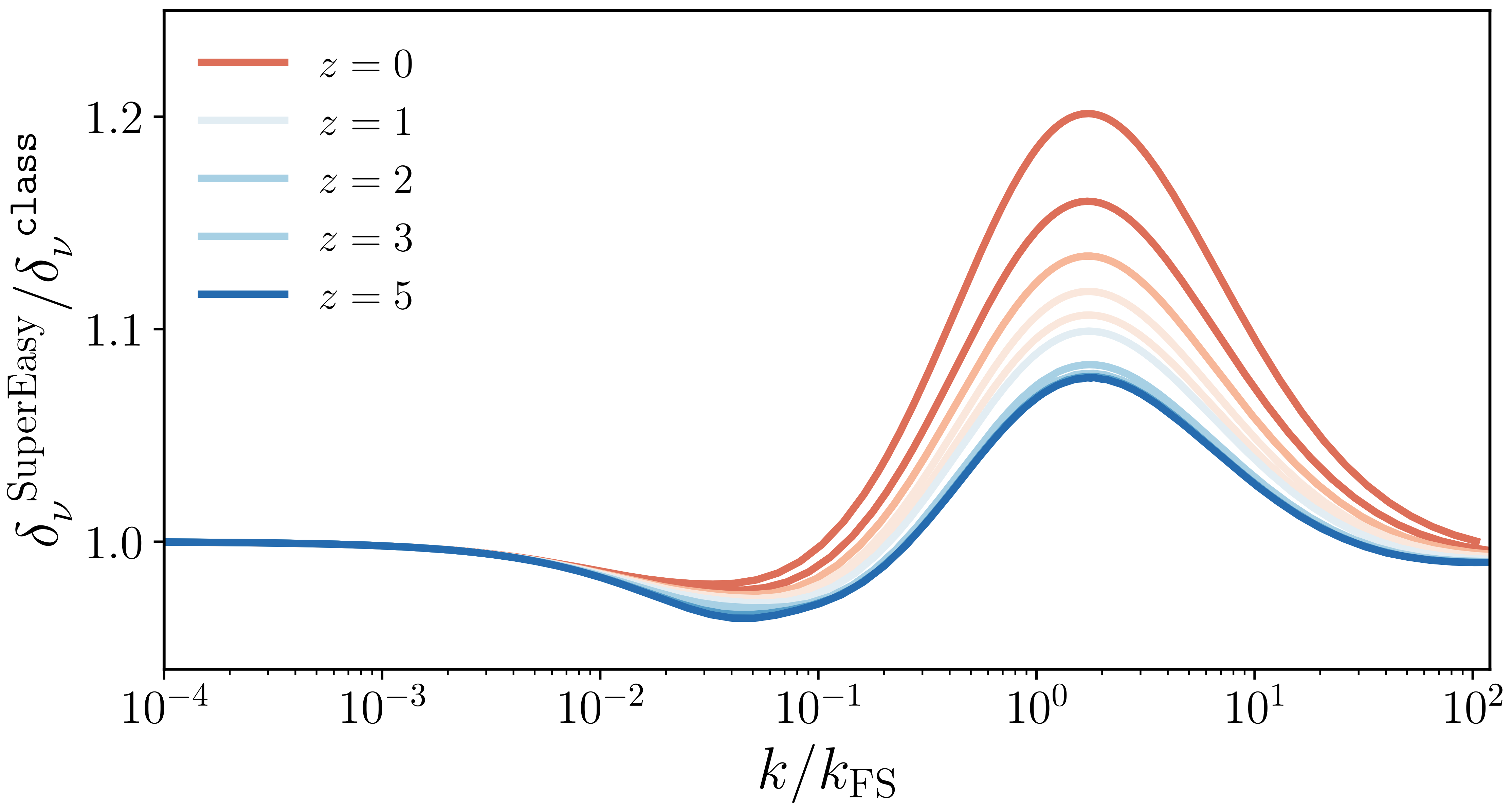
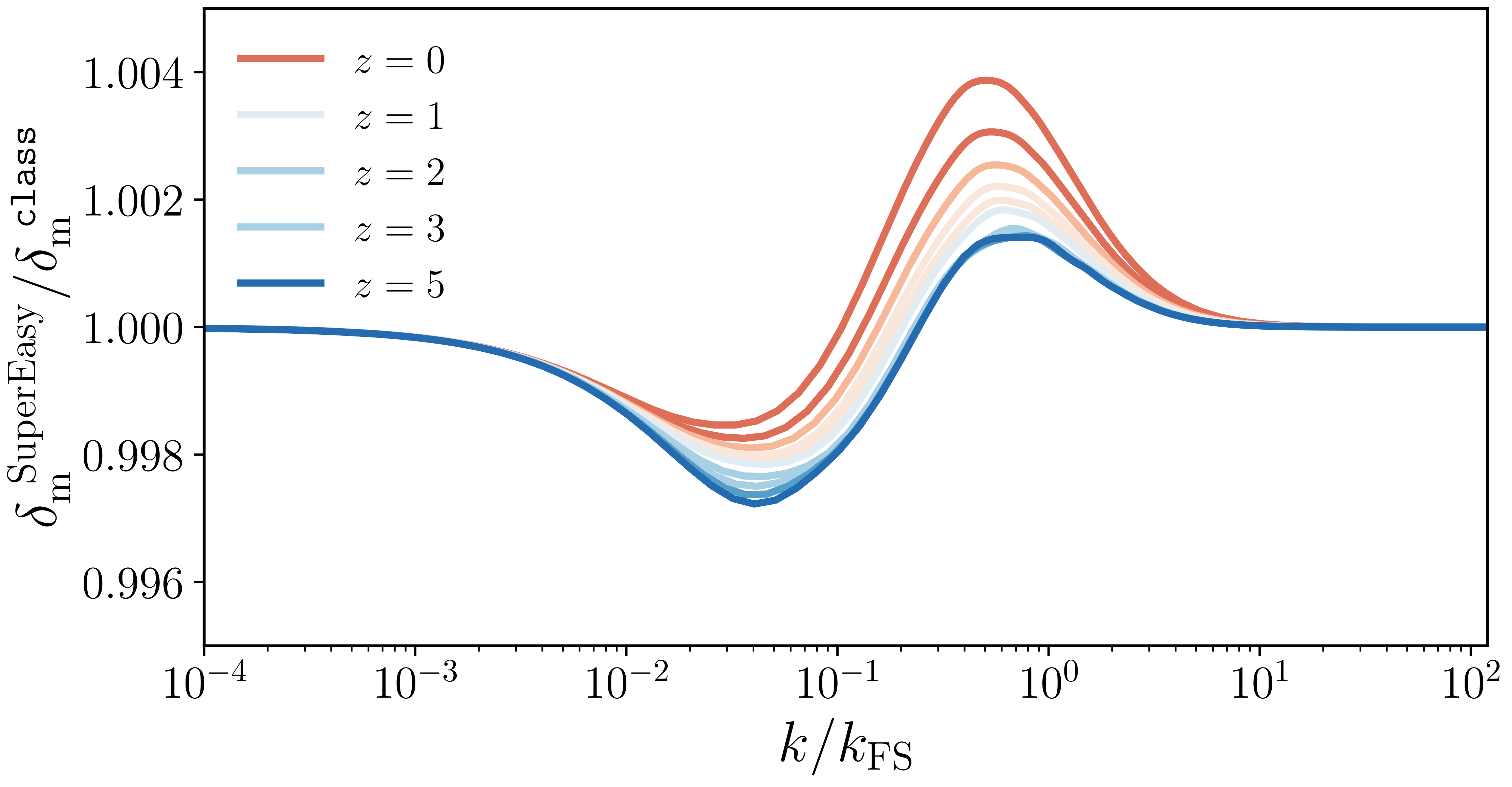
Total matter density including contributions from neutrinos

$$\delta_{\rm m}(\vec{k}, s) = \frac{[k+k_{\rm FS}(s)]^2(1-f_\nu)}{[k+k_{\rm FS}(s)]^2 - k_{\rm FS}^2(s) f_\nu}\delta_{\rm cb}(\vec{k}, s)$$





SuperEasy in N-body simulations

One-line modification to the gravitational potential,
only requires CDM density as a real-time input

↓

perfect for PM solvers of an N-body simulation

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$$k^2\Phi(\vec{k},s) = -(3/2)\mathcal{H}^2(s)\Omega_{\rm cb}(s)\tilde{g}(k,s)\delta_{\rm cb}(\vec{k},s)$$

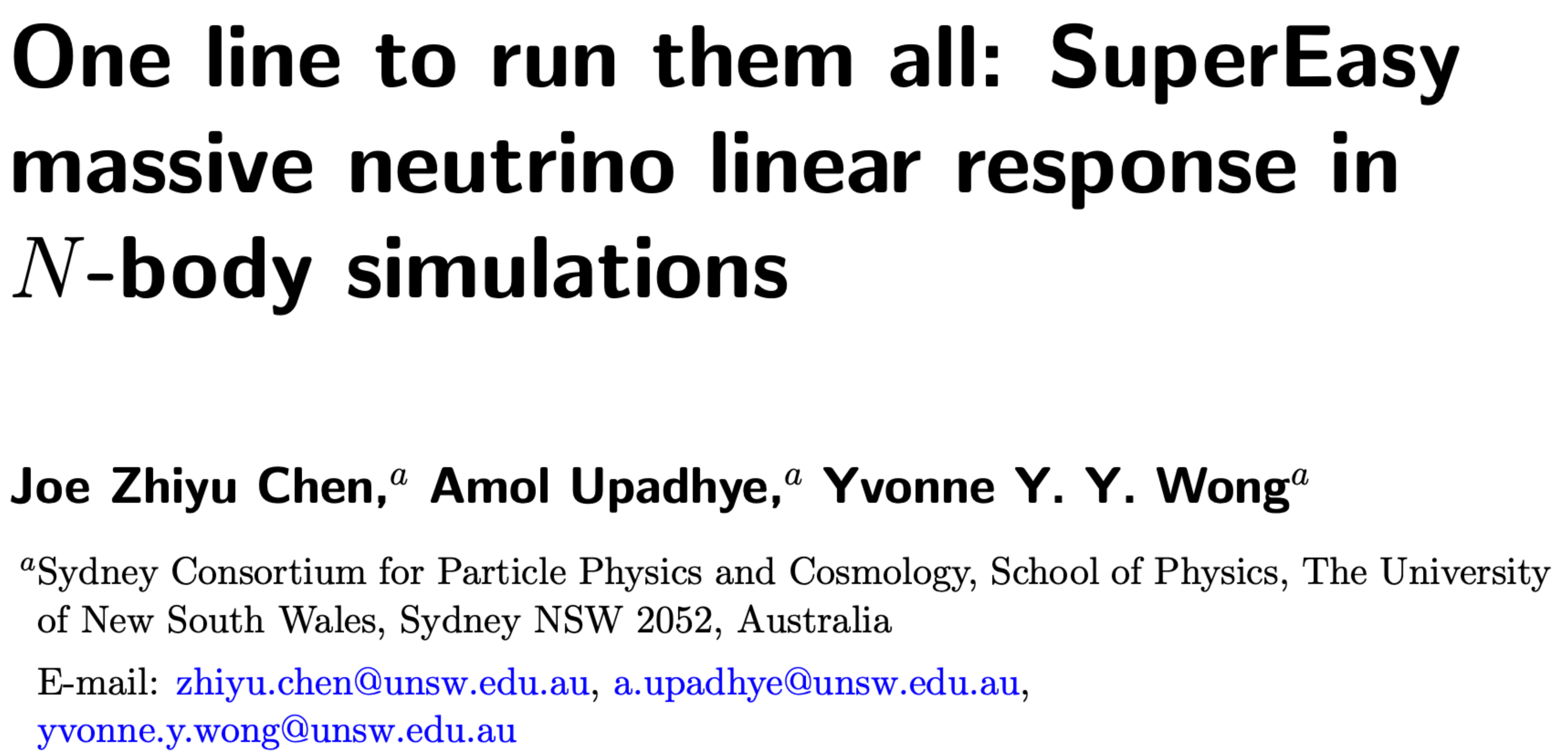
modification factor
due to neutrinos

calculated from
cold particles

$$\tilde{g}(k,s) = \frac{\left[k+k_{\rm FS}(s)\right]^2}{\left[k+k_{\rm FS}(s)\right]^2 - k_{\rm FS}^2(s)f_\nu}$$

*No additional memory or runtime
compared to CDM only*

Chen, Upadhye & Wong, 2020



One line to run them all: SuperEasy massive neutrino linear response in N-body simulations

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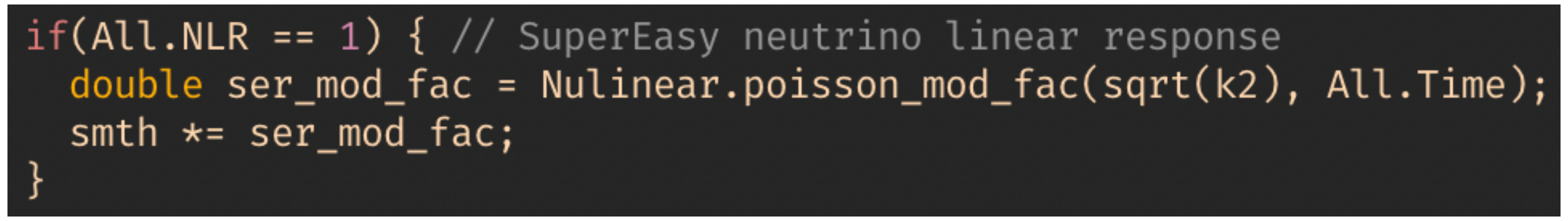
[a]Sydney Consortium for Particle Physics and Cosmology, School of Physics, The University of New South Wales, Sydney NSW 2052, Australia

