Gravitational wave spectrum from expanding string loops on domain walls: Implication to nano-hertz pulsar timing array signal

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Based on arXiv:2405.09599 w/ Y. Hamada (DESY),

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Outlines

- Introduction
- Expanding string loops on DWs
- Analytic calculation of the GW spectrum
- Summary

Introduction

Stochastic gravitational wave background

- Gravitational wave can propagate the physics earlier than CMB
- Recently, pulsar timing array exp. report the stochastic GW signals around nHz region
- The sources of GW can be super massive black hole, first order phase transition, topological objects, etc.

Calculation of GW spectrum

- The spectrum of GW is often calculated by using the numerical simulation or naive quadrupole formula

- For FOPT case, the first principle calculation is possible with some assumptions

  Jinno, Takimoto (2016), (2017)

- This first principle calculation is also possible for some specific soliton model

Expanding string loops on DWs

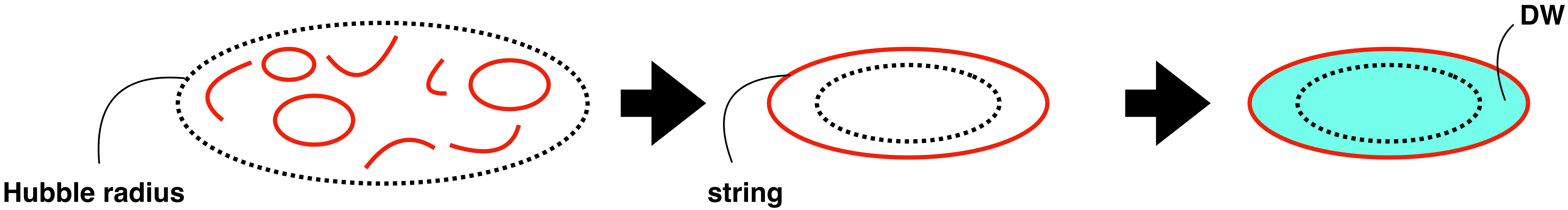
Domain walls bounded by strings

- Motivated theories (e.g., GUT) show the chain of symmetry breaking

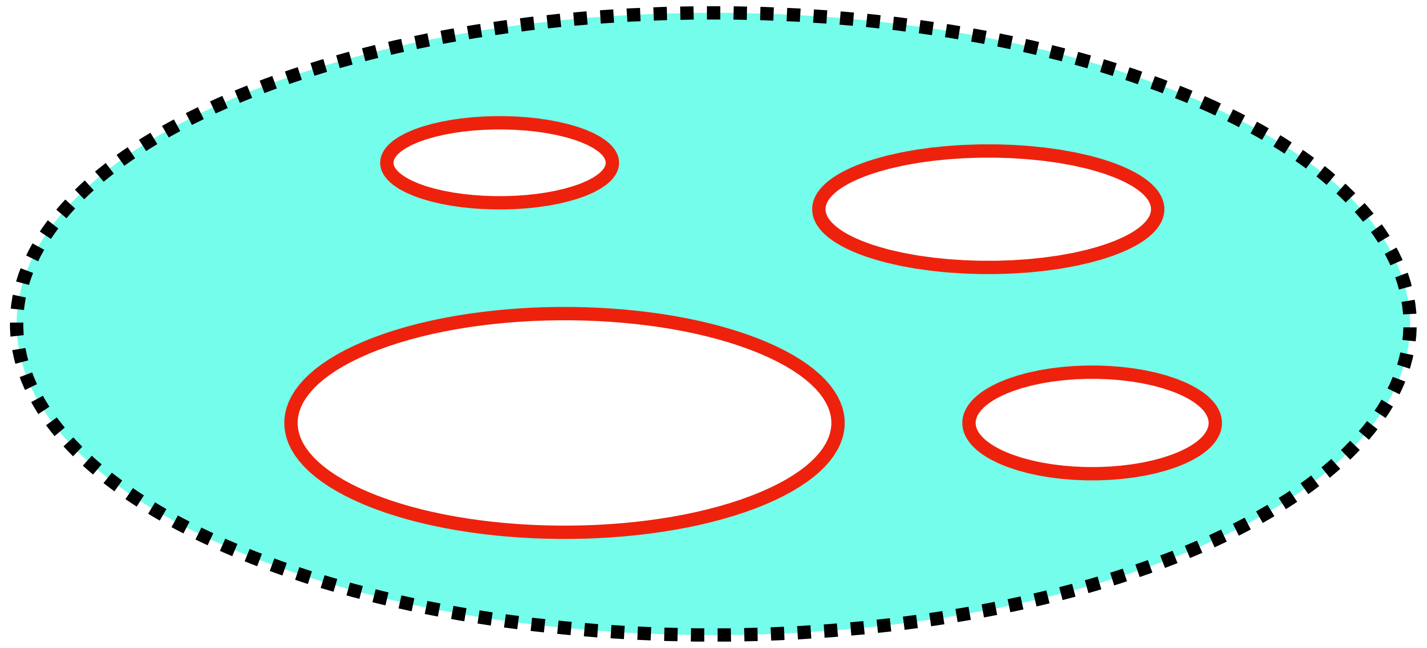
- Especially, DWs bounded by strings appear by, e.g.,

Dunsky, Ghoshal, Murayama, Sakakihara, White, 2111.08750

$$G \overset{\text{strings}}{\to} H \times Z_2 \to (\text{inflation}) \overset{\text{walls}}{\to} H$$



String loop nucleations



- **String loops are nucleated when**

Kibble, Lazarides, Shafi, '82, Preskill, Vilenkin, '92

$$E_{\rm string}(R) < E_{\rm DW}(R)$$

- **String loops expand with the released DW energy**

Dunsky, Ghoshal, Murayama, Sakakihara, White, 2111.08750

- **The nucleation rate is assumed as the Taylor expanded one**

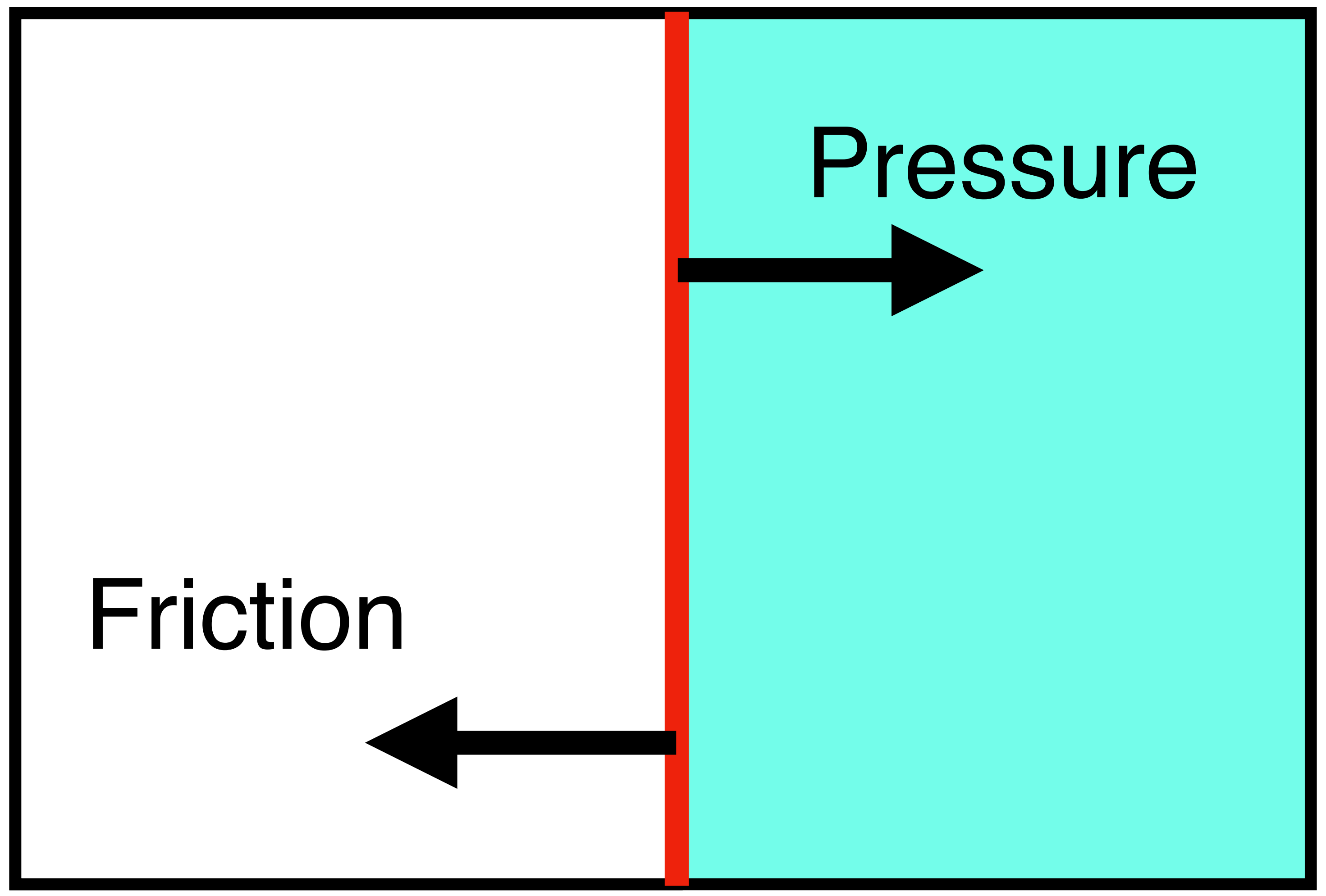
$$\Gamma(t) = \Gamma_* e^{\beta(t-t_*)} \qquad @\ t \sim t_*$$

like first-order phase transition cases

M. Turner, E. Weinberg, L. Widrow (1992), M. Kamionkowski, A. Kosowsky, M. Turner (1994)

Runaway assumption

Jinno, Takimoto (2016), (2017)



- Strings expand w/ released energy and plasma works as friction

➡ Dynamics?

