Classical Backgrounds as Coherent States and Canonical Quantization of General Relativity

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Based on work with

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Semi-Classical Picture

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{cl}} + \hat{h}_{\mu\nu} \qquad\qquad \varphi = \varphi^{\mathrm{cl}} + \delta\hat{\varphi}$$

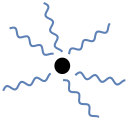
$\hat{h}_{\mu\nu}$ and $\delta\hat{\varphi}$ are quantum fluctuations on the fixed background.

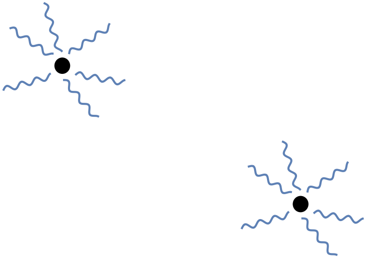
Quantum Picture

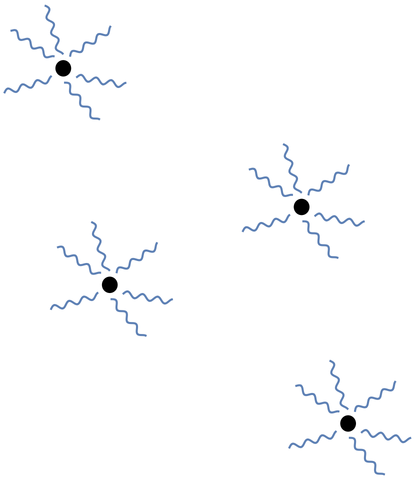
Dvali, Gomez '13

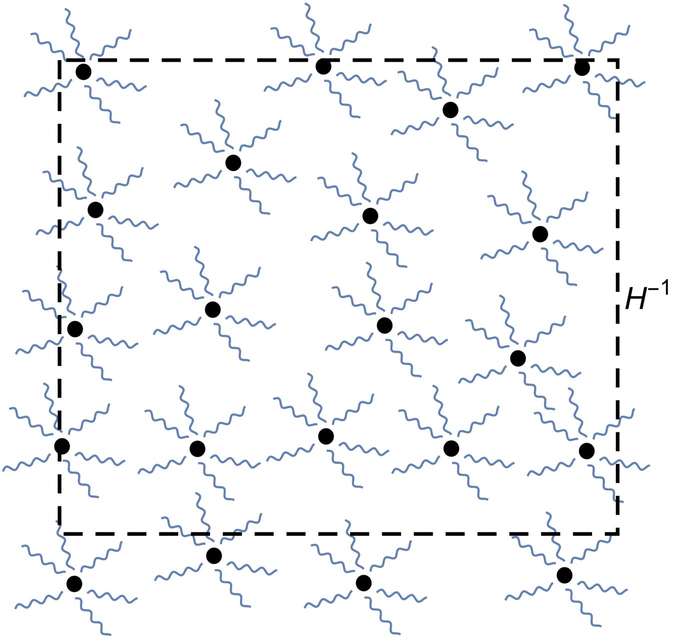
$$g_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu} \qquad\qquad \varphi = \hat{\varphi}$$

Classical background is viewed as BEC of $\hat{h}_{\mu\nu}$ and $\hat{\varphi}$.





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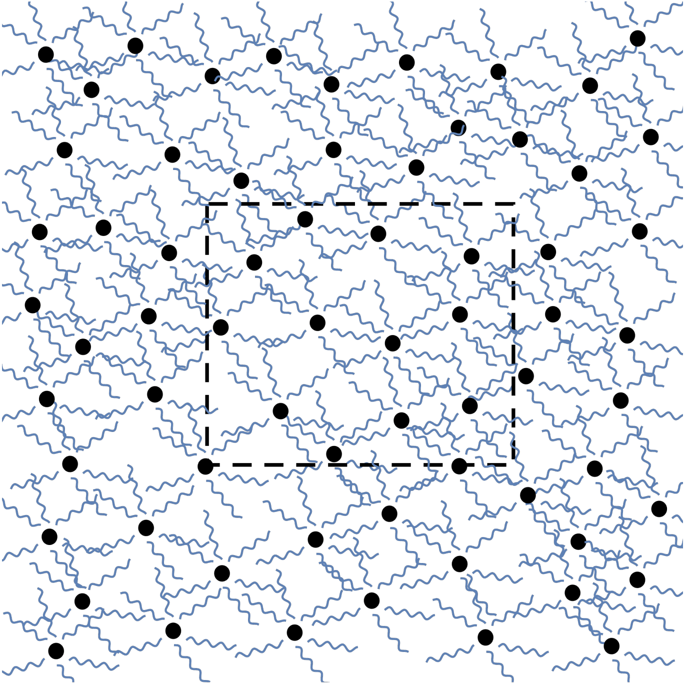