E-mail: zhiyu.chen@unsw.edu.au, a.upadhye@unsw.edu.au, yvonne.y.wong@unsw.edu.au



```
if(All.NLR == 1) { // SuperEasy neutrino linear response
  double ser_mod_fac = Nulinear.poisson_mod_fac(sqrt(k2), All.Time);
  smth *= ser_mod_fac;
}
```

`src/pm/pm_periodic.cc`

https://github.com/cppccosmo/gadget-4-cppc

SuperEasy for any HDM

Generalised SuperEasy: Interpolation at the momentum level, before the integration/sum over the momenta

$$\delta f(\vec{k},\vec{p},s) \simeq im\vec{k}\cdot\nabla_{\vec{p}}\,\bar{f}\int_{s_{\rm i}}^{s}\mathrm{d}s'\,a^2\,\Phi\,e^{-\frac{i\vec{k}\cdot\vec{p}}{m}(s-s')}$$

SuperEasy for any HDM

Generalised SuperEasy: Interpolation at the momentum level, before the integration/sum over the momenta

$$\delta f(\vec{k}, \vec{p}, s) \simeq i m \vec{k} \cdot \nabla_{\vec{p}} \bar{f} \int_{s_\mathrm{i}}^{s} \mathrm{d}s' \, a^2 \, \Phi \, e^{-\frac{i\vec{k}\cdot\vec{p}}{m}(s-s')}$$

angle average ⟶ $\langle \delta f \rangle_\mu (\vec{k}, p, s)$ $\quad \mu \equiv \hat{k} \cdot \hat{p}$

$$\delta_\mathrm{hdm}(\vec{k}, s) = \frac{1}{C} \left[\int_0^\infty \mathrm{d}p \, p^2 \, \bar{f}(p) \, \mathcal{G}(k, p, s) \right] \delta_\mathrm{m}(\vec{k}, s)$$

SuperEasy for any HDM

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angle average $\longrightarrow \langle\delta f\rangle_\mu(\vec{k},p,s)$ $\quad \mu\equiv\hat{k}\cdot\hat{p}$

$$\delta_\mathrm{hdm}(\vec{k},s) = \frac{1}{C}\left[\int_0^\infty \mathrm{d}p\,p^2\,\bar{f}(p)\,\mathcal{G}(k,p,s)\right]\delta_\mathrm{m}(\vec{k},s)$$

Momentum dependent free-streaming scale

$$k_{\mathrm{FS},p}(s)\equiv\sqrt{\frac{3}{2}}\frac{ma(s)}{p}\mathcal{H}(s)\Omega_\mathrm{m}^{1/2}(s)$$

Interpolation function

$$\mathcal{G}(k,p,s) = \frac{k_{\mathrm{FS},p}^2}{k^2+\beta k k_{\mathrm{FS},p}+k_{\mathrm{FS},p}^2}$$

$\mathcal{G}(k/k_{\mathrm{FS},p}\to 0)\to 1$ Free-streaming limit

$\mathcal{G}(k/k_{\mathrm{FS},p}\to \infty)\to \frac{k_{\mathrm{FS},p}^2}{k^2}$ Clustering limit

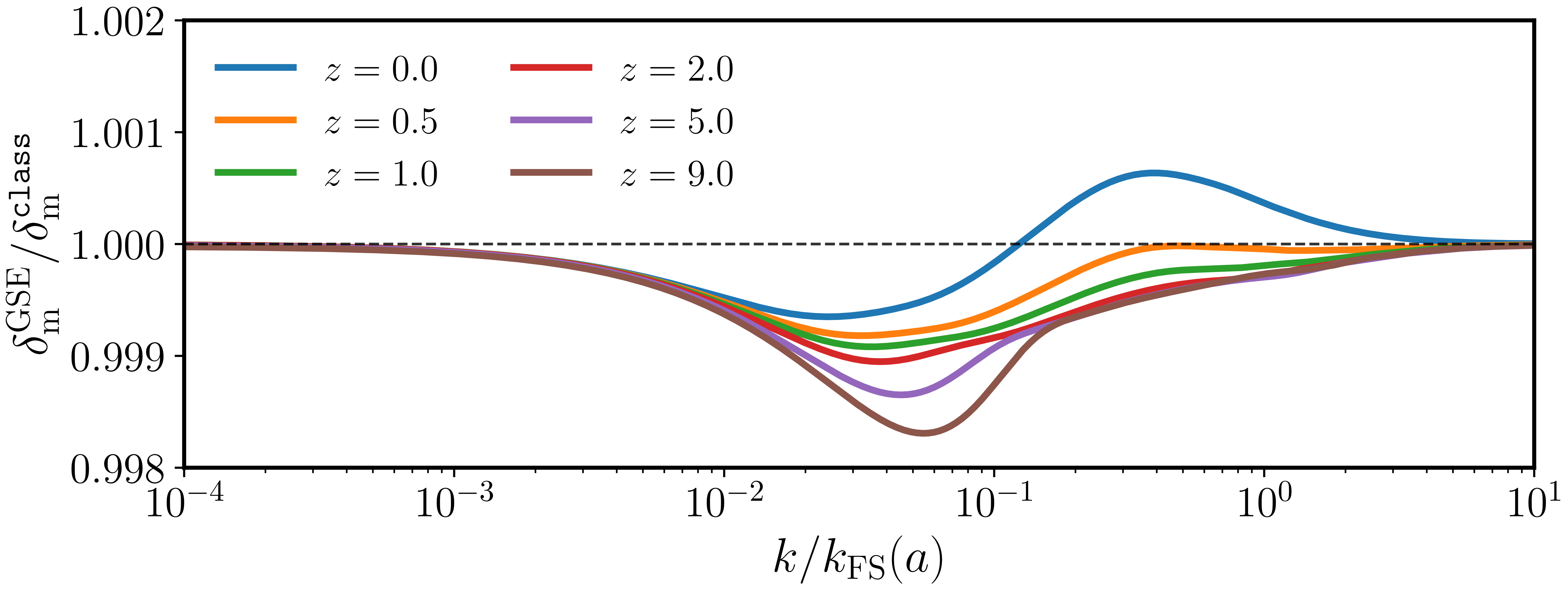
SuperEasy for any HDM

$$\int_0^\infty \mathrm{d}p\, p^2 \bar{f}(p)\, \mathcal{G}(k,p,s) \to \sum_{i=1}^{N} \left[\int_0^\infty \mathrm{d}p\, p^2 \bar{f}(p)\, \omega_i(p)\right] \mathcal{G}_i(k,s)$$

N different types of HDM, each with a free-streaming scale and a density contrast δ_{hdm_i}

Generalised SuperEasy

$$\delta_{\mathrm{m}}(k,s) \simeq \left(1 + \sum_{i=1}^{N} f_{\mathrm{h}_i} \left[\mathcal{G}_i(k,s) - 1\right] + \mathcal{O}(f_{\mathrm{h}_i}^2)\right) \delta_{\mathrm{cb}}(\vec{k},s)$$



Gauss-Laguerre binning
$N = 15$

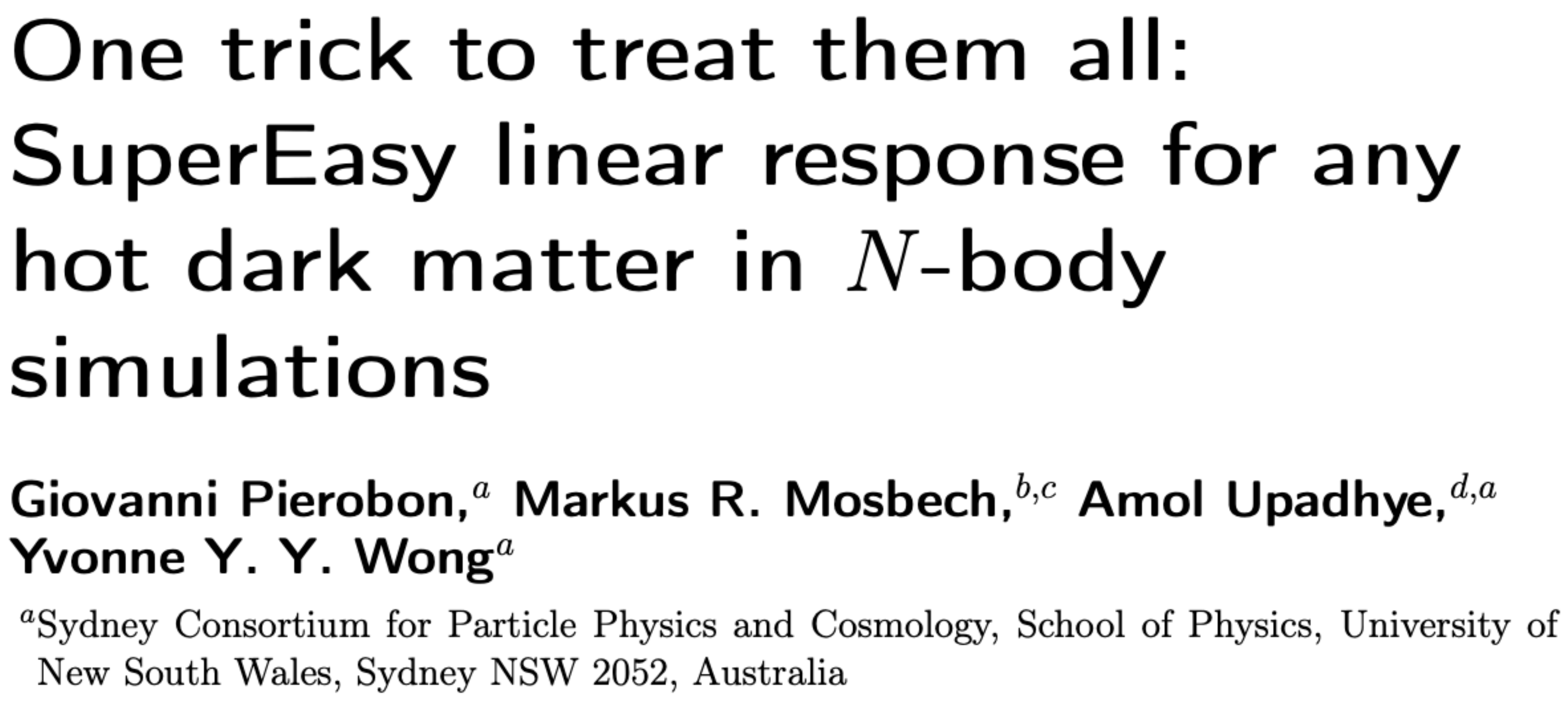
Generalised SuperEasy in N-body simulations

$$k^2\Phi(\vec{k},s) = -(3/2)\mathcal{H}^2(s)\Omega_{\rm cb}(s)\tilde{g}(k,s)\delta_{\rm cb}(\vec{k},s)$$

$$\tilde{g}(k,s) = \left(1 + \sum_{i=1}^{N_i} f_{{\rm h}_i}\left[\mathcal{G}_i(k,s) - 1\right]\right) f_{\rm cb}^{-1}$$

No additional memory or runtime compared to CDM only

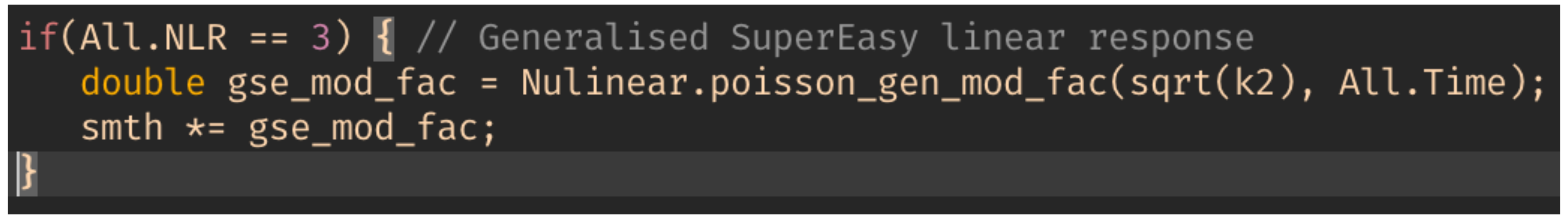
Only inputs: HDM mass, temperature and any momentum distribution



One trick to treat them all: SuperEasy linear response for any hot dark matter in N-body simulations

Giovanni Pierobon,[a] Markus R. Mosbech,[b,c] Amol Upadhye,[d,a] Yvonne Y. Y. Wong[a]