- Assuming

Released energy density from DW

$$\alpha(t_*) = \rho_{\rm re}/\rho_{\rm pla}(t_*) \gtrsim \mathcal{O}(1)$$

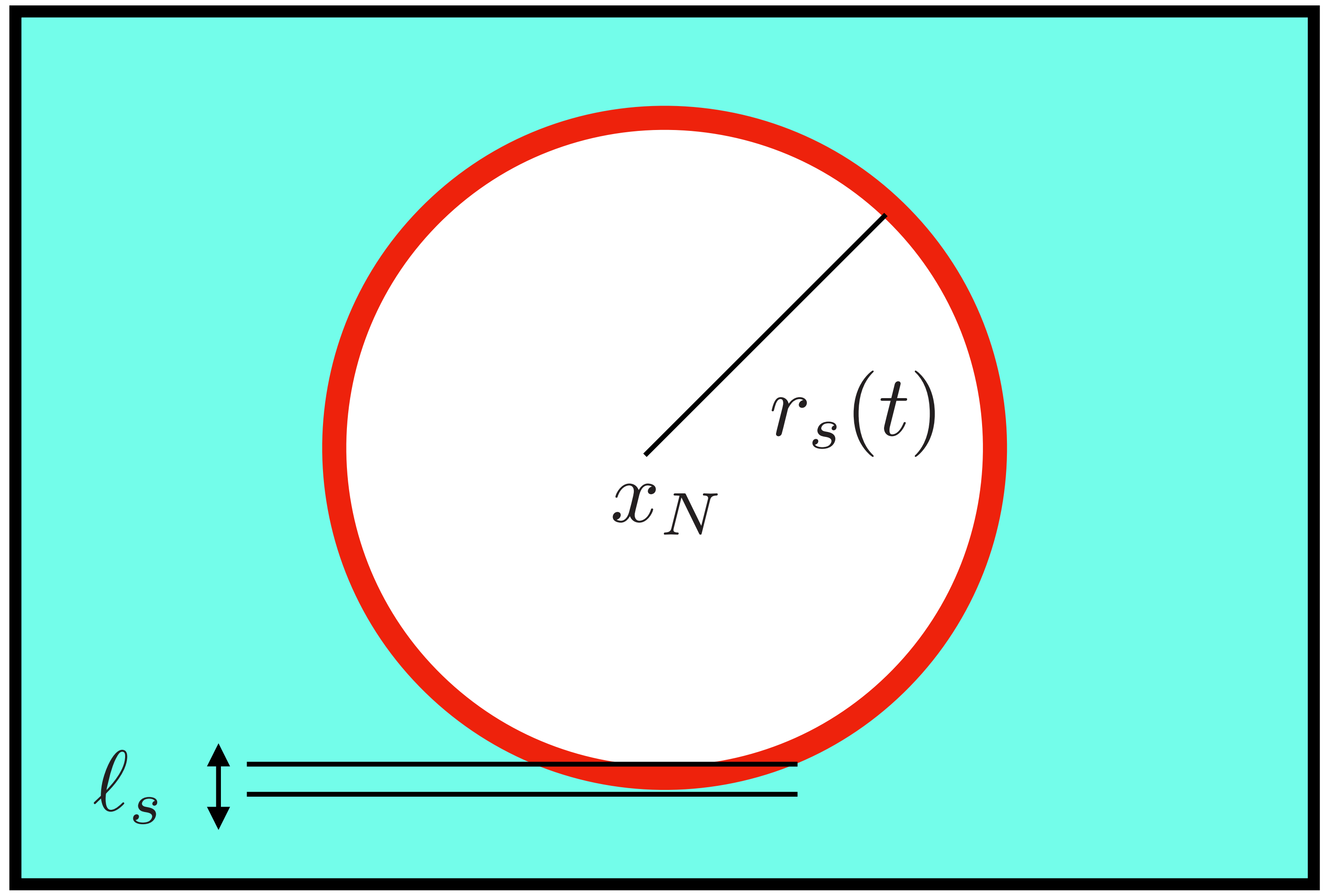
i.e., the expanding speed is the speed of light

Envelope and thin-string approximations

- **Released energy is localized at string w/ fraction κ (Thin-string approx.)**

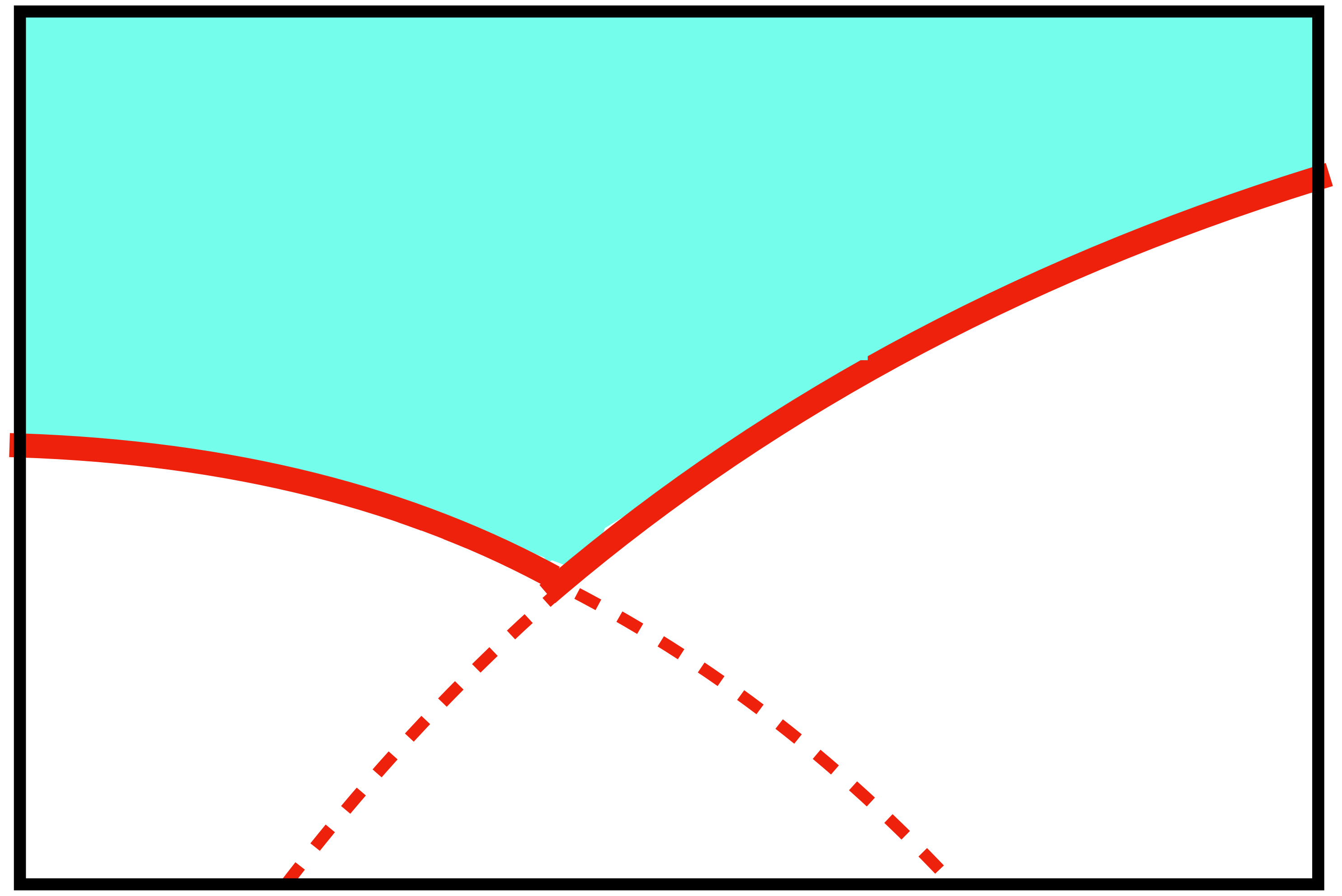
$$T_{ij}(x) = \rho^s(x)\hat{n}(x - x_N)_i\hat{n}(x - x_N)_j,$$

$$\rho^s(x) = \pi r_s(t)^2 \frac{\kappa \rho_{\rm re}}{2\pi r_s(t)\ell_s} \qquad (r_s(t) < |\vec{x} - \vec{x}_N| < r_s(t) + \ell_s)$$



- **Collided strings (dotted) are neglected (Envelope approx.)**

Kosowsky, Turner, Watkins, PRD45 ('92)



Dynamics is determined

→ Calculation **for flat DWs**

in the same way as the FOPT calculation

Jinno, Takimoto (2016)

Analytic calculation of the GW spectrum

Energy density of GW

- **Metric:** $ds^2 = -dt^2 + (\delta_{ij} + 2h_{ij})dx^i dx^j + \mathcal{O}(h^2)$

- **Einstein eq.:** $\ddot{h}_{ij}^{\rm TT}(t,\vec{k}) + \vec{k}^2 h_{ij}^{\rm TT}(t,\vec{k}) = 8\pi G \Pi_{ij}^{\rm TT}(t,\vec{k})$

$$\Rightarrow \quad h_{ij}^{\rm TT}(t,\vec{k}) = 8\pi G \int_{-\infty}^{t} dt' \underbrace{G_k(t,t')}_{\text{Green func.}} \underbrace{\Pi_{ij}^{\rm TT}(t',\vec{k})}_{\text{Energy-momentum tensor of expanding string loops}}$$

Green func. Energy-momentum tensor of expanding string loops

$O(2) \neq O(3)$ sym.

- **Energy dens.** $\rho_{\rm GW}(t) = \frac{1}{8\pi G} \langle \dot{h}_{ij}^{\rm TT}(t,\vec{x})\, \dot{h}^{{\rm TT},ij}(t,\vec{x}) \rangle_T$

$$\propto \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cos\left(k(t_1 - t_2)\right) \Pi(t_1, t_2, k)$$

$$\Pi(t_1,t_2,k) = K_{ijkl}(\hat{k}) K_{ijmn}(\hat{k}) \int d^3 r e^{i\vec{k}\cdot\vec{r}} \langle T_{kl} T_{mn} \rangle_{\rm ens}(t_1,t_2,r)$$

TT-projection — Known from dynamics — **Need to calculate**

Ensemble average

$$\langle T_{kl} T_{mn} \rangle_{\text{ens}}(t_1, t_2, r) = \int \boxed{\text{Prob. dist. for GW passing two points } (\vec{r}_1, t_1), (\vec{r}_2, t_2)} \times T_{kl}(\vec{r}_1, t_1) T_{mn}(\vec{r}_2, t_2)$$

- **The probability is given by conditions of**

(I): The points are inside DWs (~3D-Buffon's needle)

(II): No string loop is nucleated inside the past light cones (Envelope approx.)