Quantum Picture

$$g_{\mu\nu}^{\mathrm{cl}} \quad \Longleftrightarrow \quad |g_{\mathrm{cl}}\rangle \quad \text{– Coherent States}$$

with

$$\langle g_{\mathrm{cl}}|\hat{g}_{\mu\nu}|g_{\mathrm{cl}}\rangle = g_{\mu\nu}^{cl}$$

Canonical Quantization:

- Degrees of freedom: $\hat{g}_{\mu\nu}, \hat{\varphi}, \ldots$ and their conjugate momenta
- $\hat{H}$
- Set of states: $|\psi\rangle$
- Vacua: $|\Omega\rangle$

Quantization around Minkowski spacetime $\Longleftrightarrow \langle\Omega|\hat{g}_{\mu\nu}|\Omega\rangle = \eta_{\mu\nu}$

Repercussions

Dvali, Gomez '13

$|g_{\mathrm{cl}}\rangle$ – Coherent States $\longrightarrow$ Gradual Loss of Coherence

This leads to ramifications, e.g. for the beginning of inflation.

LB, Trodden '15, '22

Coherent States

QED: Glauber '63, Sudarshan '63

$$|C\rangle = e^{-\mathrm{i}\int \mathrm{d}^3x\left(\phi_{\rm cl}(x)\hat{\pi}(x)-\pi_{\rm cl}(x)\hat{\phi}(x)\right)}|\Omega\rangle$$

Sets initial conditions for all correlators:

$$\langle C|\hat{\phi}|C\rangle(t=0) = \phi_{\rm cl}(x)$$
$$\langle C|\hat{\pi}|C\rangle(t=0) = \pi_{\rm cl}(x)$$

$$\langle C|\hat{\phi}(x,0)\hat{\phi}(y,0)|C\rangle = \phi_{\rm cl}(x)\phi_{\rm cl}(y) + \langle\Omega|\hat{\phi}(x,0)\hat{\phi}(y,0)|\Omega\rangle$$
$$\langle C|\hat{\pi}(x,0)\hat{\pi}(y,0)|C\rangle = \pi_{\rm cl}(x)\pi_{\rm cl}(y) + \langle\Omega|\hat{\pi}(x,0)\hat{\pi}(y,0)|\Omega\rangle$$

(Caveat: $|\Omega\rangle$ requires background dependent squeezing for 1-loop consistency, and even non-Gaussian modification at higher loops)

LB, Zantedeschi '20, LB, Cintia, Zantedeschi '21, '23

Departure from Classicality

LB, Zantedeschi '20

Classical evolution:

$$|C_{\rm cl}\rangle(t) = e^{-{\rm i}\int {\rm d}^3x\left(\phi_{\rm cl}(x,t)\hat{\pi}(x)-\pi_{\rm cl}(x,t)\hat{\phi}(x)\right)}|\Omega\rangle$$

Quantum evolution:

$$|C\rangle(t) = e^{-{\rm i}\hat{H}t}e^{-{\rm i}\int {\rm d}^3x\left(\phi_{\rm cl}(x,0)\hat{\pi}(x)-\pi_{\rm cl}(x,0)\hat{\phi}(x)\right)}|\Omega\rangle$$

Departure:

$$|\langle C_{\rm cl}|C\rangle|^2(t) = \sum_n \alpha_n t^n$$

Background Field Method

For computational practicality:

$$\hat{\phi} = \Phi + \hat{\psi} \qquad \text{with} \qquad \Phi(x,t) \equiv \langle C|\hat{\phi}|C\rangle$$

For quartic scalar field:

$$\left(-\Box + m^2\right)\Phi(x,t) + \frac{\lambda}{3!}\Phi^3(x,t) = -\frac{\lambda}{2}\langle C|\hat{\psi}^2(x,t)|C\rangle \ldots$$

This should be supplemented with analogous equations for higher order correlation functions.