[a]Sydney Consortium for Particle Physics and Cosmology, School of Physics, University of New South Wales, Sydney NSW 2052, Australia



```
if(All.NLR == 3) { // Generalised SuperEasy linear response
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   smth *= gse_mod_fac;
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```

`src/pm/pm_periodic.cc`

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Multifluid method

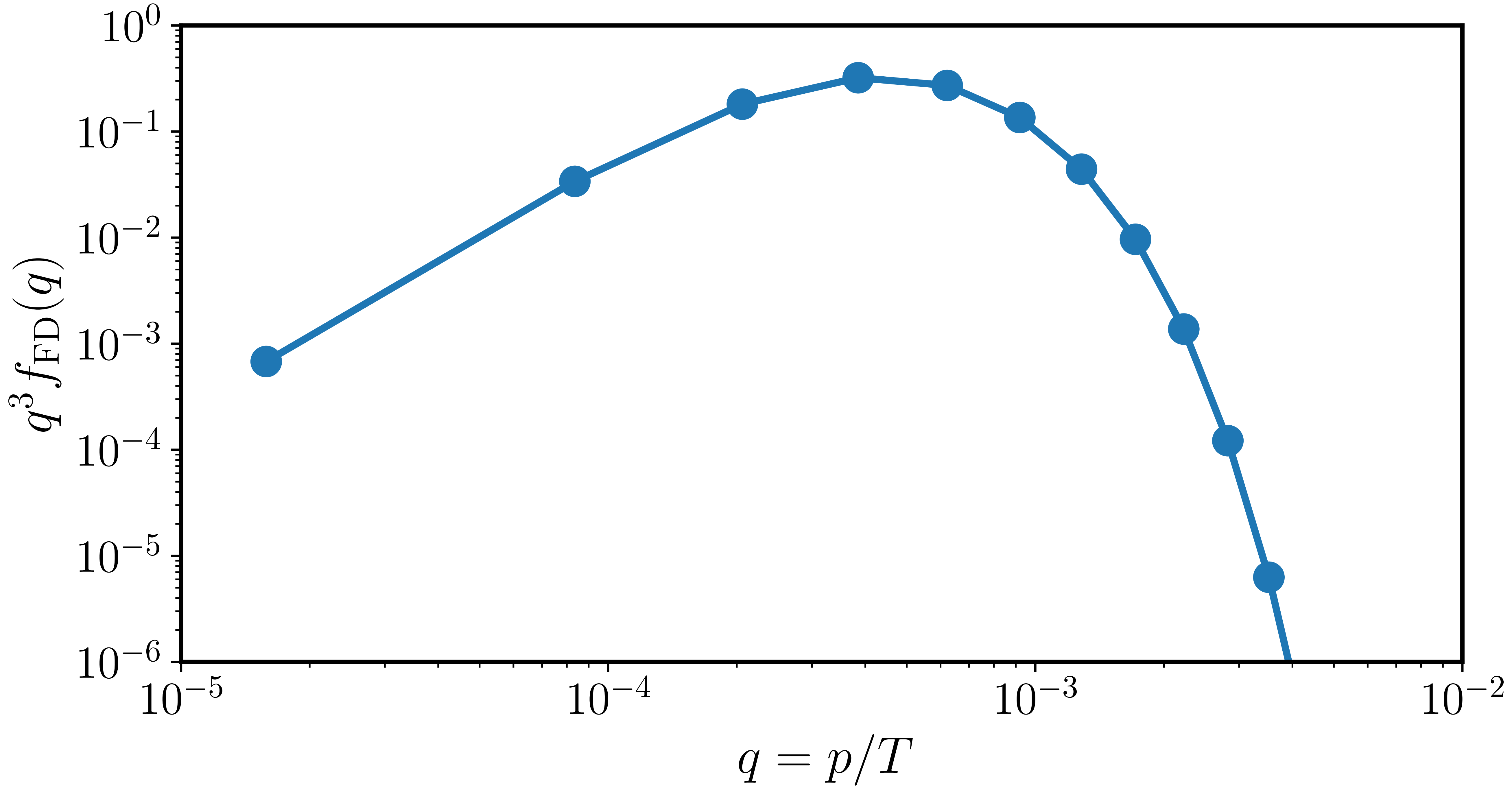
Multi-fluid **linear response**: based on perturbation theory of Dupuy & Bernardeau (2014)

Partition a single HDM fluid into N different flows, each with momenta, densities and Legendre multiple moments

$$\ell \in (0, N_\mu - 1)$$

$$\{\delta_{i,\ell}, \theta_{i,\ell}\}$$

Density, velocity perturbations



$$\delta'_{i,\ell} = \theta_{i,\ell} + \frac{kv_a}{\mathcal{H}}\left(\frac{\ell}{2\ell-1}\delta_{i,\ell-1} - \frac{\ell+1}{2\ell+3}\delta_{i,\ell+1}\right)$$