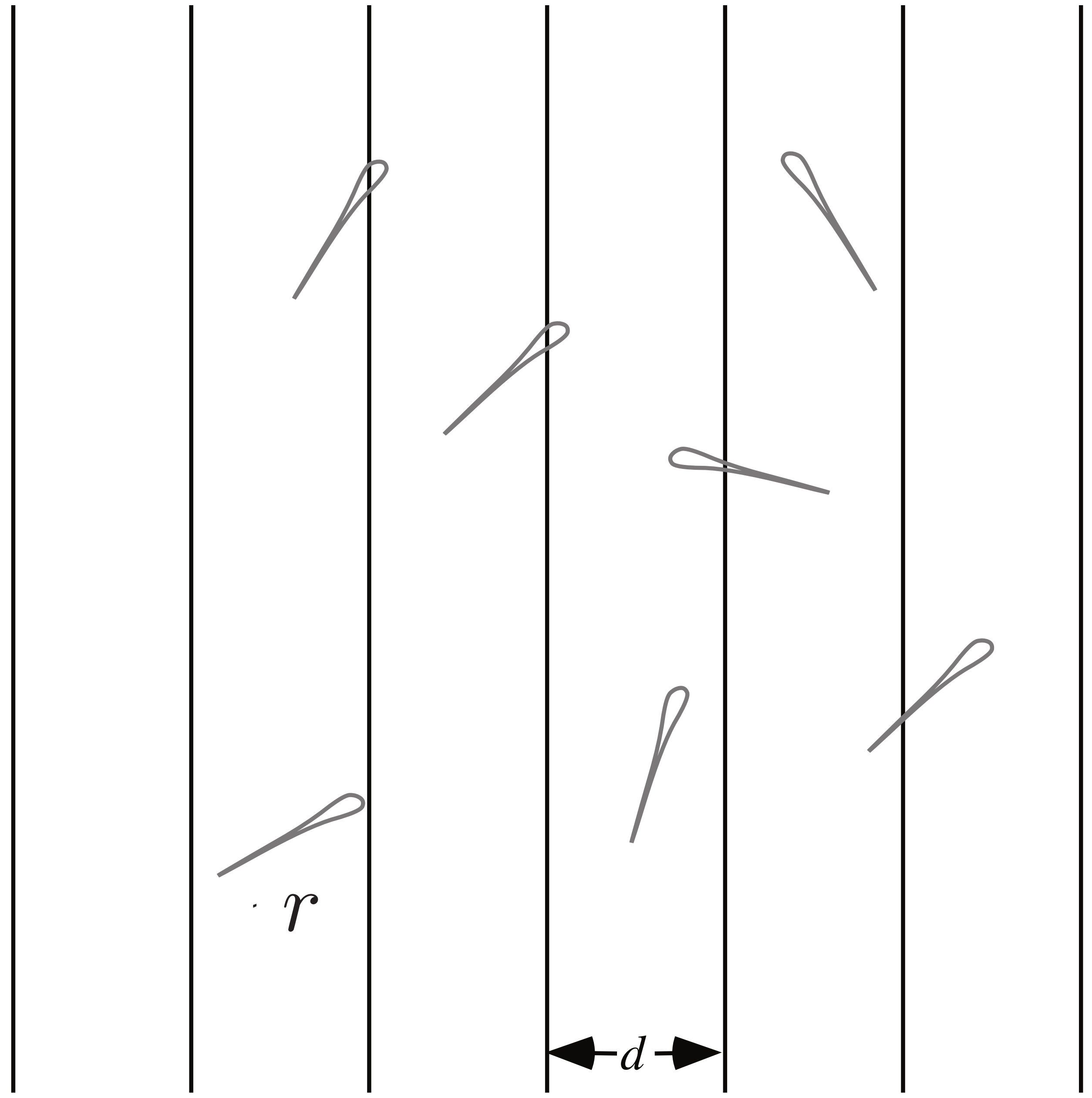
(III): The string-loop nucleation points are on the surface of past light cones (Thin-string approx.)

(I):Buffon's needle

https://mathworld.wolfram.com/BuffonsNeedleProblem.html

- **The distance of two points r must be inside DWs of width $d_{\rm DW}$**

$$r = |\vec{r}_1 - \vec{r}_2|$$



$$C^{(0)}_{\rm DW}(r) = \int_0^{\pi} d\phi \int_{\cos^{-1}(d_{\rm DW}/r)}^{\pi/2} d\theta \int_{-\frac{d_{\rm DW}}{2}+\frac{r}{2}\cos\theta}^{\frac{d_{\rm DW}}{2}-\frac{r}{2}\cos\theta} dh \, \frac{\sin\theta}{\pi d_H}$$
$$= \frac{d_{\rm DW}^2}{2 r d_H}.$$

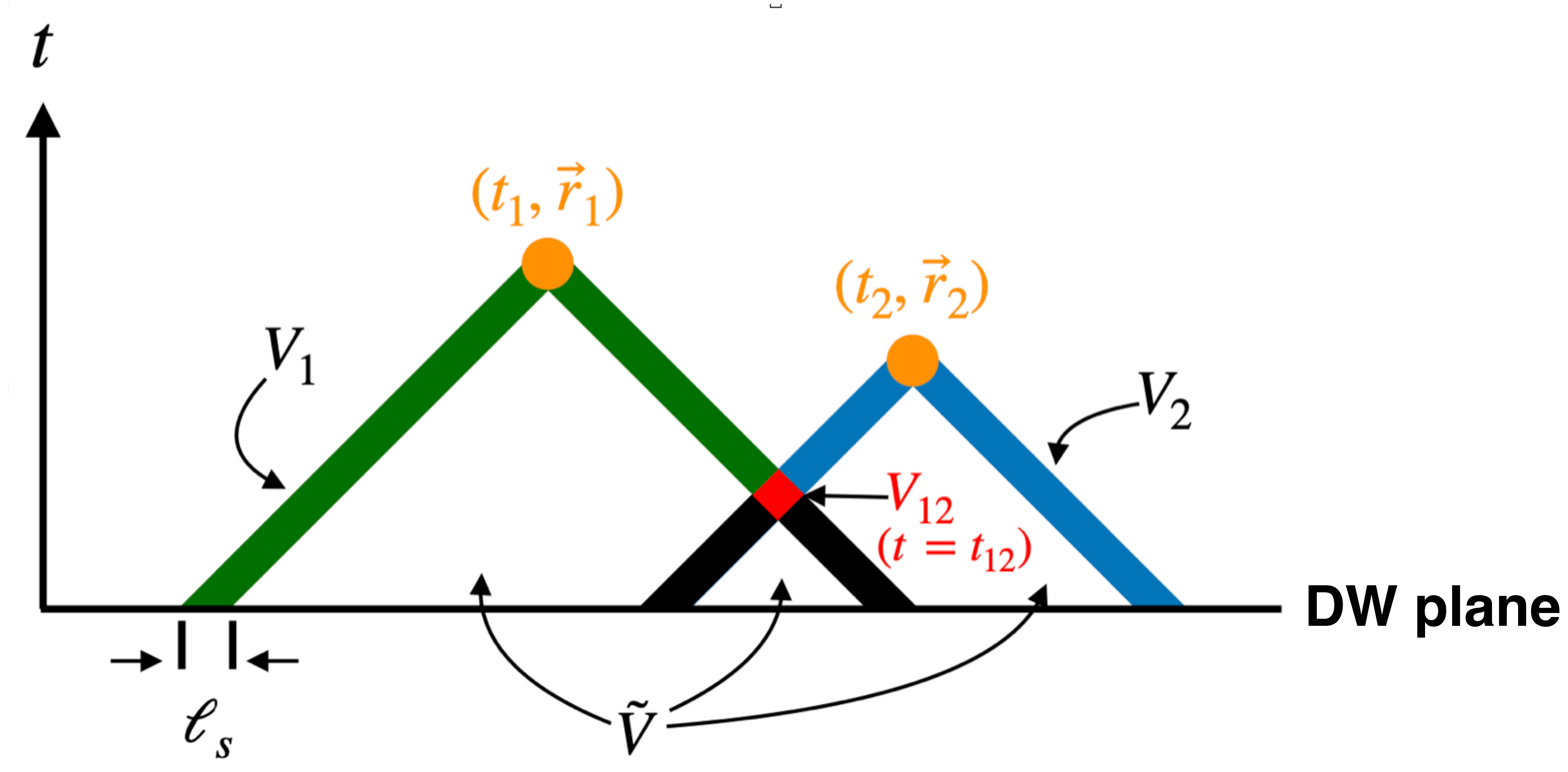
(II):DW remained in past light cones

- **No nucleation in volume $\tilde{V}$**

$$P(t_1, t_2, r) = \prod_i (1 - \Gamma(t) d\tilde{V}^i)$$
$$= e^{-I(x_1, x_2)},$$

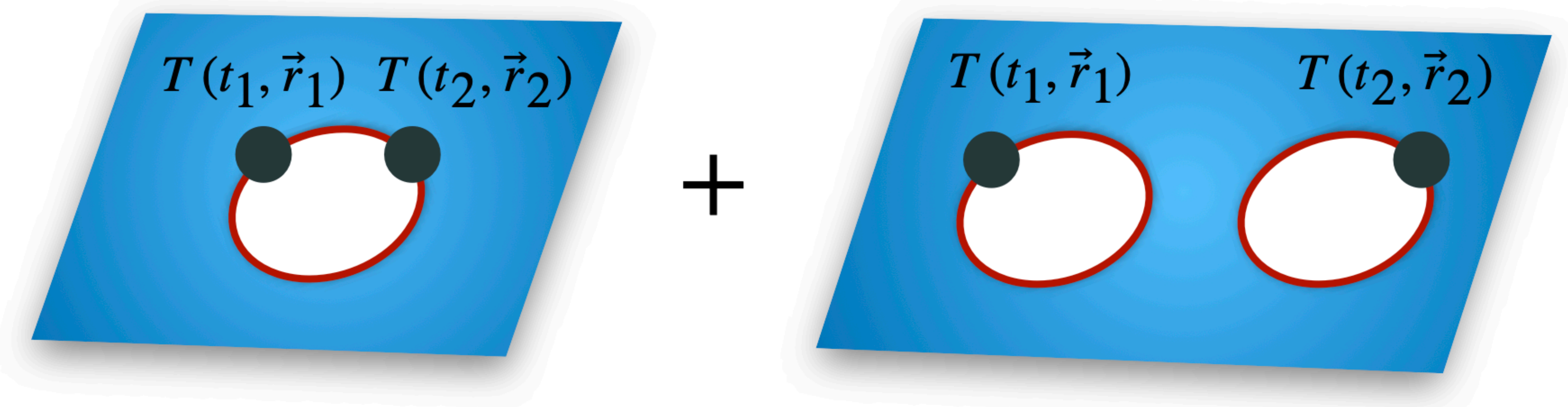
$$I(x_1, x_2) = \Gamma(T)\mathcal{I}(r, t_d)$$

Nucleation rate **Volume factor**



(III):Nucleations on past light cone surface

- **Nucleation rate times area**
- **Contributions from one and two string loops**

