Going back to move forward

Beyond the Standard Model with geometric quantization

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International Joint Workshop on the Standard Model and Beyond 2024

UNSW Sydney, 10 December 2024

Synopsis

- General relativity cannot be quantized with canonical or path integral methods
- Neither can anything else!
- Can we build a better “machine” to quantize classical fields?
- Maybe! If so, GR may quantize without issue
- Guiding idea: geometry leads the way

Outline

1. Review (?) of Hamiltonian particle theory
2. Geometric quantization: a “better” way to quantize particle theories
3. Geometric quantization for classical fields?
4. Polysymplectic (Hamiltonian) field theory
5. Geometric quantization for classical fields! (Sort of.)
6. Directions for future research

Hamiltonian particle theory

Configuration space Q (space; x^i)

Phase space T^*Q (space + momentum; x^i, p_i)

Tautological one-form $\theta = p_i dq^i$

Symplectic form $\omega = d\theta = dp_i \wedge dq^i$

Poisson structure $\Pi = \omega^{-1} = -\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$

Physically realizable trajectories $\gamma = x^i(t), p_i(t)$ must satisfy Hamilton's equations $T\gamma = \Pi(dH, -)$

Note: Time dependence is brought in "after-the-fact"

Hamiltonian particle theory highlights

Equivalent to Newton’s laws/Euler Lagrange-equations

Coordinate independent

Tautological one-form θ, Poisson structure Π : ingredients for geometric quantization!

Geometric quantization

Canonical quantization: $x^i, p_i \mapsto x^i, -i\hbar \frac{\partial}{\partial x^i}$

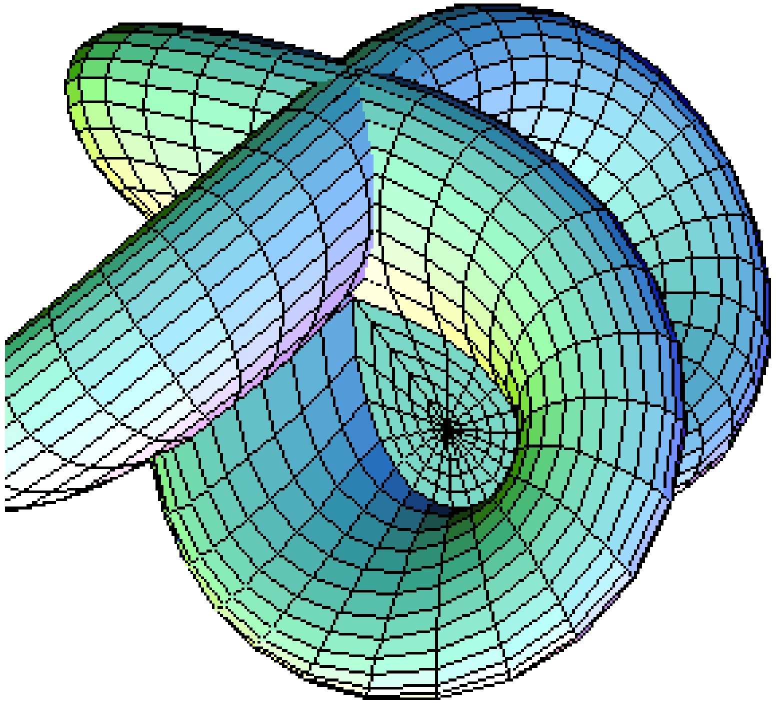
Geometric quantization: $f \mapsto f + \Pi(\mathrm{df}, \theta) + i\hbar\, \Pi(\mathrm{df}, -)$

Geometric quantization highlights

Coordinate invariant

Works for all possible configuration spaces (e.g., 2-sphere)

Traditionally has a hard time handling quadratic-in-momentum Hamiltonians (can be fixed; TM, in progress)



Geometric quantization for classical fields?