$$\theta'_{i,\ell} = -\delta^{(K)}_{i,0}\frac{k^2}{\mathcal{H}^2}\Phi - \left(1 + \frac{\mathcal{H}'}{\mathcal{H}}\right)\theta_{i,\ell} + \frac{kv_a}{\mathcal{H}}\left(\frac{\ell}{2\ell-1}\theta_{i,\ell-1} - \frac{\ell+1}{2\ell+3}\theta_{i,\ell+1}\right)$$

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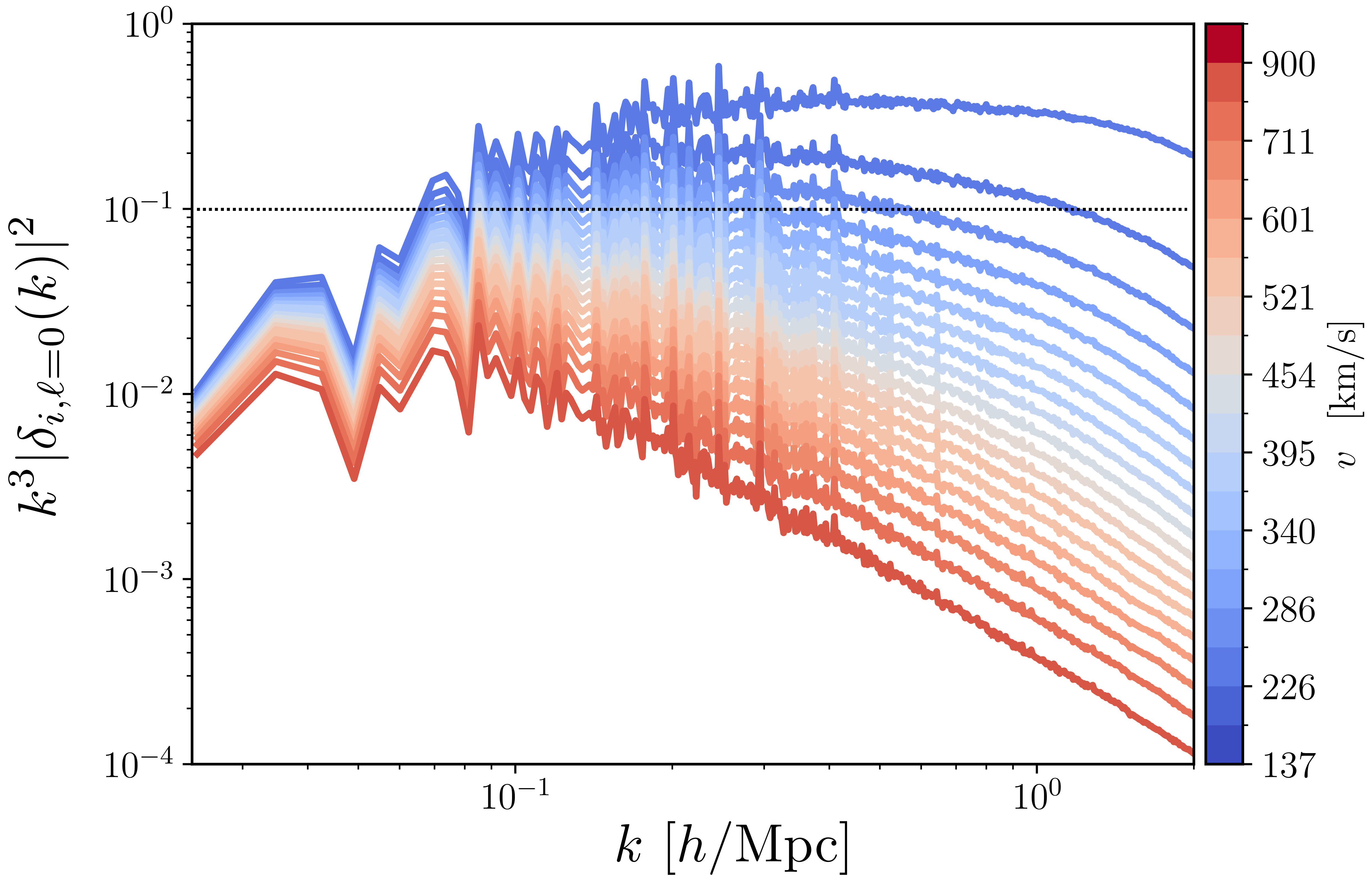
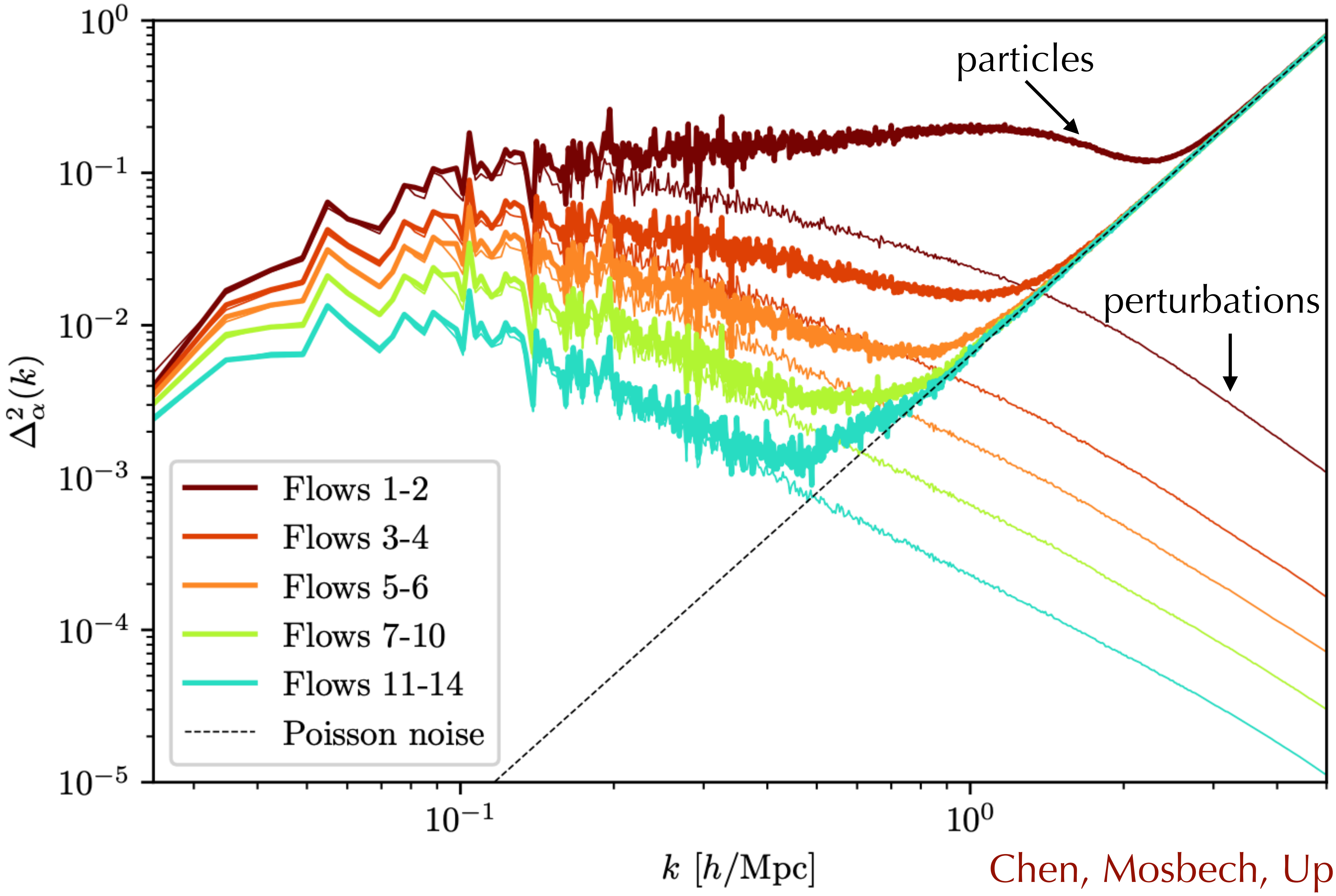


Multifluid+hybrid

Multi-fluid **linear response**: based on perturbation theory of Dupuy & Bernardeau (2014)

Hybrid method: convert perturbations of slowest flows into new set of particles



Chen, Mosbech, Upadhye & Wong, 2022

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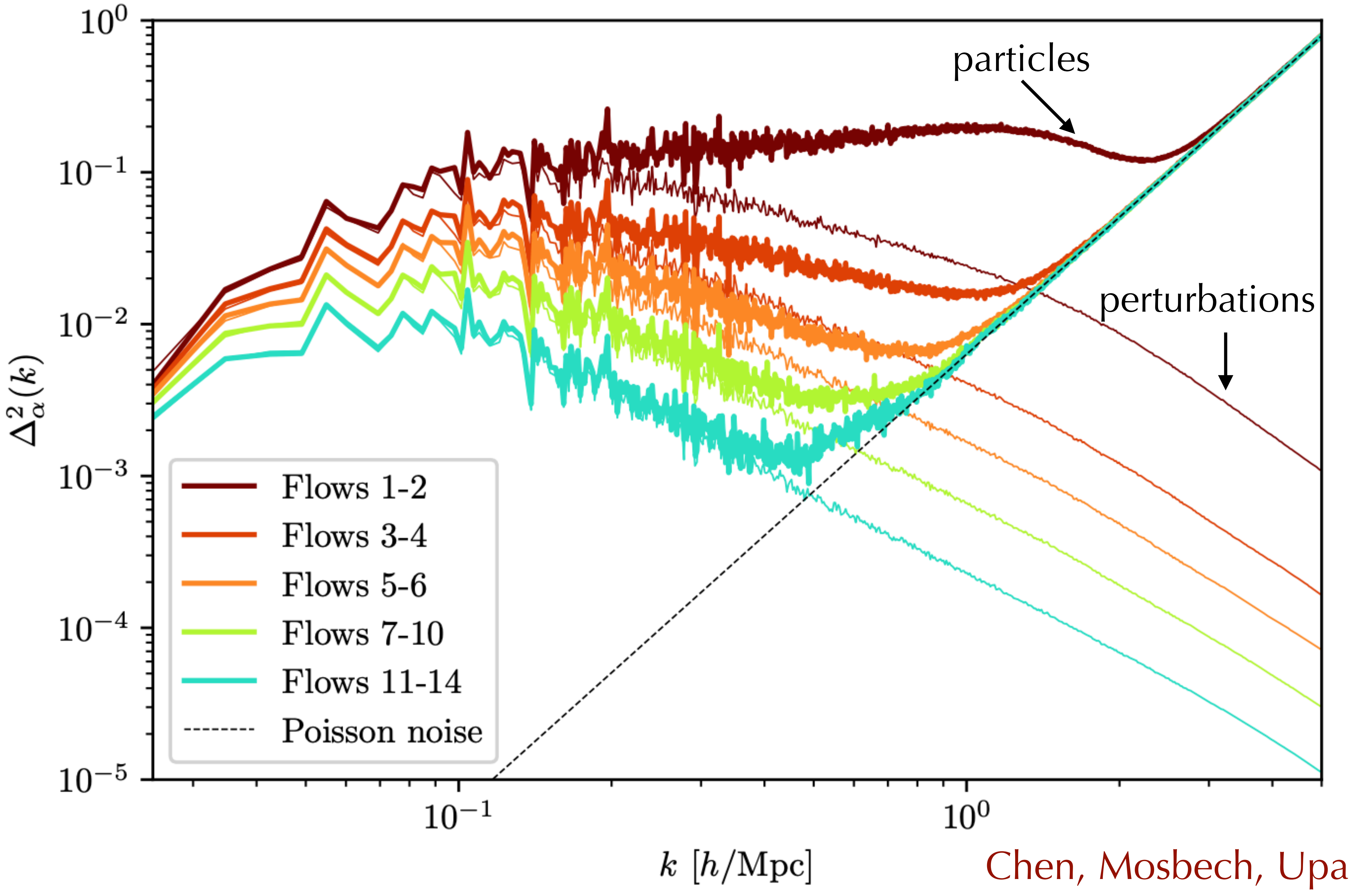
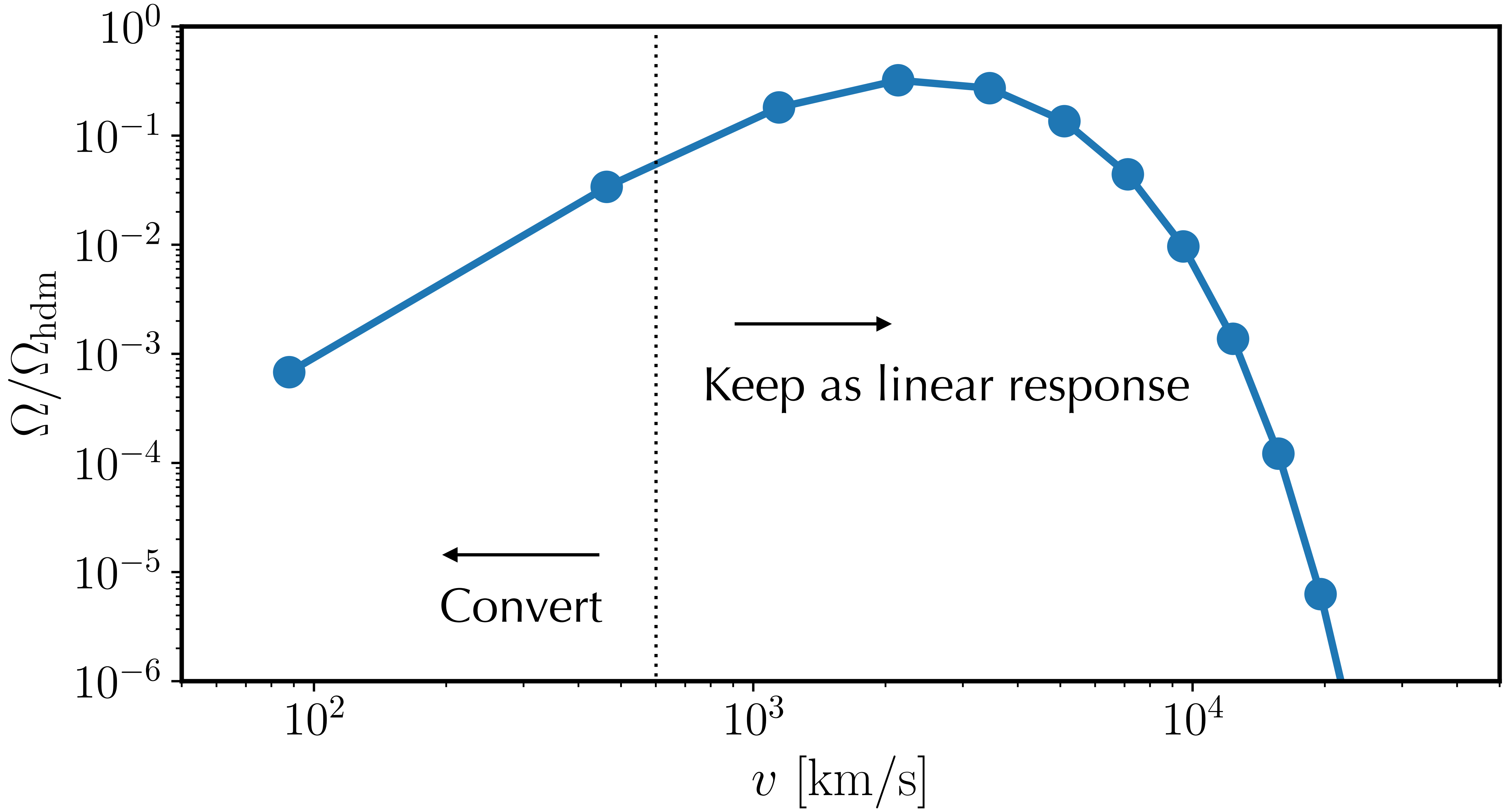
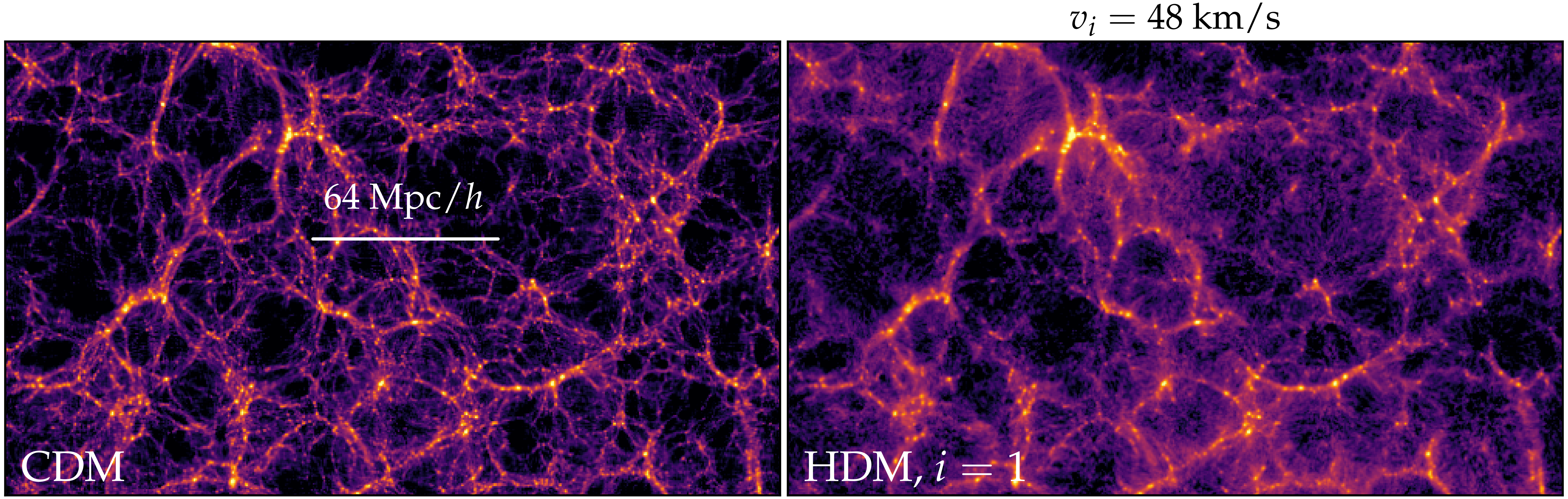




Hybrid **criterion**: only convert flows with $v \lesssim 600\ \mathrm{km/s}$

Chen, Mosbech, Upadhye & Wong, 2022

Hybrid simulations



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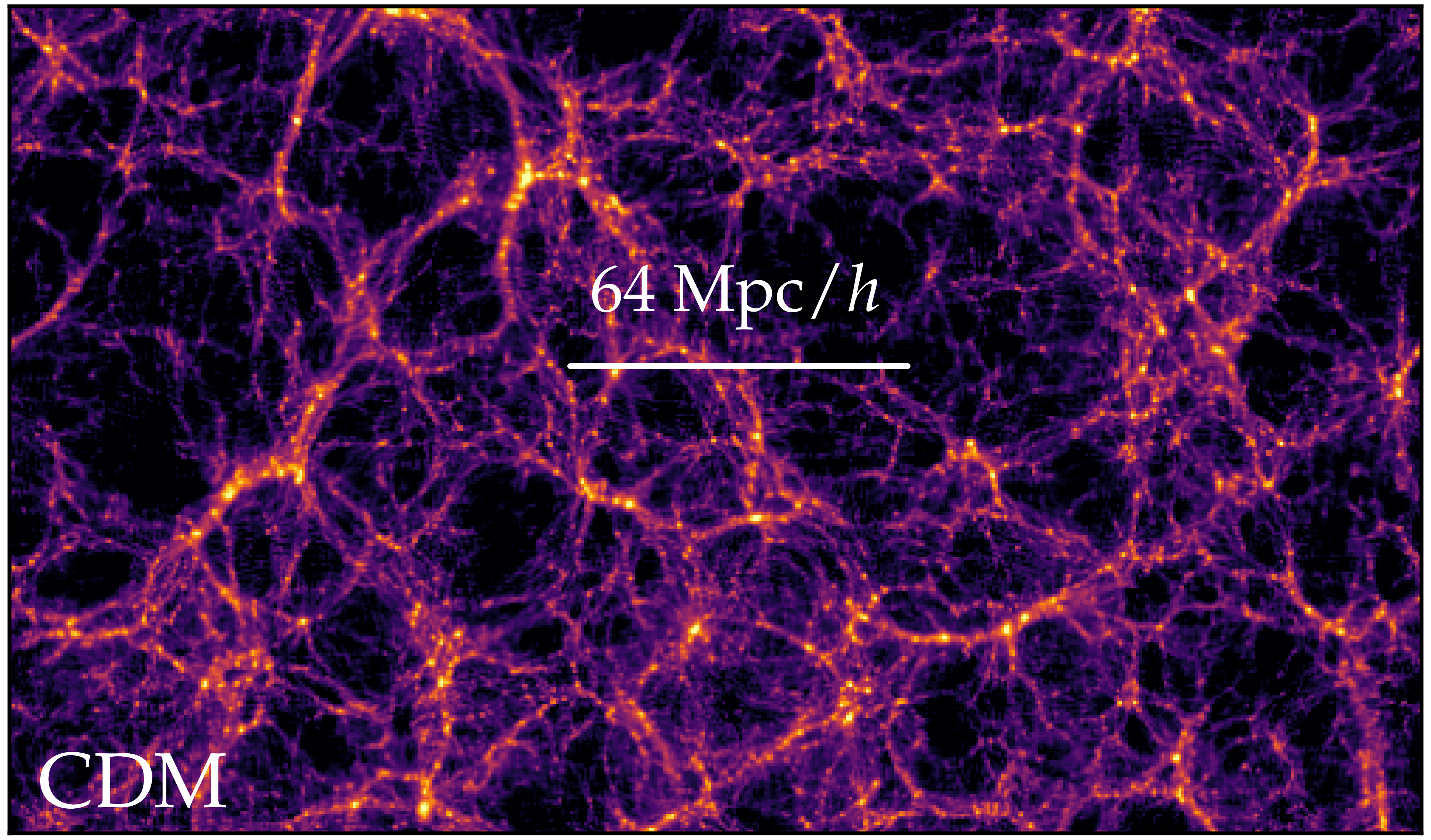
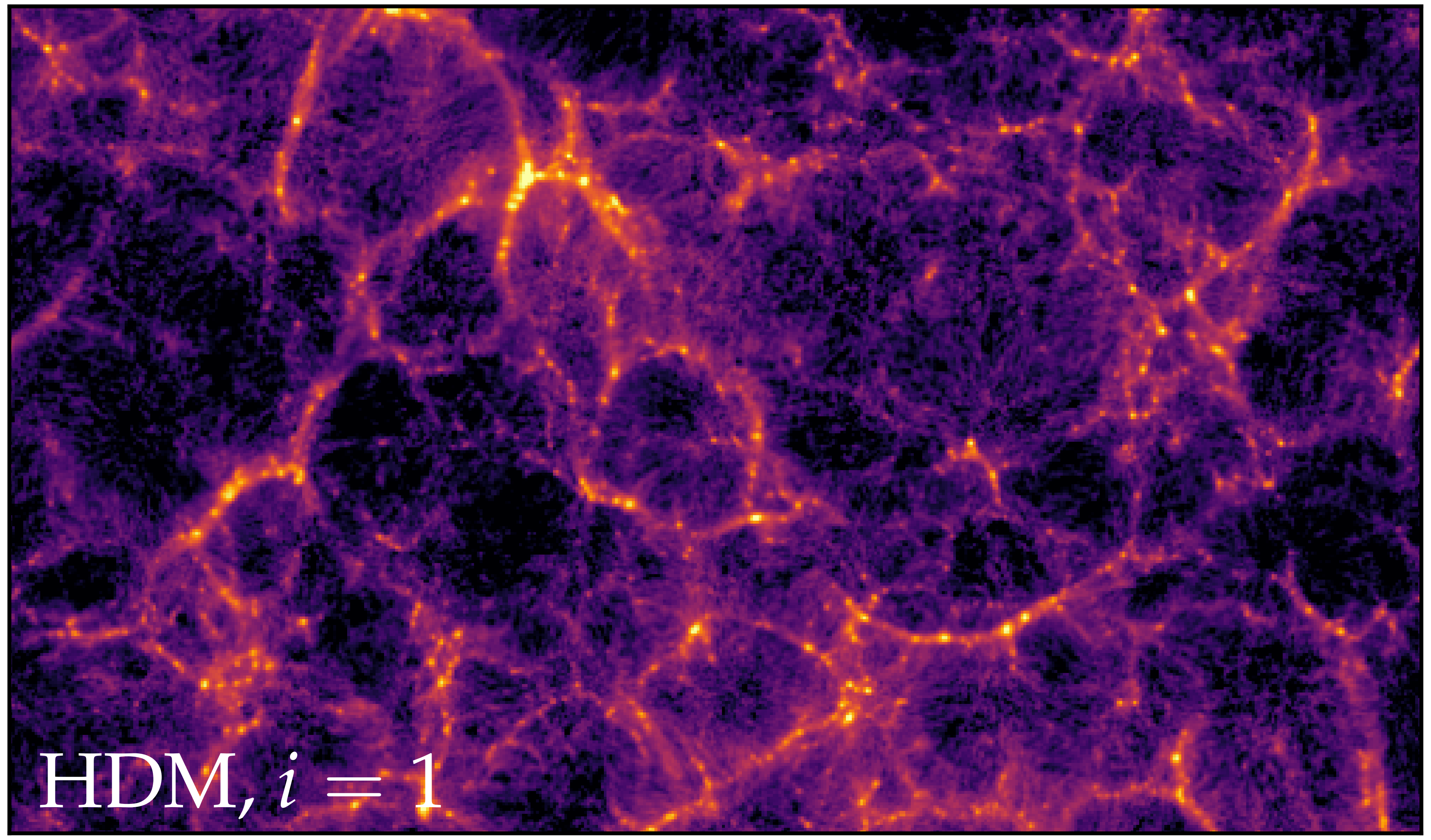