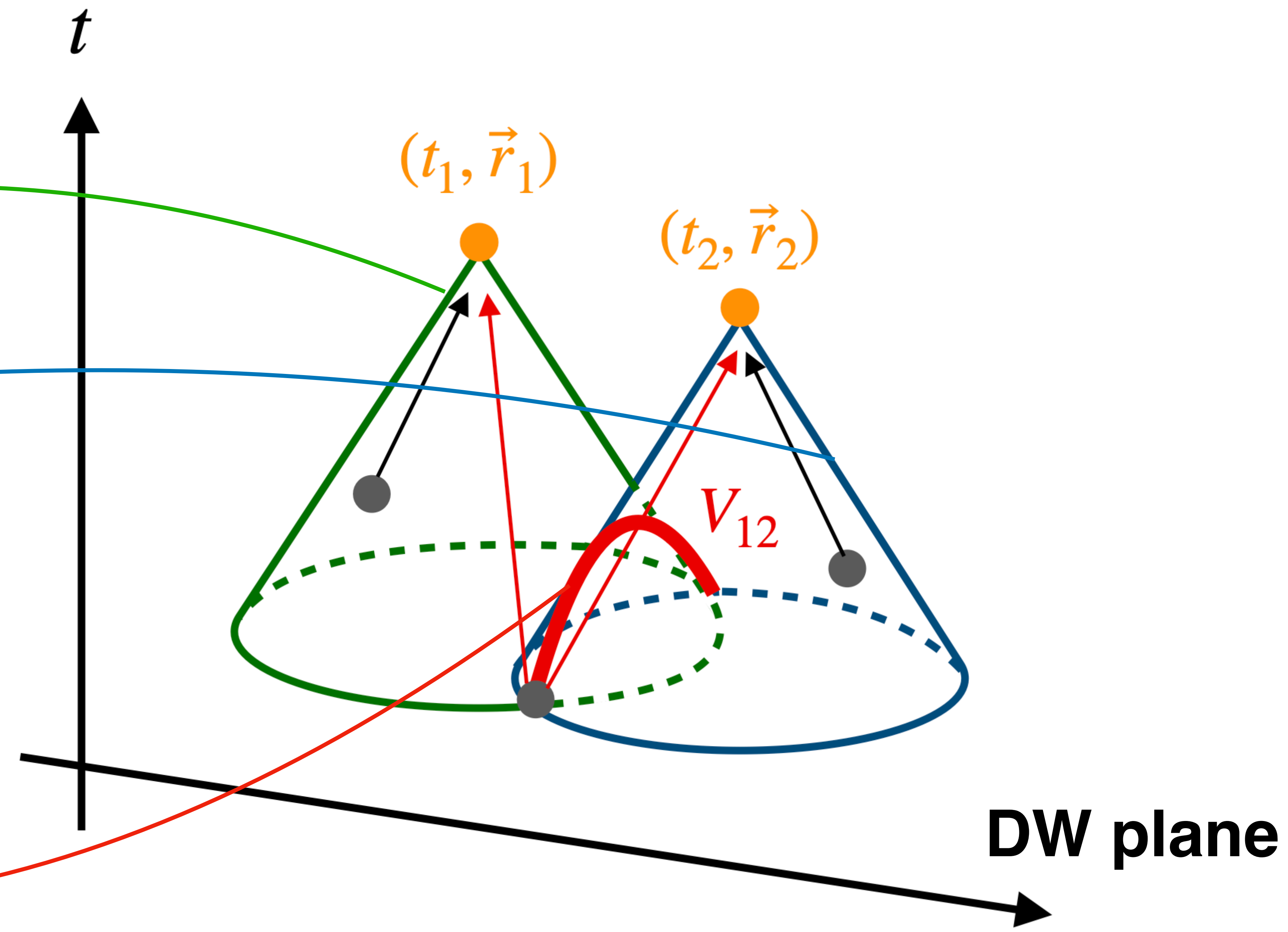


One string-loop cont. Two string-loop cont.

$$\langle T_{kl}T_{mn}\rangle^{\rm double}_{\rm ens}(t_1,t_2,r) = C_{\rm DW}(r)P(t_1,t_2,r)\int_{-\infty}^{t_1} dt_{n1}\Gamma(t_{n1})\int_{A_1} d^2z_1 T_{kl}(t_1,\vec{r})$$

$$\times \int_{-\infty}^{t_2} dt_{n2}\Gamma(t_{n2})\int_{A_2} d^2z_2 T_{mn}(t_2,\vec{r})$$

$$\langle T_{kl}T_{mn}\rangle^{\rm single}_{\rm ens}(t_1,t_2,r) = C_{\rm DW}(r)P(t_1,t_2,r)\int_{-\infty}^{t_{12}} dt_n\Gamma(t_n)\int_{R_{12}} d^2z T_{kl}(t_1,\vec{r})T_{mn}(t_2,\vec{r})$$



Dimensionless GW spectrum function

- $\Delta(k/\beta)$ **is expressed by the sphe. Bessel func.** j_0, j_1, j_2 **for TT-gauge**

Clearer for FOPT case and also true w/o envelop approx.

Jinno, Takimoto (2016) **Jinno, Takimoto (2017)**

$\langle T^{TT} T^{TT} \rangle \supset (\hat{k} \cdot \hat{r})^4$

$\cos^4$

$$\Delta^{\rm single}(k/\beta) = \frac{3}{4\pi}\beta k^3 \int_0^\infty dt_d \cos(kt_d) \int_{t_d}^\infty dr \frac{C_{\rm DW}(r)}{r^3 \mathcal{I}(r,t_d)} \left[j_0(kr) F_0' + \frac{j_1(kr)}{kr} F_1' + \frac{j_2(kr)}{(kr)^2} F_2' \right]$$

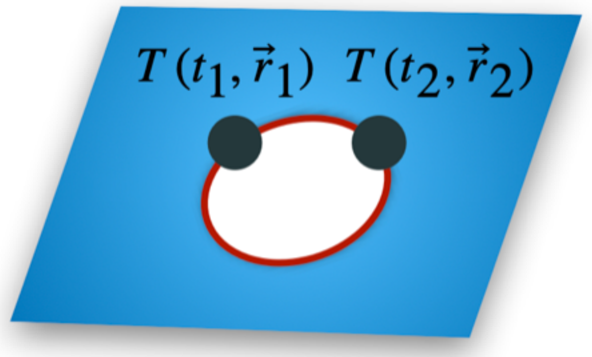
$$\begin{aligned}\Delta^{\rm double}(k/\beta) = & \frac{3}{64\pi}\beta k^3 \int_0^\infty dt_d \cos(kt_d) \int_{t_d}^\infty dr \frac{r^2 C_{\rm DW}(r)}{\mathcal{I}(r,t_d)^2} \\ & \times \Bigg[\left(j_0(kr) - 2\frac{j_1(kr)}{kr} + 3\frac{j_2(kr)}{(kr)^2} \right) \\ & \quad \times \left(g''_{a,{\rm ov}}(r,t_d) + g''_{a,{\rm non}}(r,t_d) \right) \left(g''_{a,{\rm ov}}(r,-t_d) + g''_{a,{\rm non}}(r,-t_d) \right) \\ & \quad + \left(j_0(kr) - 2\frac{j_1(kr)}{kr} - 5\frac{j_2(kr)}{(kr)^2} \right) \\ & \quad \times \left[\left(g''_{a,{\rm ov}}(r,t_d) + g''_{a,{\rm non}}(r,t_d) \right) g''_{b,{\rm ov}}(r,-t_d) + g''_{b,{\rm ov}}(r,t_d) \left(g''_{a,{\rm ov}}(r,-t_d) + g''_{a,{\rm non}}(r,-t_d) \right) \right] \\ & + \left(j_0(kr) - 2\frac{j_1(kr)}{kr} + 19\frac{j_2(kr)}{(kr)^2} \right) g''_{b,{\rm ov}}(r,t_d) g''_{b,{\rm ov}}(r,-t_d) \Bigg] .\end{aligned}$$

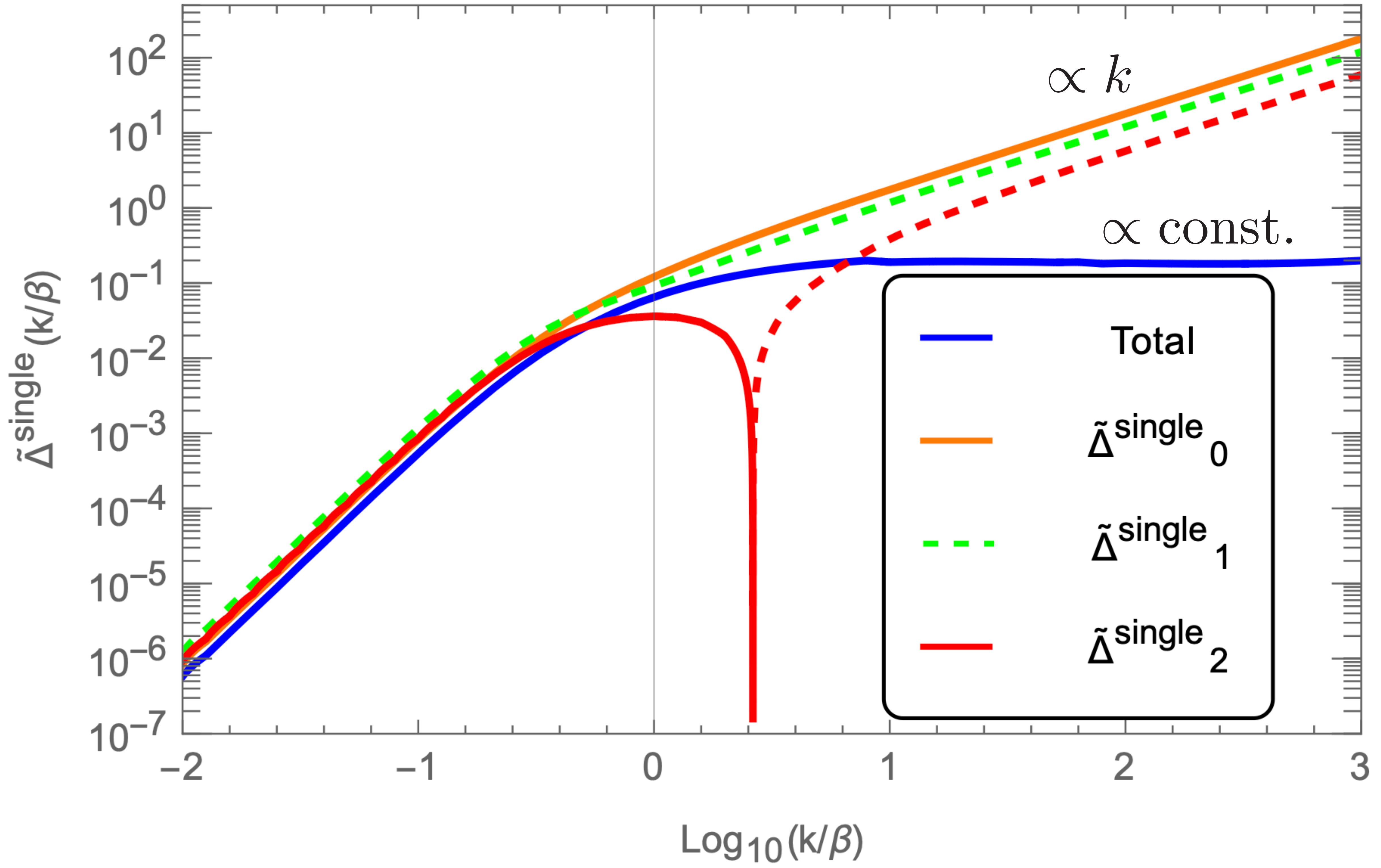
From Green func.