Hamiltonian field theory singles out time for special treatment

Treats instantaneous field configurations rather than coordinate functions

No tautological one-form

Poisson bracket is variational

Makes geometric quantization very difficult to reproduce

Introducing polysymplectic field theory

Pioneered by Christian Gunther in the 1980s

Incorporates the parameter space (spacetime) from the beginning

Natural tautological (vector-valued) one-form

Natural Poisson structure (one-form valued)

Only worked for flat spacetimes (fixed; TM 2021)

Polysymplectic field theory

Spacetime $M; x^i$

Extended configuration space $E \to M; x^\mu, \phi^I$

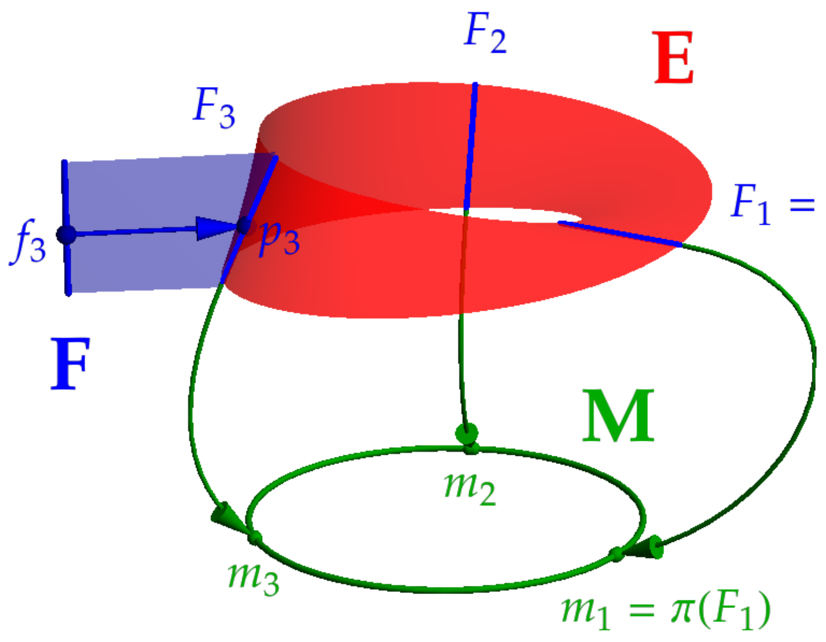
Extended phase space $P = V^*E \otimes TM;\ x^\mu, \phi^I, \pi_I^\mu$

Tautological one-form $\theta = \pi_I^\mu\, \mathrm{d}\phi^I \otimes \frac{\partial}{\partial x^\mu}$

Polysymplectic form $\omega = \mathrm{d}\pi_I^\mu \wedge \mathrm{d}\phi^I \otimes \frac{\partial}{\partial x^\mu}$

Poisson structure $\Pi = -\frac{\partial}{\partial \phi^I} \wedge \frac{\partial}{\partial \pi_I^\mu} \otimes dx^\mu$

Physically realizable fields $\gamma = \phi^I(x^\mu), \pi_I^\mu(x^\mu)$ must satisfy Hamilton's field equations $\Pi(dH, -, -) = \mathrm{T}\gamma - T\gamma_0$



Polysymplectic field theory highlights

Fully covariant

Incorporates spacetime from the beginning without space-time splitting

Natural tautological one-form and Poisson structure: the right ingredients for geometric quantization!

Geometric quantization for polysymplectic fields I

Quantization map (TM, 2024):

$f \mapsto f + \Pi(\alpha(\theta), \mathrm{df}, \mathrm{v}) + \kappa\, \Pi(-, \mathrm{df}, \mathrm{v})$

New ingredients: vector field v, one-form $\alpha, \alpha(v) = 1$

Dimensional considerations: $\kappa = \frac{i\hbar c}{V_4}$

Resulting operators for scalar field theory:

$\phi \mapsto \phi, \pi^i \mapsto \pi^i, \pi^0 \mapsto -\frac{i\hbar c}{V_3}\frac{\partial}{\partial\phi}$

Geometric quantization for polysymplectic fields II

Where is the physics?!