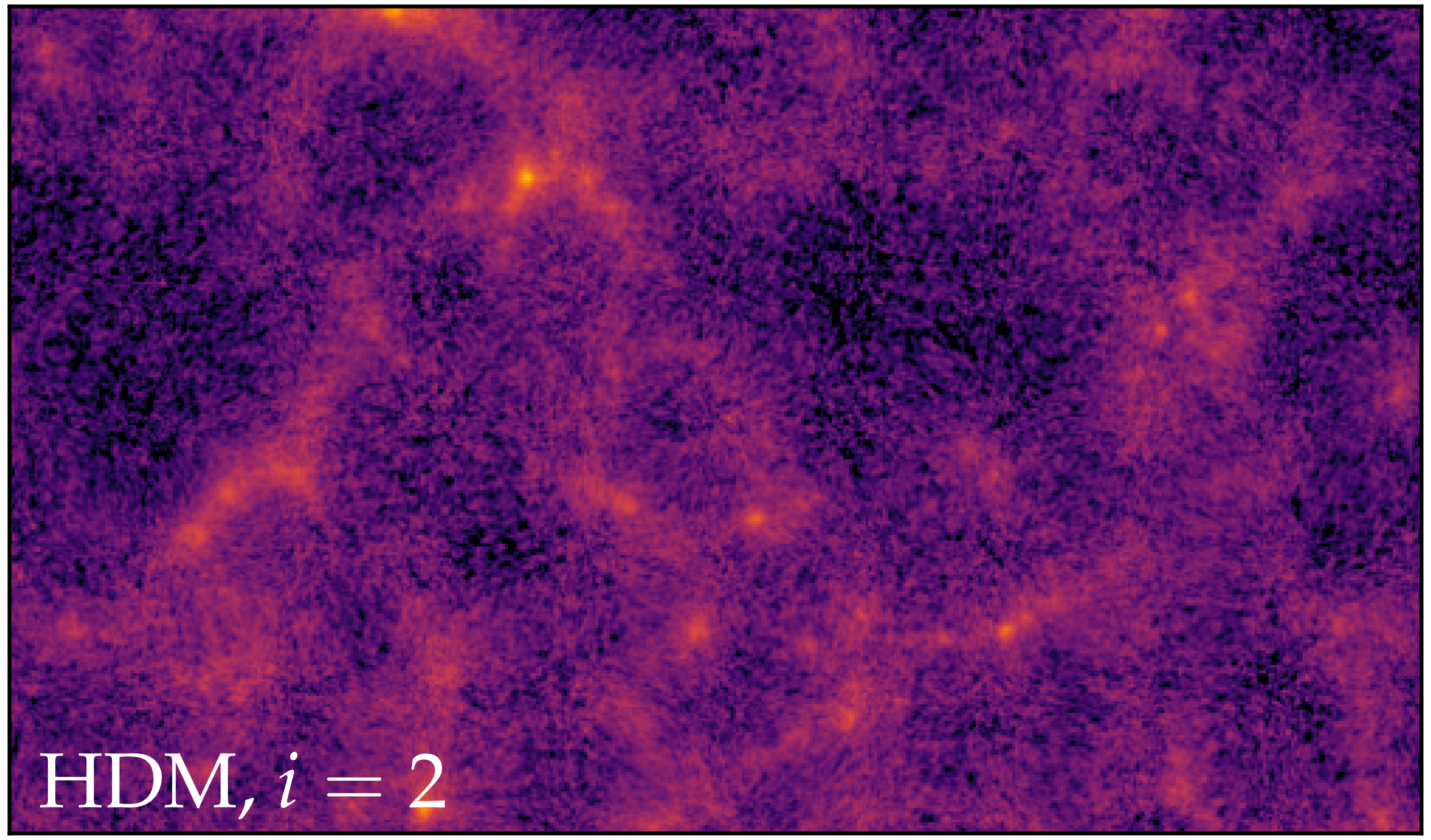


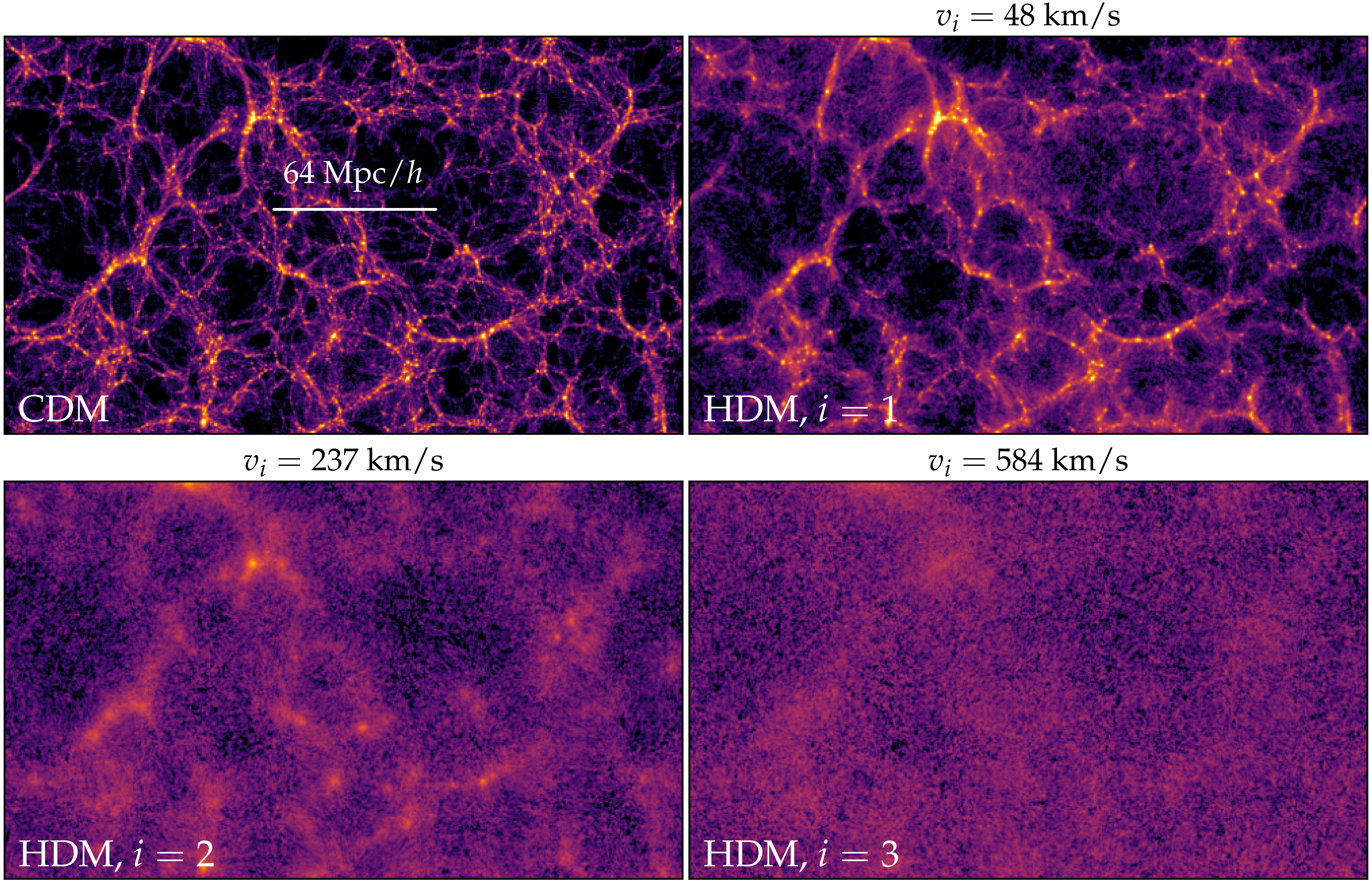
$v_i = 48$ km/s



$v_i = 237$ km/s



Hybrid simulations



Part II: applications

Mixed HDM

Consider a collection of fluids, each with mass, temperature and internal dofs

$$\rho_{\rm hdm}(\vec{x}, s) = \frac{1}{(2\pi a)^3} \sum_{\alpha=1}^{N_\alpha} m_\alpha \, g_\alpha \int_0^\infty {\rm d}^3 p \, f_\alpha^{(p)}(\vec{x}, \vec{p}, s)$$

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Free-streaming is a kinematic effect

⟶ Lagrangian velocity

$$f_\alpha^{(p)}(\vec{x}, \vec{p}, s) \to f_\alpha^{(v)}(\vec{x}, \vec{v}, s)$$

$$= \frac{1}{(2\pi a)^3} \int_0^\infty \mathrm{d}^3 v \, F^{(v)}(\vec{x}, \vec{v}, s)$$

$$F^{(v)}(\vec{x}, \vec{v}, s) = \sum_{\alpha=1}^{N_\alpha} m_\alpha^4 \, g_\alpha \, f_\alpha^{(v)}(\vec{x}, \vec{v}, s)$$

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Consider a collection of fluids, each with mass, temperature and internal dofs