(Not main story)

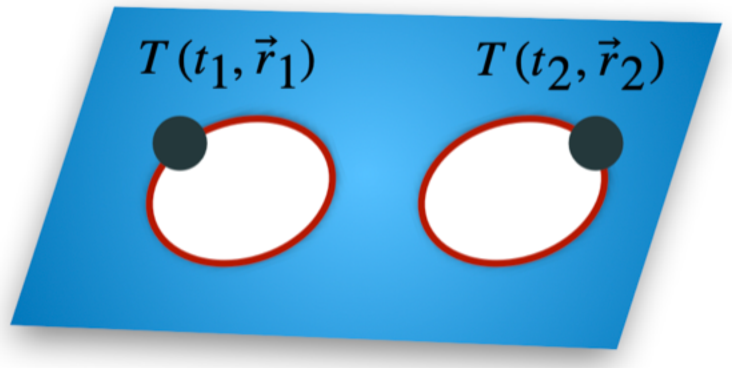
TT-gauge is not good for this calculation?

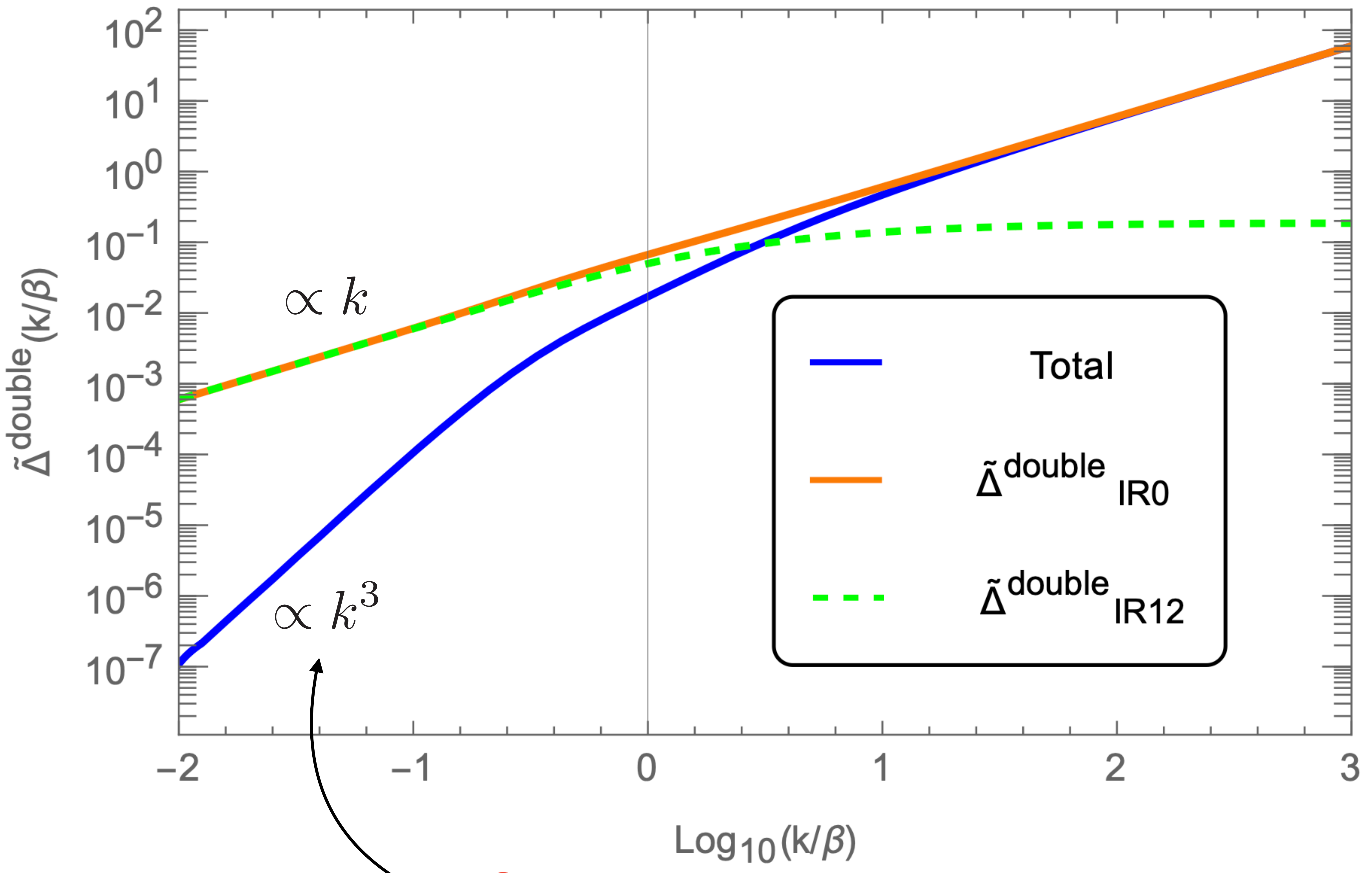
- Sums of j_0, j_1, j_2 components show cancellations





Including const. part





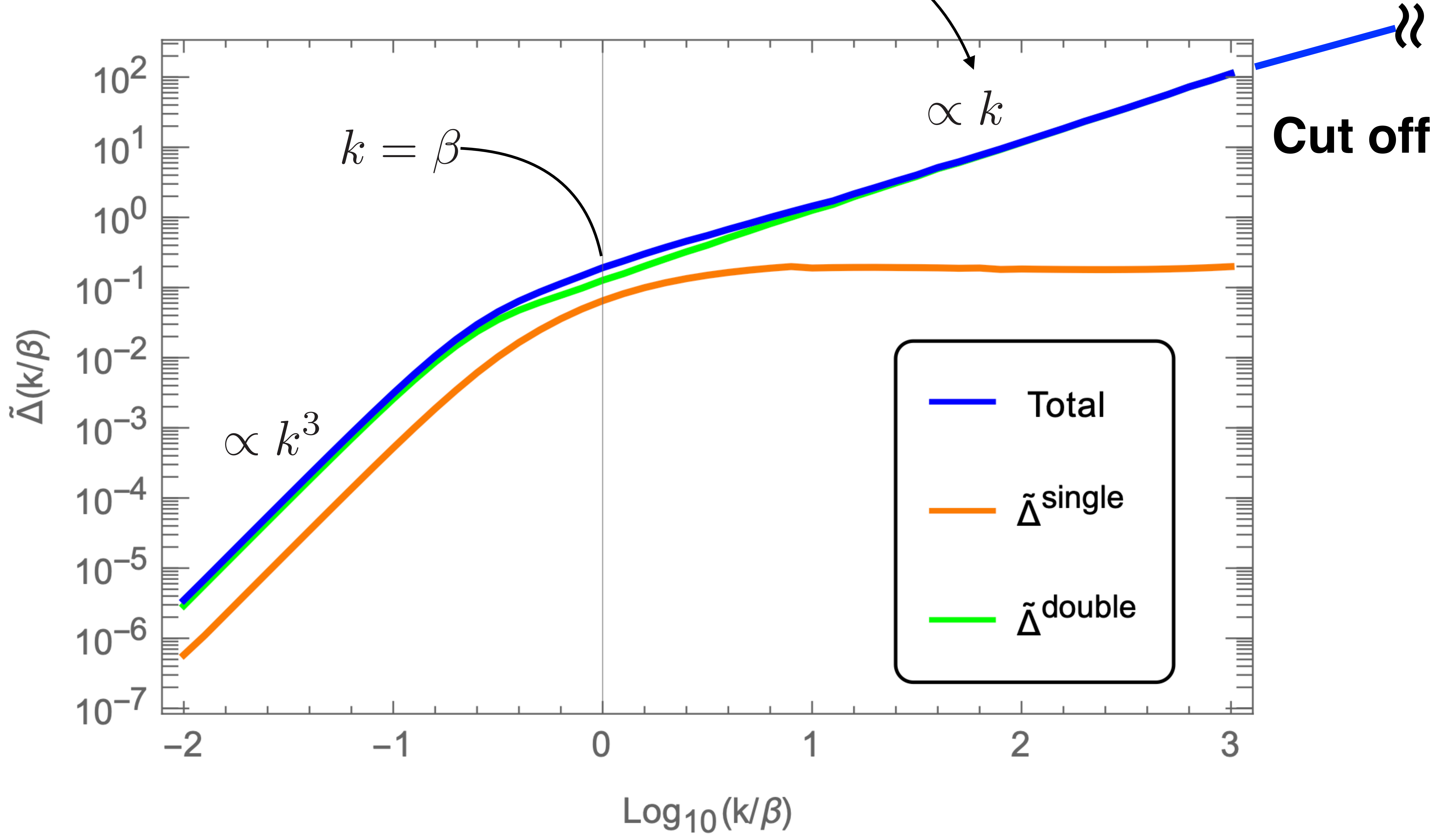
Causality shows up

Caprini, Durrer, Konstandin, Servant, 0901.1661

Total spectrum

Maybe unique for constant term of source in time

e.g., Roper Pol, Caprini, Neronov, Semikoz, [2201.05630].



