Harmonic Analysis Method to Search for Gravitational Waves with Pulsar Timing Arrays

Jonathan Nay

Theoretical Astroparticle and Cosmology Symposium in Texas

October 10, 2023

Research Advisor: Kimberly Boddy (UT Austin)
Collaborators: Chiara Mingarelli (Yale University, CCA)
Tristan Smith (Swarthmore College)

The University of Texas at Austin
Department of Physics
College of Natural Sciences

Outline

- Background and Terminology
  - Pulsar Timing Arrays (PTAs)
  - Gravitational Waves (GWs)
  - Stochastic GW Background (SGWB)
- June 2023 PTA Collaboration Results
- Harmonic Analysis Approach

PTA Collaborations

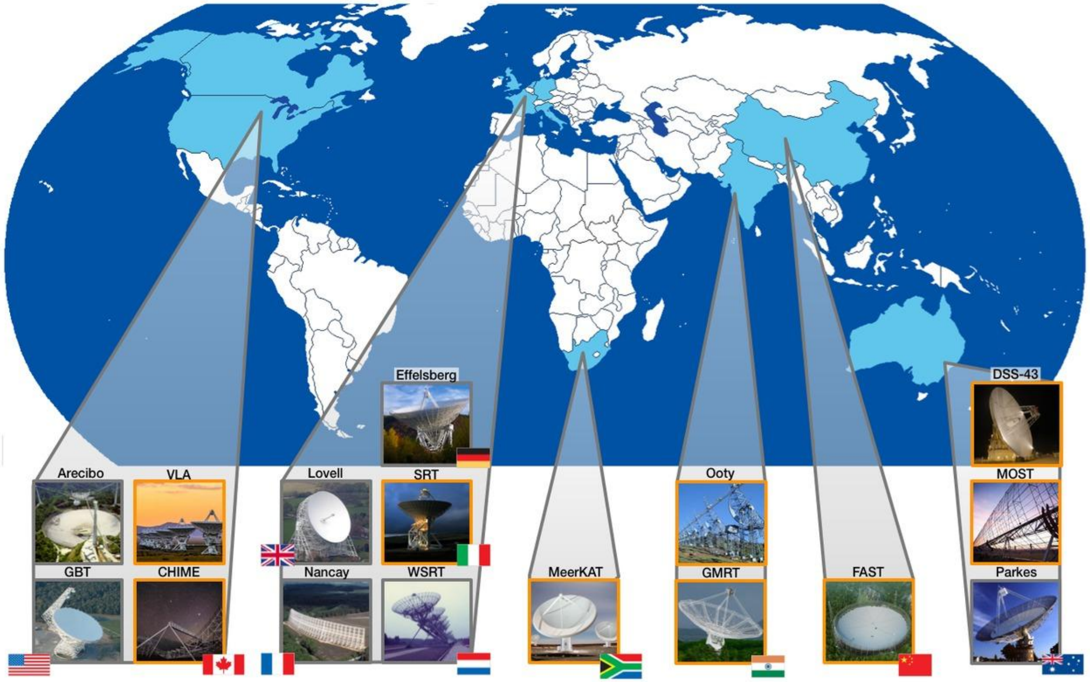


Image created by NANOGrav collaboration

- North American Nanohertz Observatory for GWs (NANOGrav)
- Parkes PTA (PPTA)
- European PTA (EPTA)
- Indian PTA (InPTA)
- Chinese PTA (CPTA)
- MeerKAT Interferometer

International PTA (IPTA) combines data from PTA collaborations

World's largest radio telescopes timing the Galaxy's best clocks!

Millisecond Pulsars (MSPs)

- MSPs used for high-precision timing
  - Older pulsars with rotational stability
  - Period accuracy as low as 10^{-14} s
- Distance from Earth to MSPs range from a few hundred pc to a few kpc
- Current number of MSPs being timed:

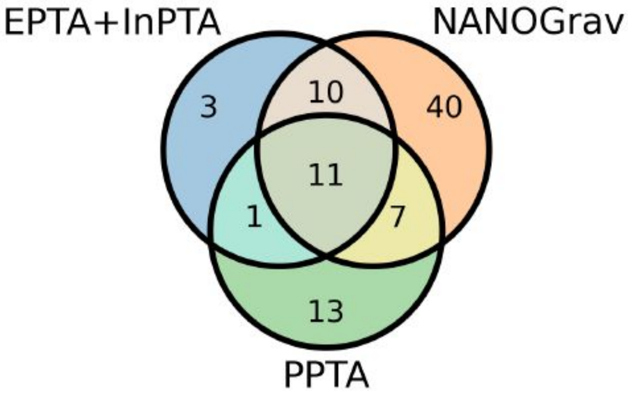
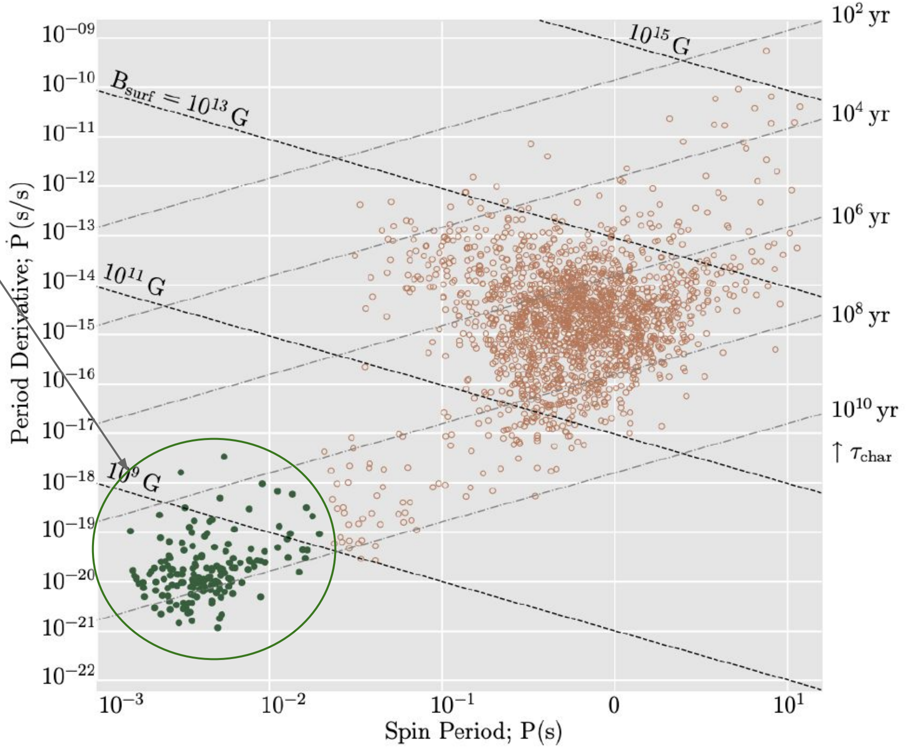


Figure 9 from G. Agazie *et al.* 9/1/2023



J. Verbiest and G. Shaifullah 2018

PTA Observable

- Pulse **Time of Arrival (TOA)**
  - Integrated radio pulses over brief observing window (to amplify signal)
  - Observing cadence for each pulsar on the order of days to months
  - Thousands of TOAs recorded per pulsar
- **Timing Residual**
  - Leftover TOA signal after subtracting pulsar's deterministic timing model:
    - Pulsar spin-down
    - Interstellar medium affects
    - Shapiro delay, etc.
  - On the order of a fraction of a microsecond

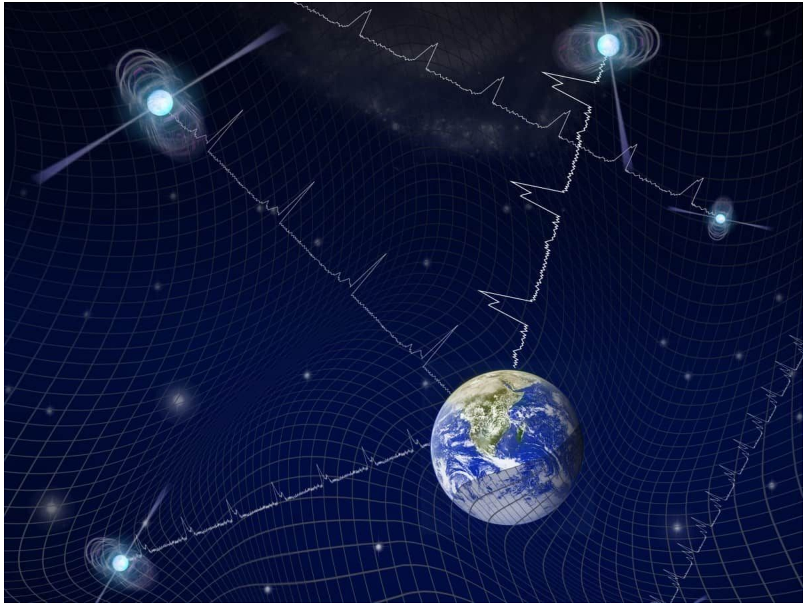


Image by T. Klein for NANOGrav

$$\vec{R}_{\rm obs} = \vec{n} + M\vec{\epsilon} + \vec{R}_{\rm RP}$$

Post-fit Timing Residual = Measurement White Noise + Unfit Deterministic Timing Signal + Red Processes (Low Frequency)

Red Processes branch into:

- Pulsar Intrinsic Red Noise (unique to each pulsar; cannot predict *a priori*)
- Nanohertz GWs (common to all pulsars; theoretical prediction)