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$$F^{(v)}(\vec{x}, \vec{v}, s) = \sum_{\alpha=1}^{N_\alpha} m_\alpha^4 \, g_\alpha \, f_\alpha^{(v)}(\vec{x}, \vec{v}, s)$$

$$= \frac{m_{\mathrm{hdm}} T_{\mathrm{hdm},0}^3}{(2\pi a)^3} \int_0^\infty \mathrm{d}^3 q \, F^{(q)}(\vec{x}, \vec{q}, s)$$

$$q \equiv (m_{\mathrm{hdm}}/T_{\mathrm{hdm},0}) \, v = (m_\alpha/T_{\alpha,0})$$

Mixed HDM

$$\rho_{\mathrm{hdm}}(\vec{x}, s) = \frac{m_{\mathrm{hdm}} T^3_{\mathrm{hdm},0}}{(2\pi a)^3} \int_0^\infty \mathrm{d}^3 q \, F^{(q)}(\vec{x}, \vec{q}, s).$$

Single fluid defined by

$$\{m_{\mathrm{hdm}},\ T_{\mathrm{hdm},0},\ F^{(q)}(\vec{x}, \vec{q}, s)\}$$

↓

Input for N-body simulations

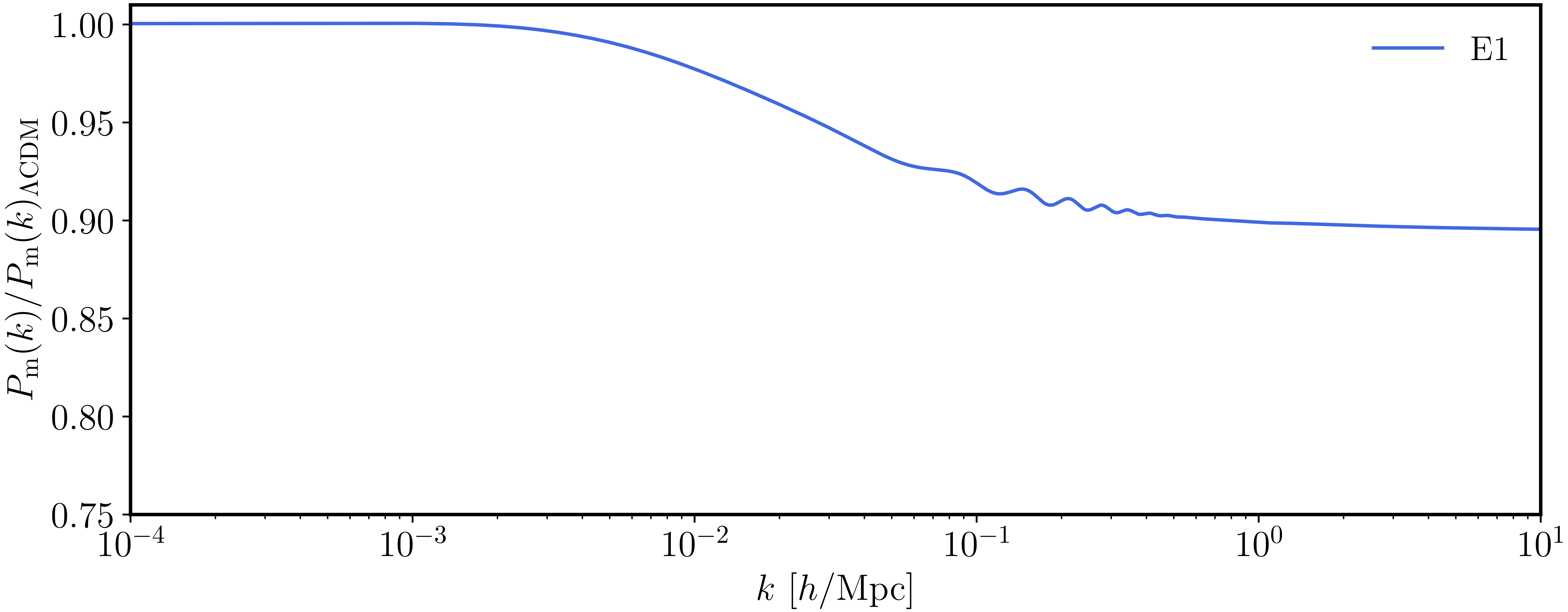
Mixed HDM models

E1: Normally-ordered SM neutrinos at minimum allowed masses

$$m_{3,2,1} = 50, 9, 0 \text{ meV}$$

$$+$$

$$m_a = 0.23 \text{ eV} \text{ QCD axion}$$



Mixed HDM models

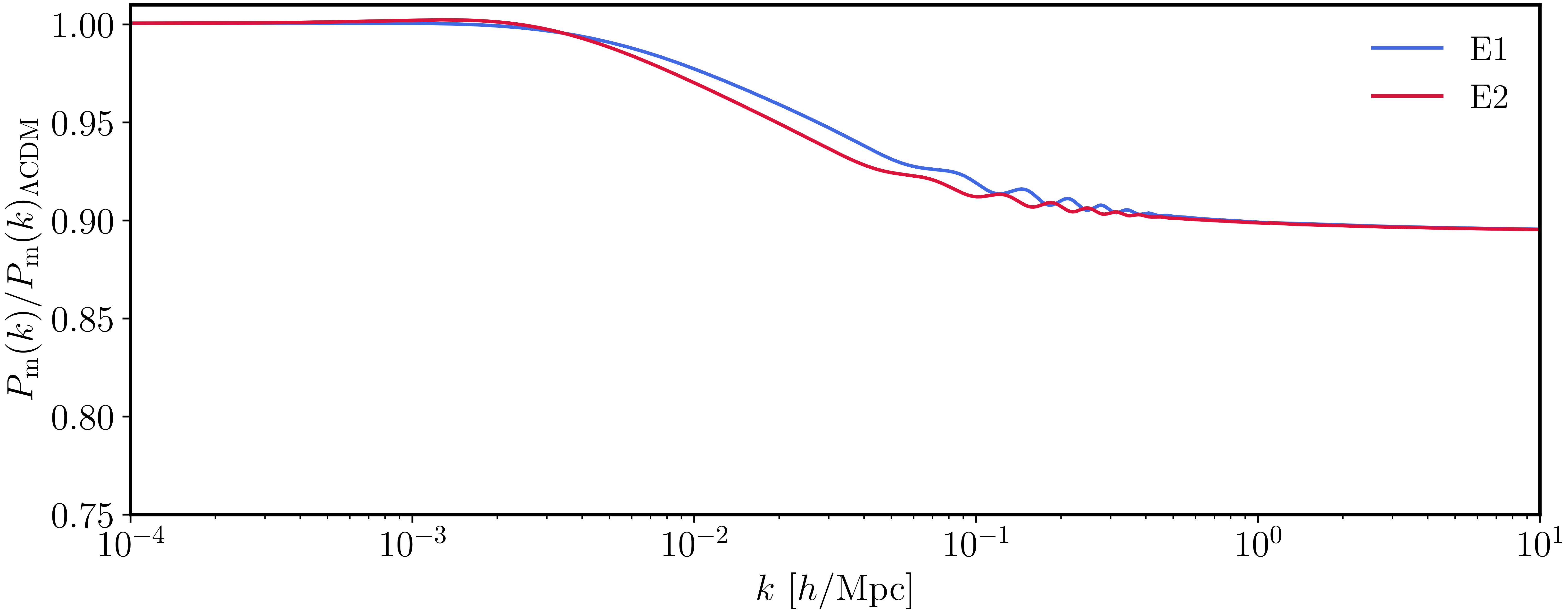
E1: Normally-ordered SM neutrinos at minimum allowed masses

$$m_{3,2,1} = 50, 9, 0 \text{ meV}$$

$$+$$

$$m_a = 0.23 \text{ eV}$$ QCD axion