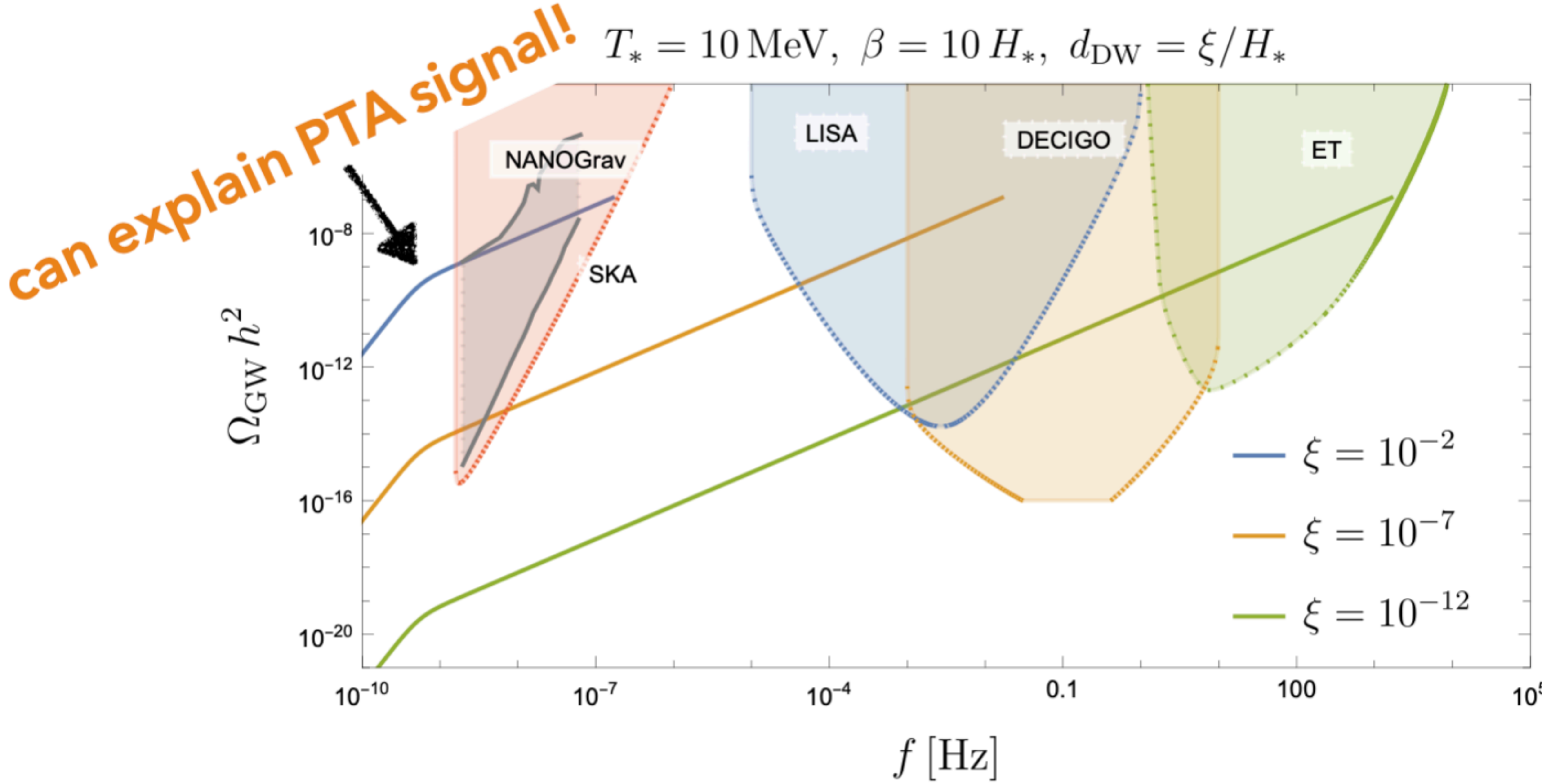
Cut off @ DW scale

or initial loop radius

Present GW spectrum

[YH-Nakano 2405.09599]

$\sigma \simeq (10^4\,\mathrm{GeV})^3$



- Assumptions: $\begin{cases} \text{non-thermal production of DW, } d_{\text{walls}} = d_H \big|_{\text{wall prod.}} \\ \text{DW not in scaling regime, } N_{\text{walls}} = d_{H_*}/d_{\text{walls}} \text{ per Hubble} \end{cases}$

https://indico.cern.ch/event/1335106/contributions/5833097/attachments/2857038/5002596/EuCAPT_hamada.pdf

Summary

- Expanding string loops emit GW due to $O(2) \neq O(3)$ symmetry
- The GW spectrum from expanding string loops is derived and fitted with the PTA signals
- This calculation neglected the Hubble expansion, sound wave, and turbulence of plasma (or plasma itself)

Appendix

Definition of spectrum function

$$\Omega_{\mathrm{GW}}(t_*, k_*) = \kappa^2 \left(\frac{H_*}{\beta}\right)^2 \left(\frac{\alpha(t_*)}{1+\alpha(t_*)}\right)^2 \Delta(k_*/\beta)\,,$$

$$\begin{aligned}\Delta(k/\beta) &= \frac{3}{8\pi G}\frac{\beta^2 \rho_{\mathrm{tot}}(t)}{\kappa^2 \rho_{\mathrm{re}}^2}\Omega_{\mathrm{GW}}(t,k),\\ &= \frac{3}{4\pi^2}\frac{\beta^2 k^3}{\kappa^2 \rho_{\mathrm{re}}^2}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cos\left(k(t_1-t_2)\right)\Pi(t_1,t_2,k)\end{aligned}$$

Spherical Bessel representation

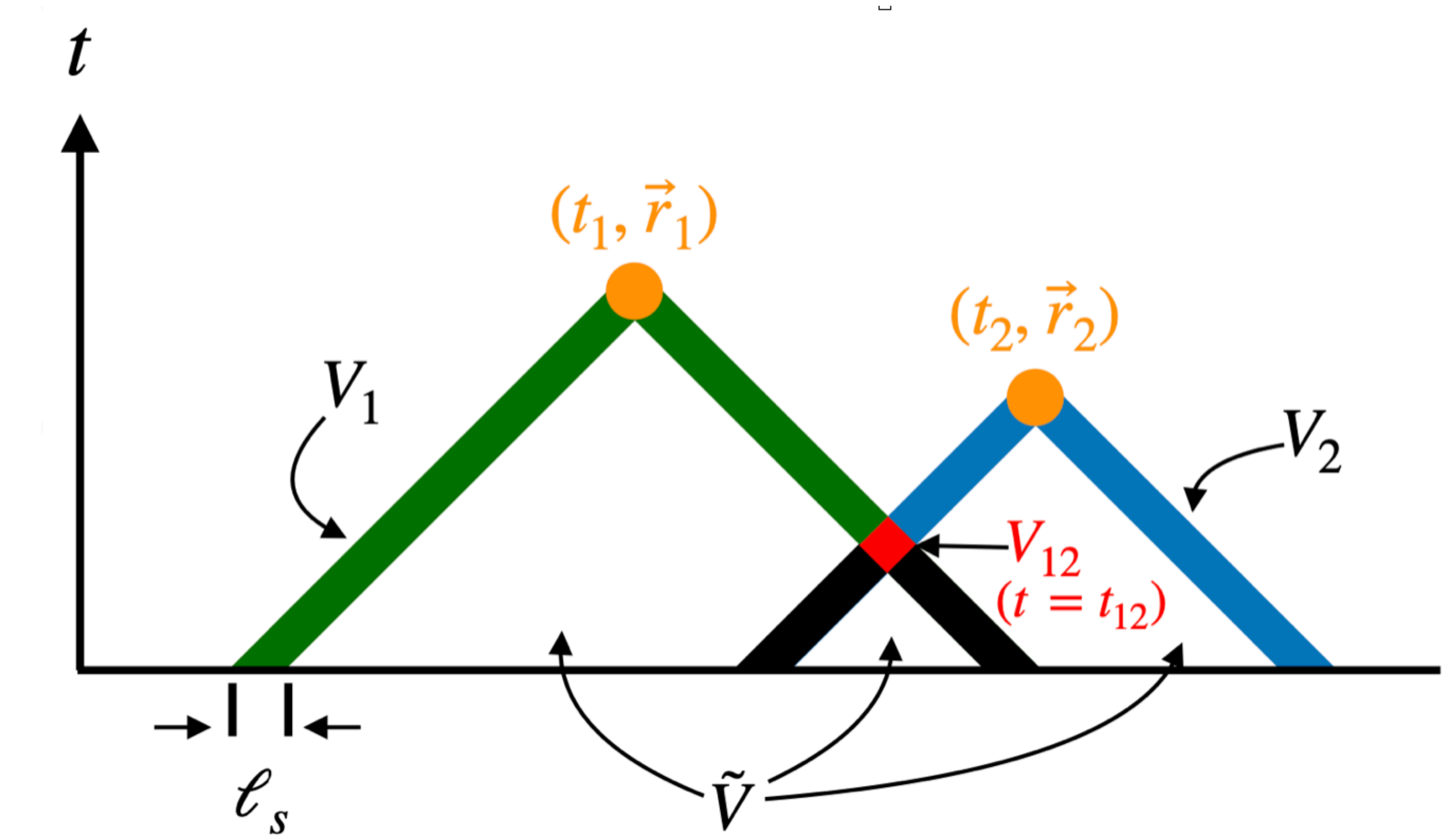
$$\langle T_{ij} T_{kl} \rangle^{(s)} = a_1 \delta_{ij} \delta_{kl} + a_2 \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$
$$+ b_1 \delta_{ij} \hat{r}_k \hat{r}_l + b_2 \delta_{kl} \hat{r}_i \hat{r}_j$$
$$+ b_3 \frac{1}{4} (\delta_{ik} \hat{r}_j \hat{r}_l + \delta_{il} \hat{r}_j \hat{r}_k + \delta_{jk} \hat{r}_i \hat{r}_l + \delta_{jl} \hat{r}_i \hat{r}_k)$$
$$+ c_1 \hat{r}_i \hat{r}_j \hat{r}_k \hat{r}_l,$$

➡

$$K_{ij,kl}(\hat{k}) K_{ij,mn}(\hat{k}) \langle T_{kl} T_{mn} \rangle^{(s)}$$
$$= 2a_2 + (1 - c_{rk}^2) b_3 + \frac{1}{2} (1 - c_{rk}^2)^2 c_1$$