This method encapsulates the semi-classical approximation, but in principle can be pushed much further.

BRST-invariant Quantization of GR

Kugo, Ojima '78

$$\mathcal{L} = \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{FP}}$$
$$\mathcal{L}_{\mathrm{EH}} = \sqrt{-g} M_{\mathrm{pl}}^2 R$$
$$\mathcal{L}_{\mathrm{GF}} = M_{\mathrm{pl}} B_\nu \partial_\mu \left(\sqrt{-g} g^{\mu\nu}\right) - \frac{1}{2}\alpha\eta^{\mu\nu} B_\mu B_\nu$$
$$\mathcal{L}_{\mathrm{FP}} = \mathrm{i}\partial_\mu \bar{c}_\nu \left(\sqrt{-g} g^{\mu\sigma}\partial_\sigma c^\nu - \partial_\sigma\left(\sqrt{-g} g^{\sigma\nu}\right) c^\mu\right)$$

Practical canonical variables are $\gamma_{ij} \equiv g_{ij}$ and $A^\mu \equiv \sqrt{-g} g^{0\mu}$

LB, Dvali, Sakhelashvili '24

Connection with ADM variables:

$$N = -\frac{\sqrt{\gamma}}{A^0}\,, \qquad \text{and} \qquad N^j = -\frac{A^j}{A^0}$$

Conjugate Momenta and Canonical Quantization

$$
\begin{aligned}
\gamma_{ij} \quad &\longleftrightarrow \quad \Pi^{ij} = -M_{\rm pl}^2\sqrt{\gamma}\left(K^{ij} - \gamma^{ij}K\right) \\
A^\mu \quad &\longleftrightarrow \quad \Pi_\nu = M_{\rm pl}B_\nu - \mathrm{i}(\partial_\mu \bar{c}_\nu)c^\mu \\
\bar{c}_\mu \quad &\longleftrightarrow \quad \Pi_{\bar{c}}^\nu = \mathrm{i}\left(A^\sigma\partial_\sigma c^\nu - \partial_\sigma(\sqrt{-g}g^{\sigma\nu})c^0\right) \\
c^\mu \quad &\longleftrightarrow \quad \Pi^c_\nu = -\mathrm{i}A^\mu\partial_\mu\bar{c}_\nu
\end{aligned}
$$

Equal-time (anti-)commutation relations:

$$
\begin{aligned}
\left[\hat{\gamma}_{ij}(x), \hat{\Pi}^{k\ell}(y)\right] &= \frac{\mathrm{i}}{2}\left(\delta_i^k\delta_j^\ell + \delta_i^\ell\delta_j^k\right)\delta^{(3)}(x-y) \\
\left[\hat{A}^\mu(x), \hat{\Pi}_\nu(y)\right] &= \mathrm{i}\delta^\mu_\nu\delta^{(3)}(x-y) \\
\left\{\hat{c}^\mu(x), \hat{\Pi}^c_\nu(y)\right\} &= \mathrm{i}\delta^\mu_\nu\delta^{(3)}(x-y) \\
\left\{\hat{\bar{c}}_\nu(x), \hat{\Pi}^\mu_{\bar{c}}(y)\right\} &= \mathrm{i}\delta^\mu_\nu\delta^{(3)}(x-y)
\end{aligned}
$$