E2: Normally-ordered SM neutrinos with masses such that the *linear* total matter power spectrum on small scales matches that of **E1**



Mixed HDM models

E1: Normally-ordered SM neutrinos at minimum allowed masses

$$m_{3,2,1} = 50, 9, 0 \text{ meV}$$

$$+$$

$$m_a = 0.23 \text{ eV}$$ **QCD axion**

E3: Normally-ordered SM neutrinos at minimum allowed masses

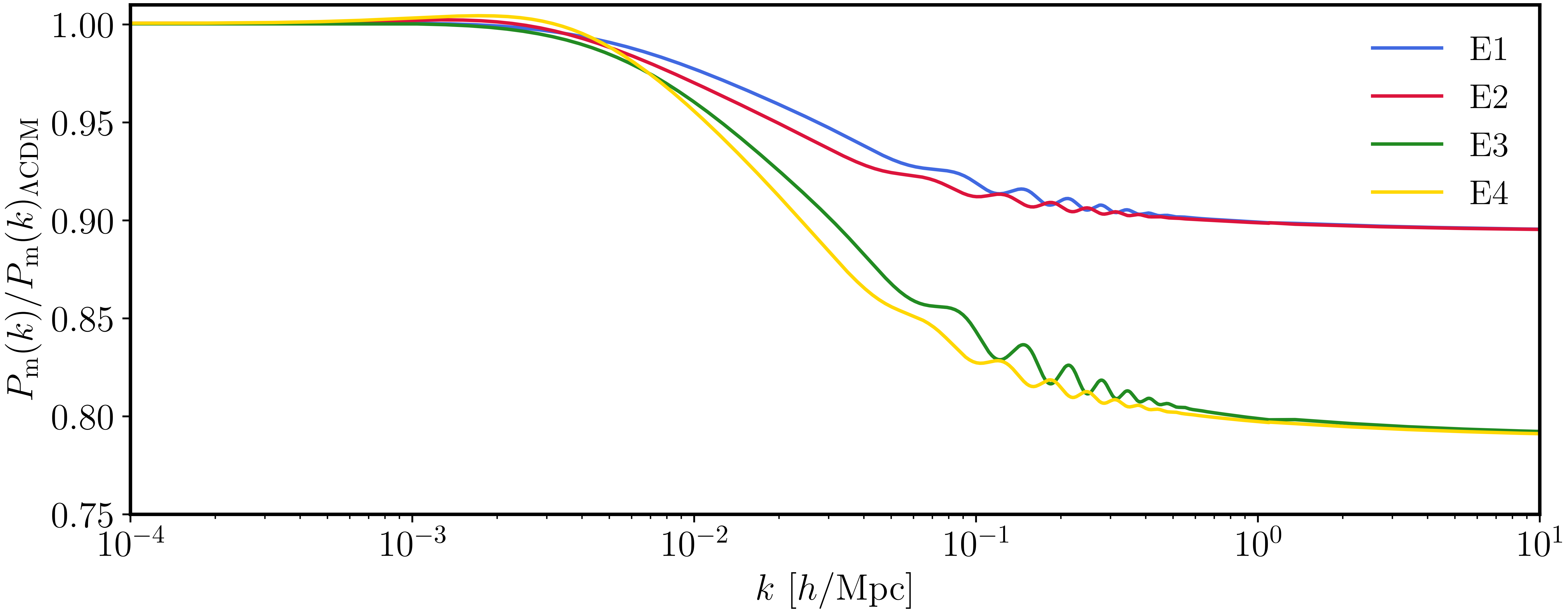
$$m_{3,2,1} = 50, 9, 0 \text{ meV}$$

$$+$$

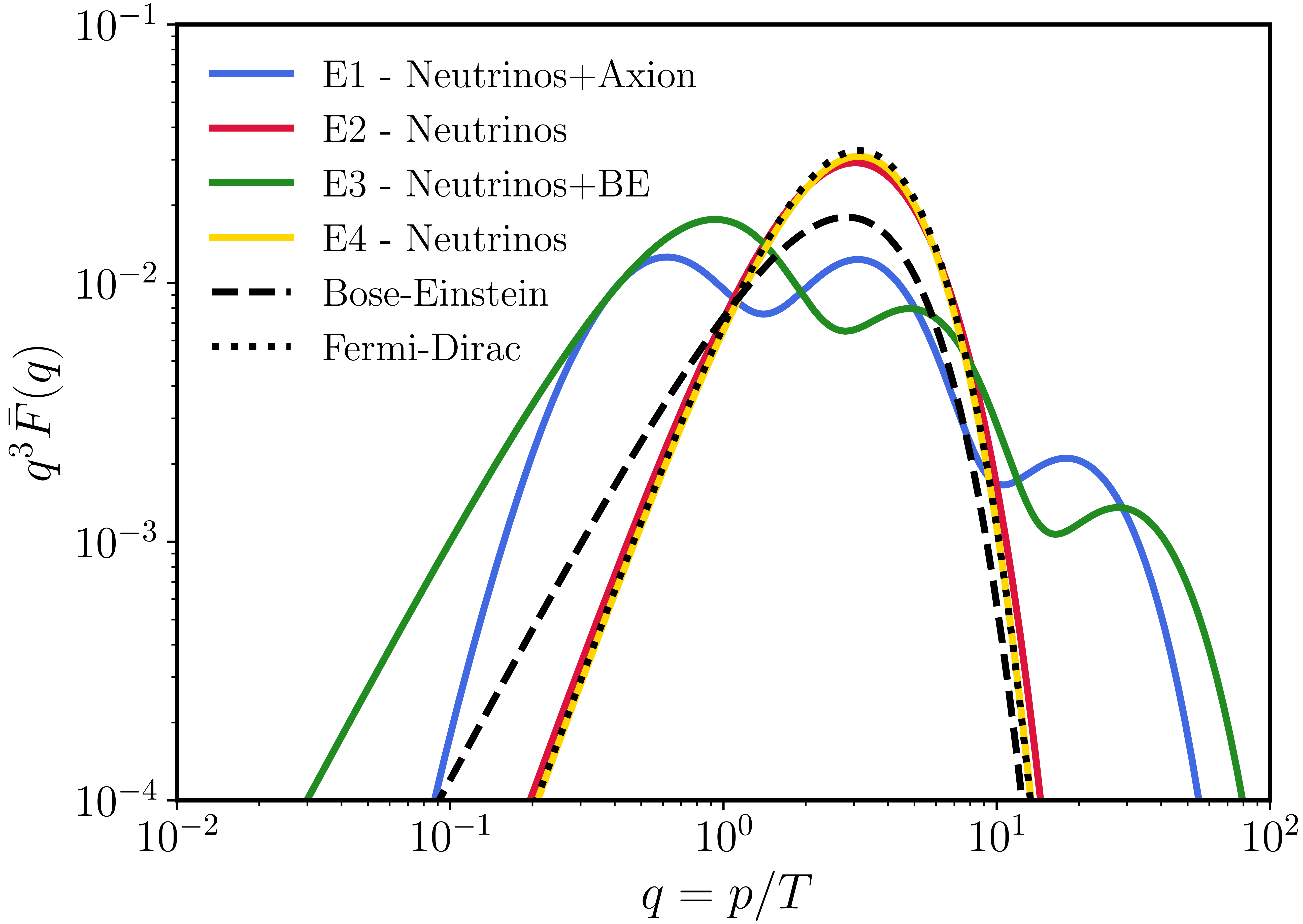
$$m_a = 0.23 \text{ eV}$$ **boson**

E2: Normally-ordered SM neutrinos with masses such that the *linear* total matter power spectrum on small scales matches that of **E1**

E4: Normally-ordered SM neutrinos with masses such that the *linear* total matter power spectrum on small scales matches that of **E3**



Mixed HDM models



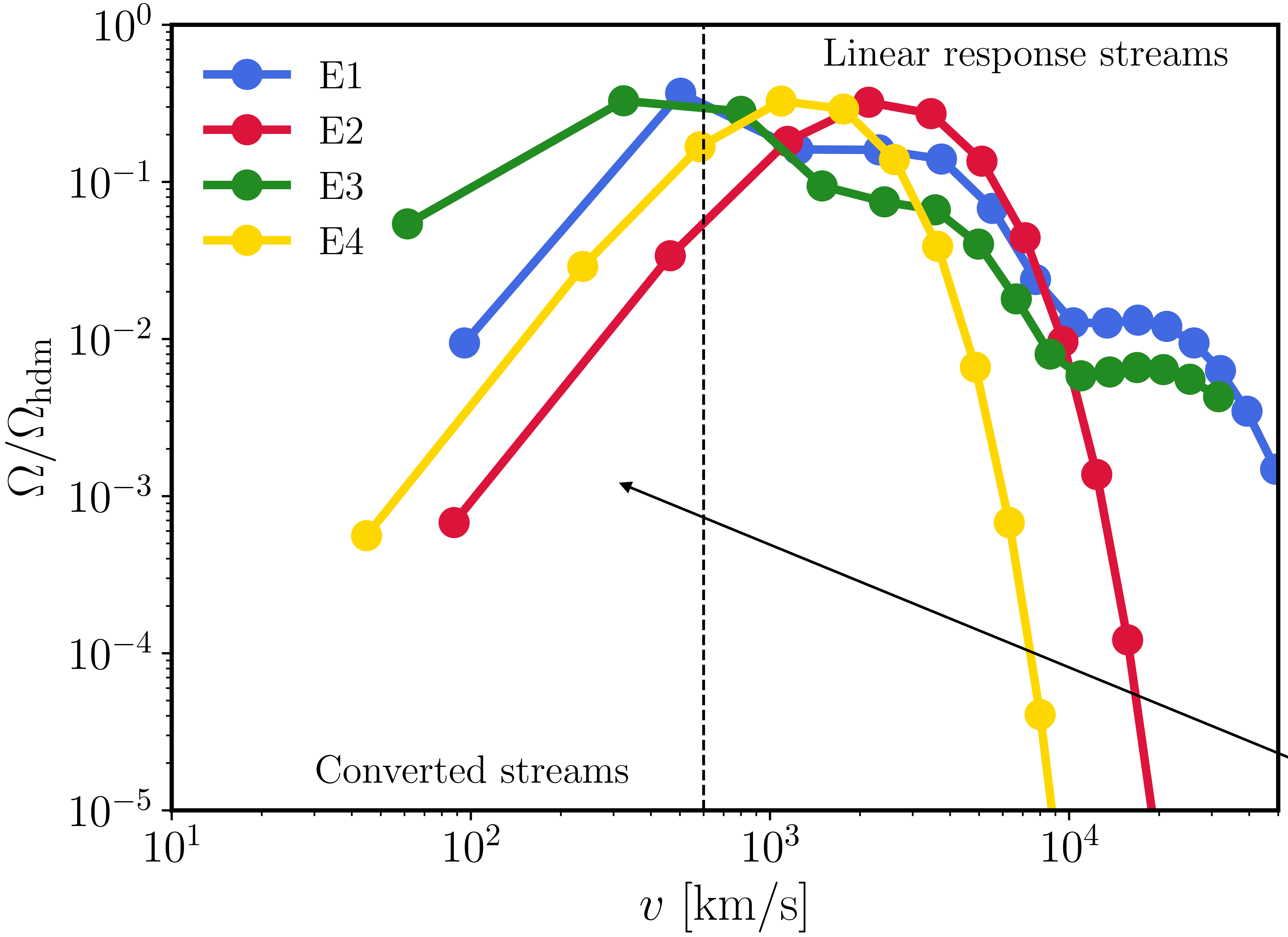
E1: $m_{\rm hdm} = 48$ meV, $T_{\rm hdm,0} = 1.88$ K

E2: $m_{\rm hdm} = 54$ meV, $T_{\rm hdm,0} = 1.95$ K

E3: $m_{\rm hdm} = 74$ meV, $T_{\rm hdm,0} = 1.88$ K