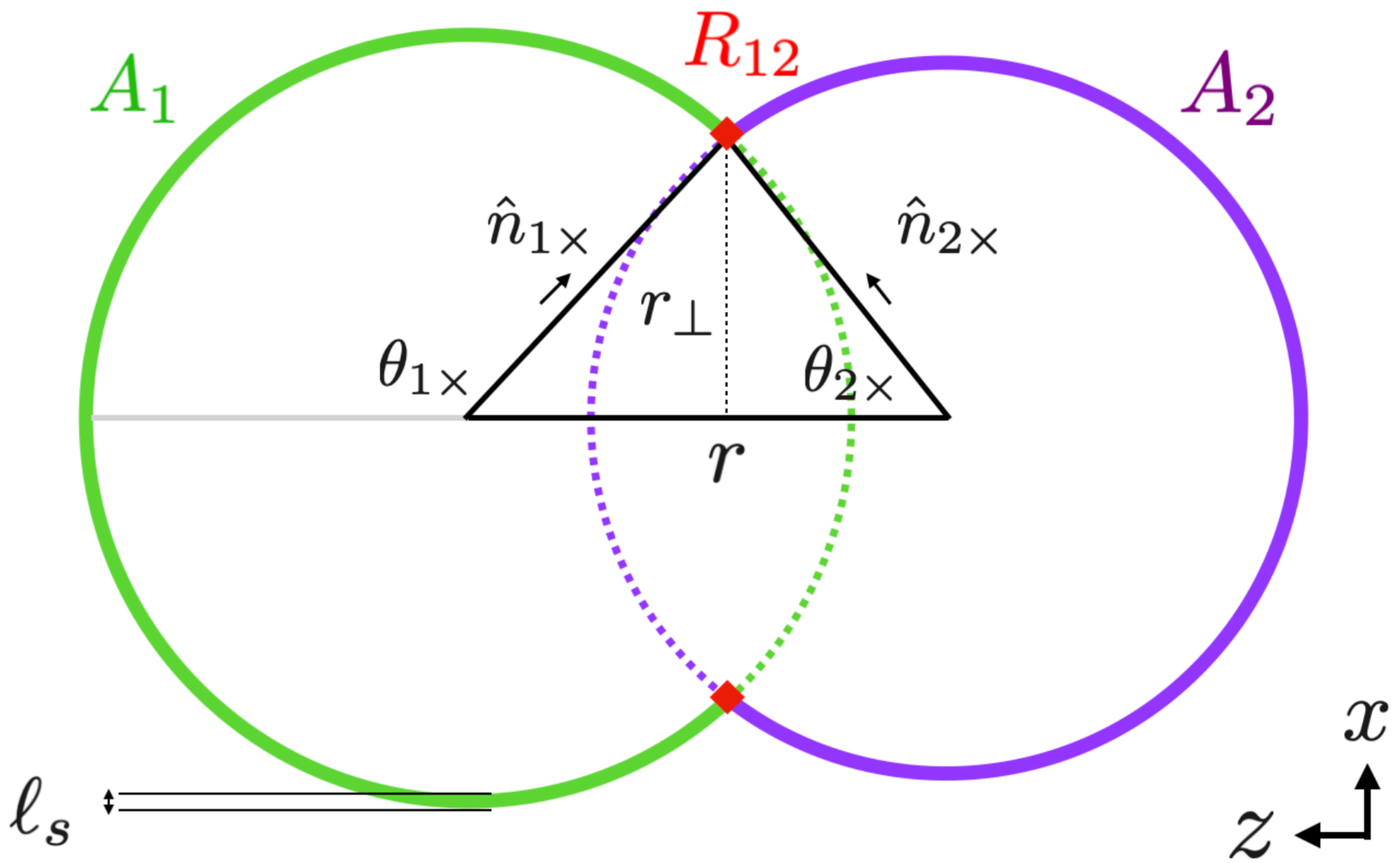
Jinno, Takimoto (2016)

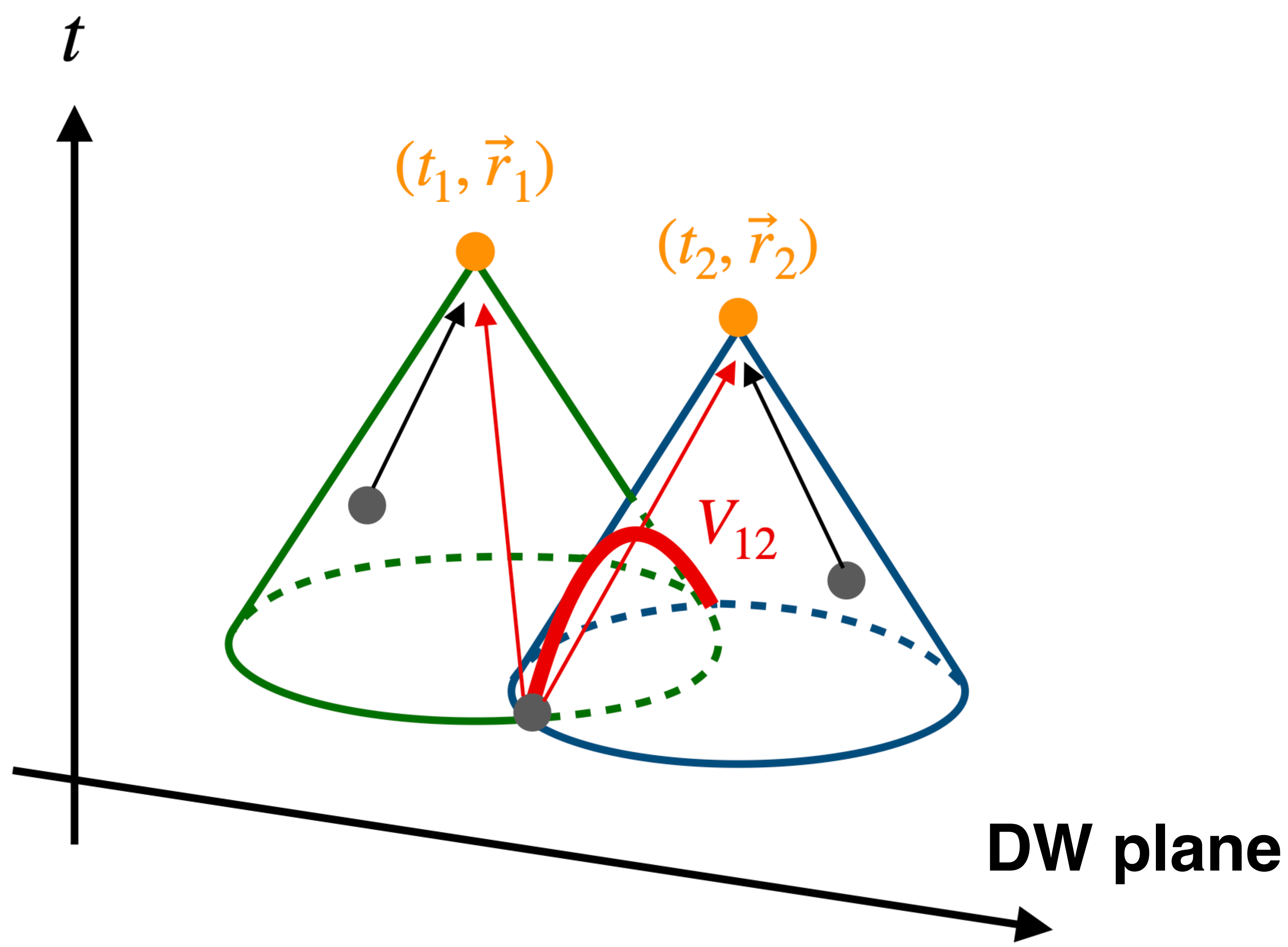
$O(2)$ Symmetric effect



$$\begin{cases} \text{overlapped regime:} & -\infty < t < t_{12} \\ \text{non-overlapped regime:} & t_{12} < t < t_{1(2)} \end{cases}$$

- **No contribution when $O(3)$ sym.**
- **Exist only for double-loop contribution**
- **Dominant contribution for spectrum**
- **Same DW-plane correlation**





Expression (single)

$$\mathcal{I}(r,t_d) = \frac{1}{2\beta^3}\left[8\pi\cosh\left(\frac{\beta t_d}{2}\right) + \beta^2 r\sqrt{r^2-t_d^2}K_1\left(\frac{\beta r}{2}\right)\right] - \frac{1}{4}\int_{-\infty}^{-r/2} dt_T\, e^{\beta t_T}\left[(2t_T-t_d)^2\left(\pi-\cos^{-1}\left(\frac{r^2-2t_dt_T}{r(2t_T-t_d)}\right)\right) + (2t_T+t_d)^2\left(\pi-\cos^{-1}\left(\frac{r^2+2t_dt_T}{r(2t_T+t_d)}\right)\right)\right]$$

$$\Delta^{\text{single}}(k/\beta) = \frac{3}{4\pi}\beta k^3\int_0^\infty dt_d\cos(kt_d)\int_{t_d}^\infty dr\,\frac{C_{\text{DW}}(r)}{r^3\mathcal{I}(r,t_d)}\left[j_0(kr)F_0' + \frac{j_1(kr)}{kr}F_1' + \frac{j_2(kr)}{(kr)^2}F_2'\right]$$

$$F_0'(r,t_d) = \frac{3r^3(r^2-t_d^2)^{3/2}}{2\beta^2}K_2\left(\frac{\beta r}{2}\right),$$

$$F_1'(r,t_d) = -\frac{r^3\sqrt{r^2-t_d^2}}{\beta^2}\left[2\beta r(r^2-t_d^2)K_1\left(\frac{\beta r}{2}\right) + 3(r^2-5t_d^2)K_2\left(\frac{\beta r}{2}\right)\right],$$

$$F_2'(r,t_d) = \frac{r^2}{2\beta^3\sqrt{r^2-t_d^2}}\left[\beta r\{9r^4-90r^2t_d^2+105t_d^4+2\beta^2r^2(r^2-t_d^2)^2\}K_0\left(\frac{\beta r}{2}\right) + 4\{9r^4-90r^2t_d^2+105t_d^4+\beta^2r^2(r^2-t_d^2)(r^2-5t_d^2)\}K_1\left(\frac{\beta r}{2}\right)\right],$$

Expression (double)

$$
\begin{aligned}
\mathcal{I}(r,t_d) =& \frac{1}{2\beta^3}\left[8\pi\cosh\left(\frac{\beta t_d}{2}\right)+\beta^2 r\sqrt{r^2-t_d^2}K_1\left(\frac{\beta r}{2}\right)\right] \\
& -\frac{1}{4}\int_{-\infty}^{-r/2} dt_T\, e^{\beta t_T}\left[(2t_T-t_d)^2\left(\pi-\cos^{-1}\left(\frac{r^2-2t_d t_T}{r(2t_T-t_d)}\right)\right)\right. \\
& \left.+(2t_T+t_d)^2\left(\pi-\cos^{-1}\left(\frac{r^2+2t_d t_T}{r(2t_T+t_d)}\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{\text{double}}(k/\beta) =& \frac{3}{64\pi}\beta k^3 \int_0^\infty dt_d \cos(kt_d)\int_{t_d}^\infty dr\, \frac{r^2 C_{\text{DW}}(r)}{\mathcal{I}(r,t_d)^2} \\
& \times\left[\left(j_0(kr)-2\frac{j_1(kr)}{kr}+3\frac{j_2(kr)}{(kr)^2}\right)\right. \\
& \quad \times\left(g''_{a,\text{ov}}(r,t_d)+g''_{a,\text{non}}(r,t_d)\right)\left(g''_{a,\text{ov}}(r,-t_d)+g''_{a,\text{non}}(r,-t_d)\right) \\
& \quad +\left(j_0(kr)-2\frac{j_1(kr)}{kr}-5\frac{j_2(kr)}{(kr)^2}\right) \\
& \quad \times\left[\left(g''_{a,\text{ov}}(r,t_d)+g''_{a,\text{non}}(r,t_d)\right)g''_{b,\text{ov}}(r,-t_d)+g''_{b,\text{ov}}(r,t_d)\left(g''_{a,\text{ov}}(r,-t_d)+g''_{a,\text{non}}(r,-t_d)\right)\right] \\
& \left.+\left(j_0(kr)-2\frac{j_1(kr)}{kr}+19\frac{j_2(kr)}{(kr)^2}\right)g''_{b,\text{ov}}(r,t_d)g''_{b,\text{ov}}(r,-t_d)\right]. \qquad (3.65)
\end{aligned}
$$