Physicality Condition:	$\hat{Q}\lvert phys\rangle = 0$

Einstein-Hilbert Hamiltonian

$$H_{\mathrm{EH}} = \int \mathrm{d}^3x \left[-\frac{1}{A^0}\mathcal{H} + \frac{A^i}{A^0}\mathcal{P}_i \right]$$

$$\mathcal{H} \equiv \frac{1}{2M_{\mathrm{pl}}^2}\left(\gamma_{ik}\gamma_{j\ell} + \gamma_{i\ell}\gamma_{jk} - \gamma_{ij}\gamma_{k\ell}\right)\Pi^{ij}\Pi^{kl} - M_{\mathrm{pl}}^2\gamma R^{(3)}$$

$$\mathcal{P}_i \equiv -2\gamma_{ik}\partial_j\Pi^{kj} - (2\partial_k\gamma_{ji} - \partial_i\gamma_{jk})\Pi^{jk}$$

Minkowski Vacuum

$$\langle\Omega|\hat{\gamma}_{ij}|\Omega\rangle = \delta_{ij}$$
$$\langle\Omega|\hat{A}^0|\Omega\rangle = -1$$
$$\langle\Omega|\hat{A}^j|\Omega\rangle = 0$$
$$\langle\Omega|\hat{B}_\mu|\Omega\rangle = 0$$

The last equality, in fact, holds for any physical state $\hat{Q}|phys\rangle = 0$, due to

$$\hat{B}_\nu = \{\hat{Q}, \hat{\bar{c}}_\nu\}$$

Classical States

LB, Dvali, Sakhelashvili '24

$$\left\langle f\left(\hat{h}, \hat{\Pi}, \ldots\right)\right\rangle = f\left(\langle\hat{h}\rangle, \langle\hat{\Pi}\rangle, \ldots\right) + \mathcal{O}(\hbar)$$

Puzzling Attempt

DeWitt '67

Classical Constraints:

$$\mathcal{H} = 0 \qquad \longrightarrow \qquad \hat{\mathcal{H}}|phys\rangle = 0$$
$$\mathcal{P}_i = 0 \qquad \longrightarrow \qquad \hat{\mathcal{P}}_i|phys\rangle = 0$$

Excluding the boundary contribution:

$$H_{\mathrm{EH}} = 0 \qquad \longrightarrow \qquad \hat{H}_{\mathrm{EH}}|phys\rangle = 0$$

Boundary contribution gives the ADM mass, but unclear how the boundary Hamiltonian alone can reproduce an adequate bulk dynamics.

Successful Quantization

- $\lim_{\hbar \to 0} \langle phys|\hat{H}|phys\rangle = 0$, excluding the boundary contribution.

- $\hat{H}|phys\rangle \neq 0$, for states with non-trivial bulk dynamics.

Constraints in BRST-invariant Quantization

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$$\partial_t \hat{\Pi}_\mu = \mathrm{i}[\hat{H}, \hat{\Pi}_\mu]$$

$$\partial_t \hat{B}_0 = -\frac{1}{(\hat{A}^0)^2}\hat{\mathcal{H}} + \frac{\hat{A}^i}{(\hat{A}^0)^2}\hat{\mathcal{P}}_i + \dots$$
$$\partial_t \hat{B}_j = -\frac{1}{\hat{A}^0}\hat{\mathcal{P}}_j + \dots$$

The expectation value in physical states, corresponding to classical configurations, reduces to:

$$\mathcal{H}\left(\langle\hat{\gamma}_{k\ell}\rangle, \langle\hat{\Pi}_{mn}\rangle\right) + \mathcal{O}(\hbar) = 0$$
$$\mathcal{P}_j\left(\langle\hat{\gamma}_{k\ell}\rangle, \langle\hat{\Pi}_{mn}\rangle\right) + \mathcal{O}(\hbar) = 0$$

Bulk Hamiltonian

Classical limit:

$$\langle \hat{H} \rangle = \int d^3x \Big[-\frac{1}{\langle \hat{A}^0 \rangle} \mathcal{H}\left(\langle \hat{\gamma}_{k\ell} \rangle, \langle \hat{\Pi}_{mn} \rangle \right) + \frac{\langle \hat{A}^i \rangle}{\langle \hat{A}^0 \rangle} \mathcal{P}_i \left(\langle \hat{\gamma}_{k\ell} \rangle, \langle \hat{\Pi}_{mn} \rangle \right) + \mathcal{O}(\hbar) \Big]$$

This does not lead to

$$\hat{H} |phys\rangle \stackrel{?}{=} 0$$

The physicality condition is

$$\hat{Q} |phys\rangle = 0,$$

which is not equivalent to the previous expression.