E4: $m_{\rm hdm} = 105$ meV, $T_{\rm hdm,0} = 1.95$ K

Mixed HDM models



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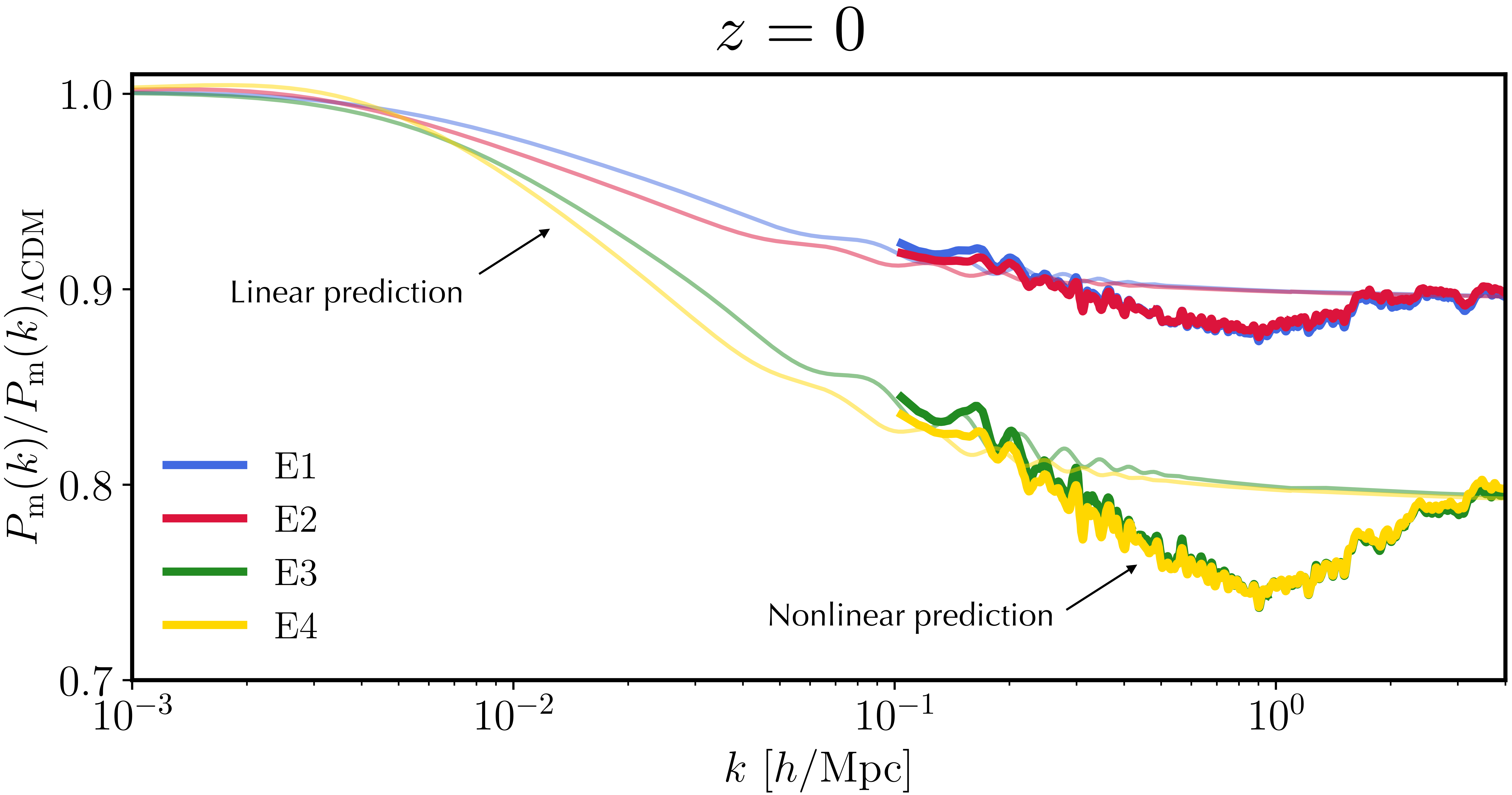
15 Gauss-Laguerre bins

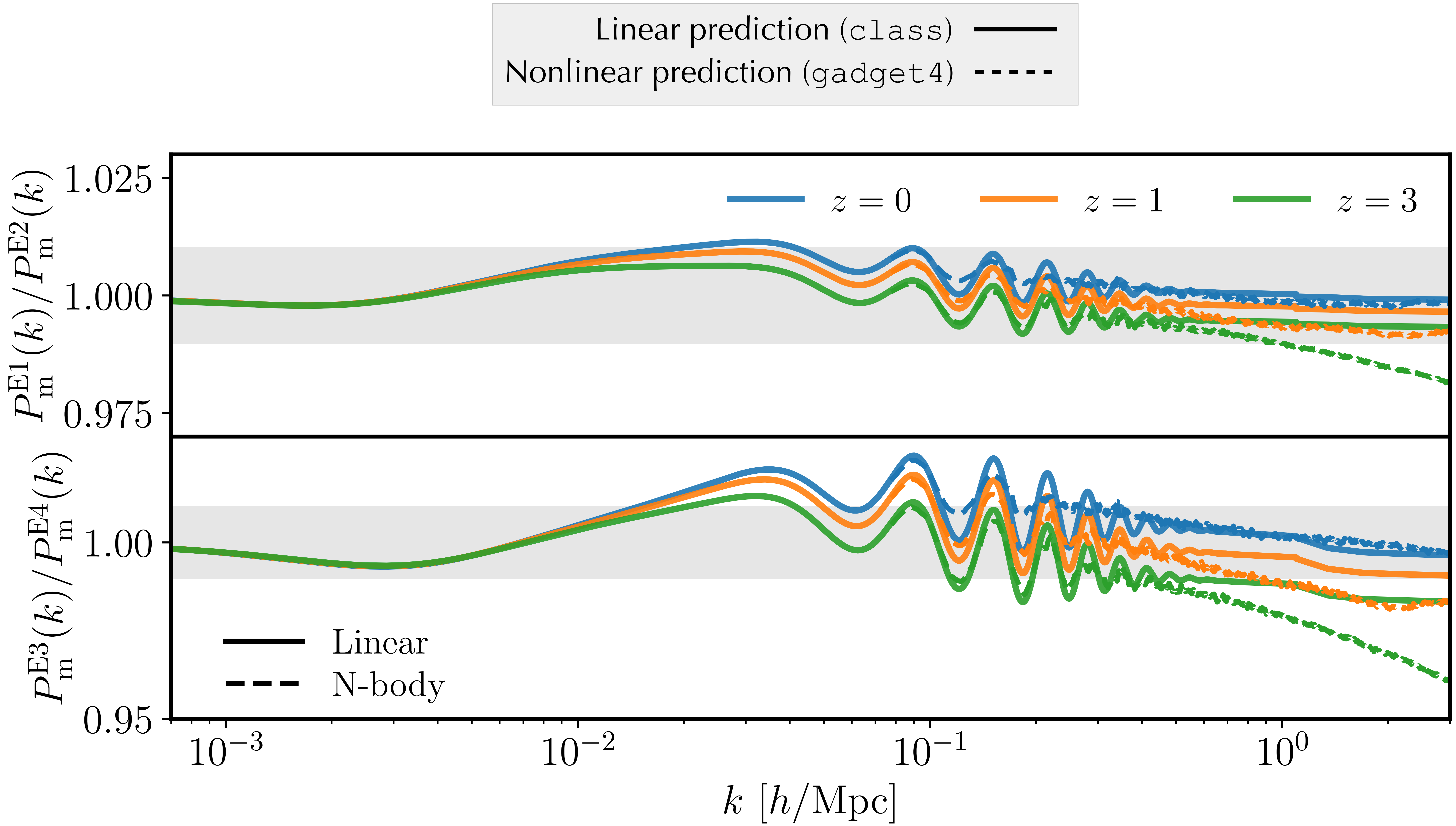
$$\int_0^\infty {\rm d}q\, q^2\, \bar{F}(q) \simeq \sum_{i=1}^{n} W_i\, q_i^2\, e^{q_i}\, \bar{F}(q_i)$$

Slowest flows are converted
into particles
for hybrid simulations

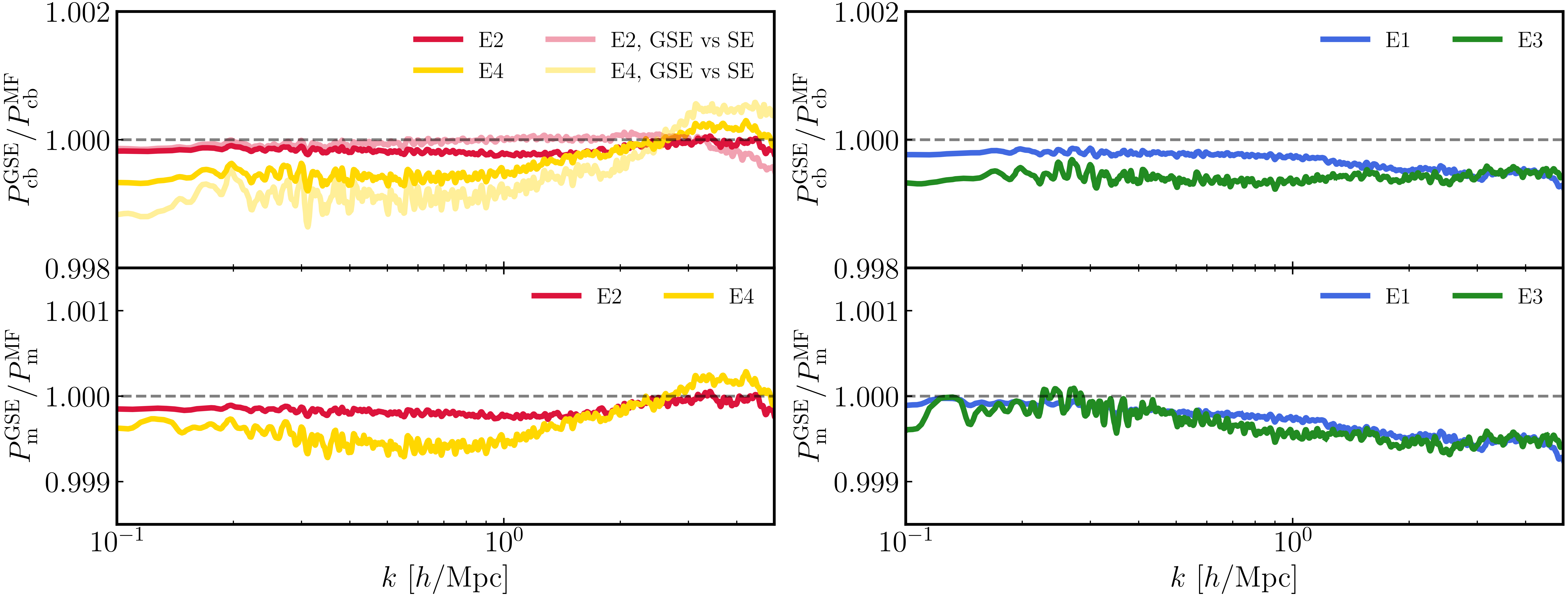
Linear vs nonlinear predictions



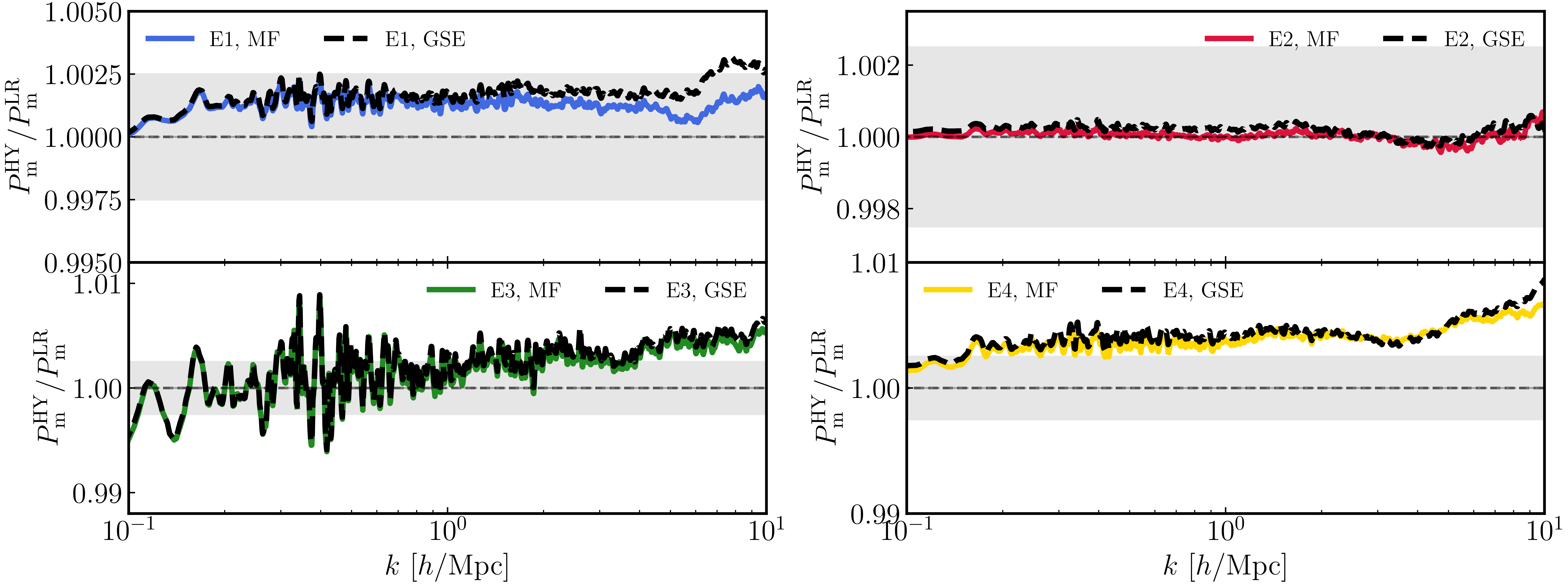
Linear vs nonlinear predictions



Convergence plots



Convergence plots



Summary

An updated version of a SuperEasy method to include HDM in non-linear modelling

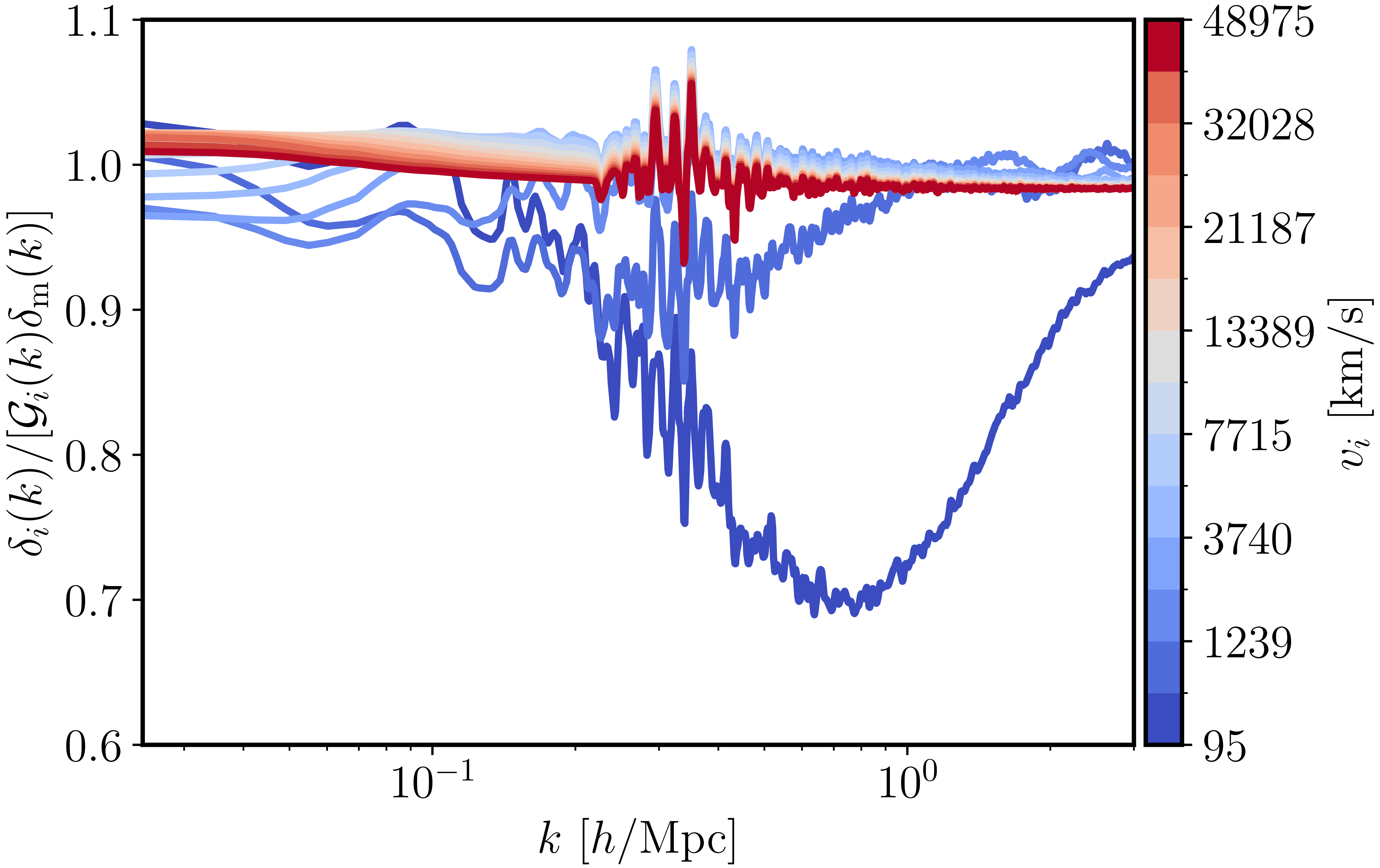
For small, unconstrained HDM masses it works reliably

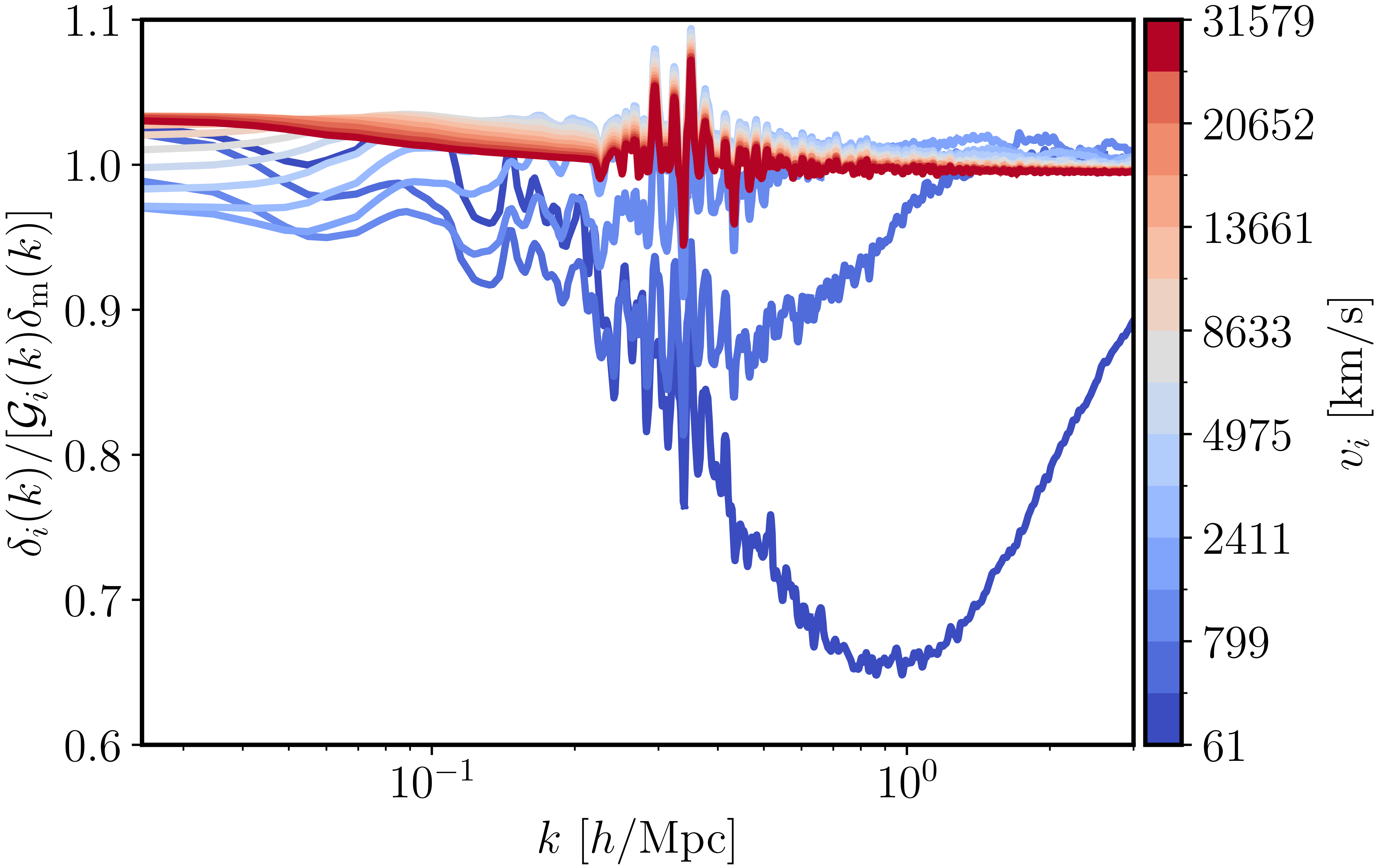
Potential to discern between neutrinos and BSM non-cold dark matter

Public code

https://github.com/cppccosmo/gadget-4-cppc

Multifluid vs SuperEasy





Hybrid enhancement