$$
\begin{aligned}
g''_{a,\text{ov}}(r,t_d) \equiv& \int_{-\infty}^{-r/2} dt_{1T}\, e^{\beta t_{1T}} g'_a(r,t_d,t_{1T}), \\
g''_{a,\text{non}}(r,t_d) \equiv& \int_{-r/2}^{t_d/2} dt_{1T}\, e^{\beta t_{1T}} g'_a(r,t_d,t_{1T}) \\
=& \frac{\pi}{\beta^3}e^{-\beta r/2}\left[8(-1+e^{\beta(r+t_d)/2})-\beta(r+t_d)\left\{4+\beta(r+t_d)\right\}\right], \\
g''_{b,\text{ov}}(r,t_d) \equiv& \int_{-\infty}^{-r/2} dt_{1T}\, e^{\beta t_{1T}} g'_b(r,t_d,t_{1T}) \\
=& \frac{\sqrt{r^2-t_d^2}}{\beta}\left[rK_1\left(\frac{\beta r}{2}\right)+t_d K_2\left(\frac{\beta r}{2}\right)\right],
\end{aligned}
$$

$$
g''_{a,\text{non}}(r,t_d)g''_{a,\text{non}}(r,-t_d) = g_{\text{IR}}(r,t_d)+g_{\text{conv}}(r,t_d),
$$

where

$$
\begin{aligned}
g_{\text{IR}}(r,t_d) =& 64\pi^2, \\
g_{\text{conv}}(r,t_d) =& \pi^2 e^{-r}\left[(8+4r+r^2)^2-2r(4+r)t_d^2+t_d^4\right] \\
& -16\pi^2 e^{-\frac{r}{2}}\left((8+4r+r^2+t_d^2)\cosh\frac{t_d}{2}-2(2+r)t_d\sinh\frac{t_d}{2}\right)
\end{aligned}
$$

$$
\Delta^{\text{double}}(k/\beta)\Big|_{g_{\text{IR}}\ \text{part}} = \Delta^{\text{double}}_{\text{IR0}}(k/\beta)+\Delta^{\text{double}}_{\text{IR12}}(k/\beta)
$$

$$
\begin{aligned}
\Delta^{\text{double}}_{\text{IR0}}(k/\beta) =& 3\pi\beta k^3\int_0^\infty dt_d\cos(kt_d)\int_{t_d}^\infty dr r^2 C_{\text{DW}}(r)\left(\frac{1}{\mathcal{I}(r,t_d)^2}-\frac{1}{\mathcal{I}(\infty,t_d)^2}\right)j_0(kr) \\
& +\frac{3N_{\text{walls}}d_{\text{DW}}^2 H_* k}{32\pi}\left(1+\frac{2\pi k/\beta}{\sinh\frac{2\pi k}{\beta}}\right). \qquad (3.79)
\end{aligned}
$$

$$
\Delta^{\text{double}}_{\text{IR12}}(k/\beta) = 3\pi\beta k^3\int_0^\infty dt_d\cos(kt_d)\int_{t_d}^\infty dr\,\frac{r^2 C_{\text{DW}}(r)}{\mathcal{I}(r,t_d)^2}\left[-2\frac{j_1(kr)}{kr}+3\frac{j_2(kr)}{(kr)^2}\right]
$$

IR divergence

$$\Delta_{\rm IR0}^{\rm double}(k/\beta) \overset{r\to\infty}{\to} \frac{3N_{\rm walls} d_{\rm DW}^2 H_* k}{32\pi} \left[\left(1 + \frac{2\pi k/\beta}{\sinh\frac{2\pi k}{\beta}}\right)(1-\cos(kL)) + \beta^{-1} k \chi(k/\beta)\sin(kL)\right]$$

FOPT result

2. ANALYTIC DERIVATION OF THE GW SPECTRUM

- Final expression $\Delta(k/\beta) \equiv \frac{3}{8\pi G}\frac{\beta^2 \rho_{\rm tot}}{\kappa^2 \epsilon_*^2}\Omega_{\rm GW}(k)$ $\Omega_{\rm GW}(k) \equiv \frac{1}{\rho_{\rm tot}}\frac{d\rho_{\rm GW}}{d\ln k}$

contribution to $\rho_{\rm GW}$ from each ln k

single	$\Delta^{(s)} = \frac{k^3}{12\pi}\int_0^\infty dt_d \int_{t_d}^\infty dr\, \frac{e^{-r/2}\cos(kt_d)}{r^3\mathcal{I}(t_d,r)} \times \left[j_0(kr)F_0 + \frac{j_1(kr)}{kr}F_1 + \frac{j_2(kr)}{k^2r^2}F_2 \right]$	$F_0 = 2(r^2-t_d^2)^2(r^2+6r+12)$, $F_1 = 2(r^2-t_d^2)\left[-r^2(r^3+4r^2+12r+24) + t_d^2(r^3+12r^2+60r+120)\right]$, $F_2 = \frac{1}{2}\left[r^4(r^4+4r^3+20r^2+72r+144) - 2t_d^2r^2(r^4+12r^3+84r^2+360r+720) + t_d^4(r^4+20r^3+180r^2+840r+1680)\right]$
double	$\Delta^{(d)} = \frac{k^3}{96\pi}\int_0^\infty dt_d \int_{t_d}^\infty dr\, \frac{e^{-r}\cos(kt_d)}{r^4\mathcal{I}(t_d,r)^2} \times \frac{j_2(kr)}{k^2r^2}G(t_d,r)G(-t_d,r)$	$G(t_d,r) = (r^2-t_d^2)\left[(r^3+2r^2)+t_d(r^2+6r+12)\right]$

contains many polynomials, exponentials, and Bessel functions, but just that

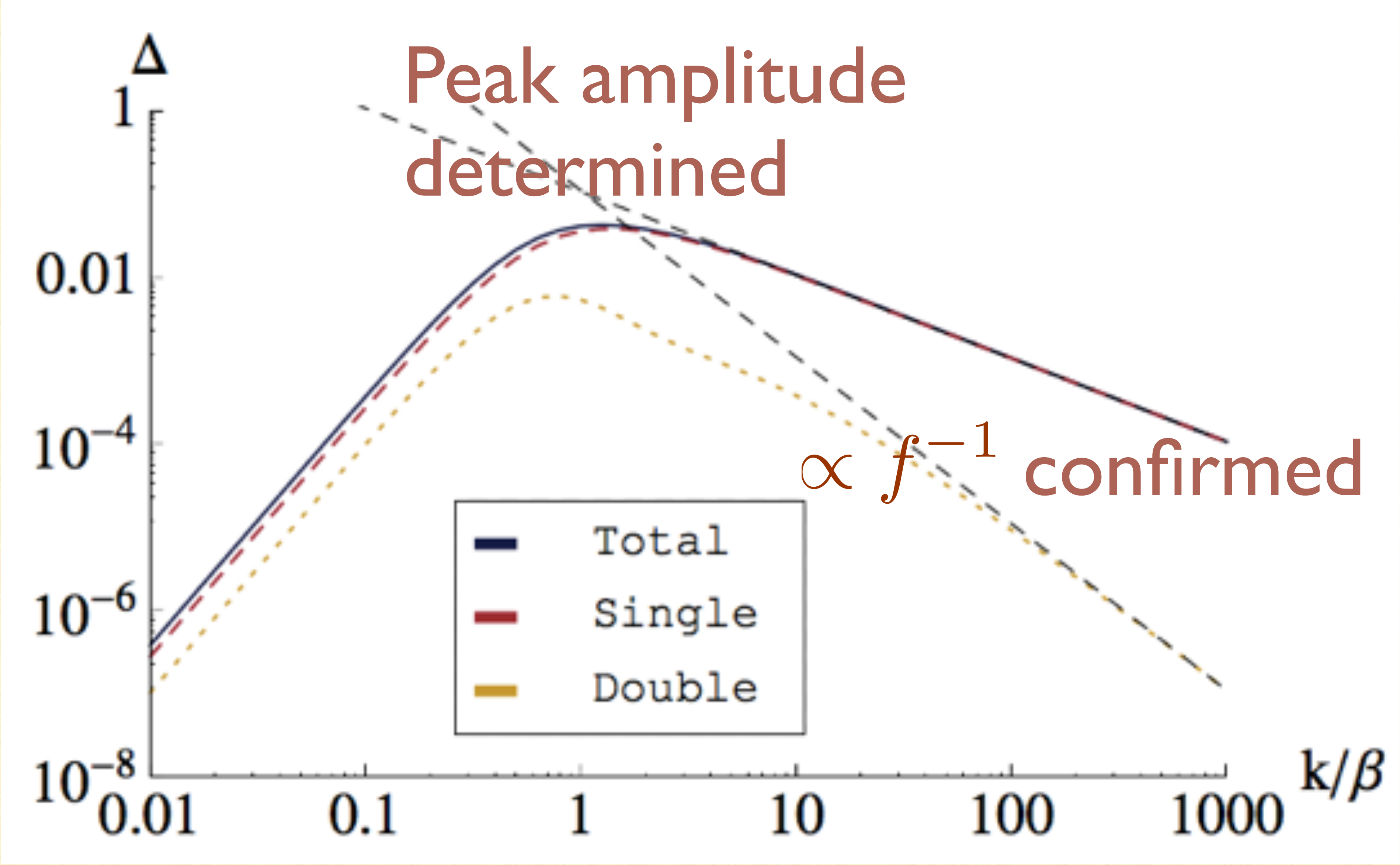
https://www2.yukawa.kyoto-u.ac.jp/~ppp.ws/PPP2016/slides/takimoto.pdf

FOPT result

2. ANALYTIC DERIVATION OF THE GW SPECTRUM

- Result

  - Consistent with numerical simulation within factor ~2



$$\Delta(k/\beta) \equiv \frac{3}{8\pi G} \frac{\beta^2 \rho_{\rm tot}}{\kappa^2 \epsilon_*^2} \Omega_{\rm GW}(k)$$

$$\Omega_{\rm GW}(k) \equiv \frac{1}{\rho_{\rm tot}} \frac{d\rho_{\rm GW}}{d\ln k}$$

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