Where is the space-time dependence?!

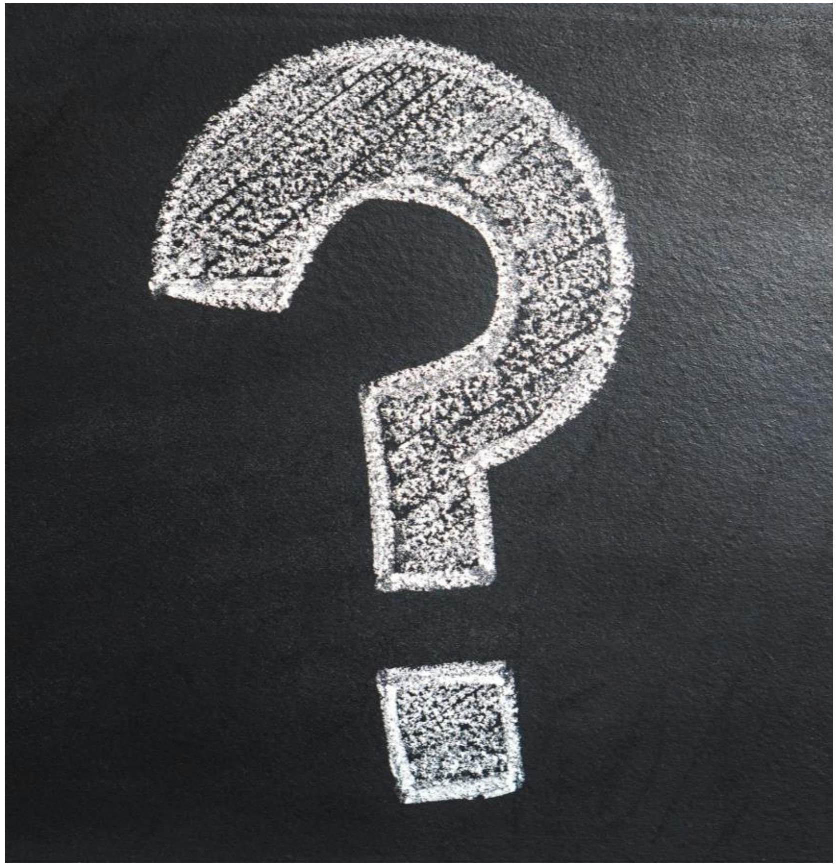
Total momentum operators for scalar field theory:

$P^i \mapsto -i\hbar \pi^i \frac{\partial}{\partial \phi}$

Naturally act on extended phase space "meta-state" $\Psi(x^\mu, \phi, \pi^i)$

Pullback by classical (complex) KG solution section

$\gamma(x^\mu) = Ae^{ig_{\mu\nu}k^\mu x^\nu}$ to get Q(P^i)(Ψ) = $\hbar k^i \gamma\, \gamma^* (\frac{\partial \Psi}{\partial \phi})$



Geometric quantization for polysymplectic fields III

Eigenvalue equation?

Yes! When $\Psi \propto e^{\frac{\phi\bar{\phi}}{|A|^2}}$

And the same process works for the total energy operator with massless fields!

Geometric quantization for polysymplectic fields highlights

Fully covariant

No need for renormalization

No obvious “red flags” to prevent it from working with GR!

Future work

Better understand the physical meaning of the meta-states

Energy eigenstates for massive particles (TM, in progress)

Spin ½ particles

Spin 1 particles

GR?

Conclusions

Geometric quantization of classical fields is hard, but can be done for free scalar fields

The resulting quantum field theory looks quite different from canonical QFT, with some tantalizing benefits

Future work will tell if we really can move forward by going back