Demonstration on QED

$$\hat{Q}_{\mathrm{QED}} = \int \mathrm{d}^3x \left[\hat{c}\left(g\hat{\rho} - \partial_j \hat{E}_j\right) + \hat{B}\hat{\Pi}_{\bar{c}} + \partial_j\left(\hat{c}\hat{E}_j\right)\right]$$

BRST constraint does not entail

$$\left(g\hat{\rho} - \partial_j \hat{E}_j\right)|phys\rangle \stackrel{?}{=} 0$$

In fact, we have

$$\left(g\hat{\rho} - \partial_j \hat{E}_j\right)|phys\rangle = -\mathrm{i}\{\hat{Q}_{QED}, \hat{\Pi}_c\}|phys\rangle$$

Gauss' law is recovered as the matrix element

$$\langle phys'|\left(g\hat{\rho} - \partial_j \hat{E}_j\right)|phys\rangle = -\mathrm{i}\langle phys'|\{\hat{Q}_{QED}, \hat{\Pi}_c\}|phys\rangle = 0$$

Dynamical Equation

$$\partial_t \hat{\Pi}^{mn} = \frac{2}{M_{\rm pl}^2} \frac{1}{\hat{A}^0} \left(\hat{\Pi}^{mk} \hat{\Pi}_k^n - \frac{1}{2} \hat{\Pi}^{mn} \hat{\Pi} \right) - \frac{M_{\rm pl}^2}{A^0} \hat{\gamma} \left(\hat{\gamma}^{mn} \hat{R}^{(3)} + \hat{R}^{(3)mn} \right) + \dots$$

From this, we recover the classical Einstein's equation for the expectation values, with

$$\lim_{\hbar \to 0} \partial_t \langle \hat{\Pi}^{mn} \rangle \neq 0 \, ,$$

assuming the disconnected contribution to the right hand side to be non-vanishing.

Implications for Cosmology

$$\Delta\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)$$

$$\langle\Omega|\hat{g}_{\mu\nu}|\Omega\rangle = \eta_{\mu\nu}\,, \qquad \langle\Omega|\hat{\phi}|\Omega\rangle = 0$$

There should also exist a state that corresponds to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2$$

Classical de Donder gauge is violated:

$$\partial_\mu\left(\sqrt{-g_{\mathrm{cl}}}g_{\mathrm{cl}}^{\mu\nu}\right) = 0$$

However, we could have instead introduced

$$\partial_\mu\left(g_{\mathrm{cl}}^{\mu\nu}\right) = 0$$

QFT in FLRW

Our approach will recover the standard QFT in curved spacetime as an $\hbar$-expansion of the background field method.

Caveat: the gauge-fixing condition introduced upon quantization, enforces the gauge for perturbations. For example, the recovery of FRW slicing would require

$$\mathcal{L}'_{\mathrm{GF}} = M_{\mathrm{pl}} B_\nu \partial_\mu g^{\mu\nu} - \frac{1}{2}\alpha \eta^{\mu\nu} B_\mu B_\nu$$

The change of gauge for perturbations is possible in the path-integral (after quantization and selection of the background), but only for the correlators of gauge-invariant variables.

Summary

We performed BRST-invariant quantization of GR as an EFT on Minkowski vacuum.

Classical configurations are viewed as coherent states built over this vacuum.

Promotion of constraint to dynamical equations for auxiliary fields was shown to be consistent with the classical time-evolution.

The framework introduces a *master* gauge-fixing condition upon quantization, which automatically fixes the slicing of the background geometry.

Possible ramifications for loop corrections to cosmological correlation functions.