Bayesian Analysis of Timing Residuals

- White noise is gaussian:

$$\text{Likelihood} = \frac{1}{\sqrt{2\pi \det(N)}} e^{-\frac{1}{2}\vec{n}^T N^{-1} \vec{n}} \qquad \vec{n} = \vec{R}_{\text{obs}} - M\vec{\epsilon} - \vec{R}_{\text{RP}} \qquad N := \langle \vec{n}\vec{n}^T \rangle$$

- Analytically marginalize over $M\vec{\epsilon}$ and $\vec{R}_{\text{RP}}$ to move unknown parameters into a combined covariance matrix C

$$\text{Likelihood} \propto \frac{1}{\sqrt{2\pi \det(C)}} e^{-\frac{1}{2}\vec{R}_{\text{obs}}^T C^{-1} \vec{R}_{\text{obs}}} \qquad C = N + \tilde{C}_{\text{RP}}$$

- Red-process power spectra generally modelled as a power law

$$S_h(f) \propto A^2 \, f^{-\gamma} \xrightarrow{\text{Wiener-Khinchin Theorem}} (C_{\text{RP}})_{ij} = \int_{f_L}^{f_H} S_h(f) \cos\big(2\pi f (t_i - t_j)\big) \, \mathrm{d}f$$

A = dimensionless amplitude
γ = spectral index

Hyperparameters of the red-process covariance

$f_H \approx 1/\text{obs cadence (days)}$
$f_L \approx 1/\text{obs timespan (years)}$

- Maximize likelihood to obtain best-fit red-process hyperparameters

GW Strain on Space-time

- Linearized theory, de Donder gauge, free-space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \Box \bar{h}_{\mu\nu} = -16\pi\, T_{\mu\nu} = 0$$

- Transverse traceless gauge (2 polarizations)

$$h_{ij}^{\rm TT}(t,\vec{x}) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int_{S^2} d\hat{\Omega}\, \tilde{h}_A(f,\hat{\Omega})\, \underbrace{\epsilon_{ij}^{A}(\hat{\Omega})}_{\text{Polarization Tensor}}\, e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x})}$$

- Stochastic GW background (SGWB) ensemble average

$$\left\langle \tilde{h}_A(f,\hat{\Omega})\tilde{h}_{A'}(f',\hat{\Omega}') \right\rangle = \frac{1}{8\pi}\delta^2(\hat{\Omega}-\hat{\Omega}')\,\delta(f-f')\,\delta_{AA'}\,\underbrace{S_h(f)}_{\text{One-sided Power Spectral Density (PSD)}}$$

  - *Gaussian*
    - *Isotropic*
      - *Stationary*
        - *Unpolarized*

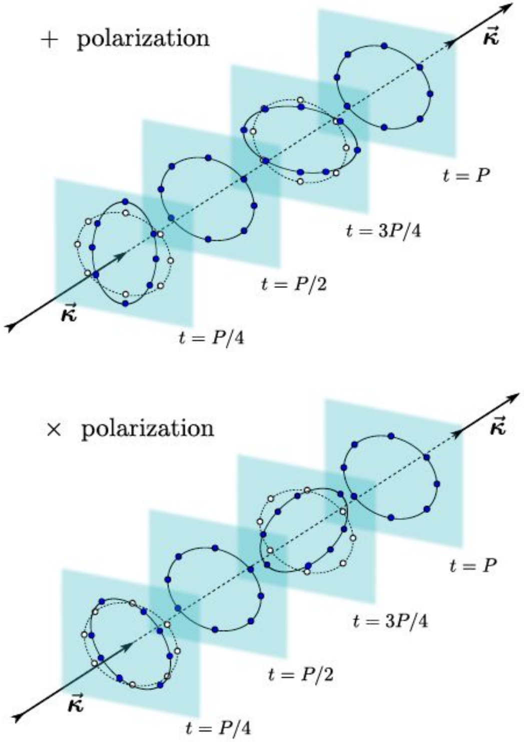


Figure 2 of N. Bishop and L. Rezzolla (2016)

SGWB Effect on Timing Residual

- SGWB-induced shift of single pulse from pulsar 'a' (Maggiore 2018)

$$z_a(t) := \underbrace{\frac{\Delta P_a}{P_a}} = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \int_{S^2} d\hat{\Omega}\, \tilde{h}_A(f, \hat{\Omega})\, \underbrace{F_a^A(\hat{\Omega})}\, e^{-2\pi i f t} \left(1 - e^{-2\pi i f t_a (1+\hat{\Omega}\cdot\hat{p}_a)}\right)$$

Fractional change in pulsar spin period

t_a = pulse propagation time
$\hat{p}_a$ = direction to pulsar a

$$F_a^A(\hat{\Omega}) := \frac{1}{2} \frac{p_a^i\, p_a^j\, \epsilon_{ij}^A(\hat{\Omega})}{1 + \hat{\Omega}\cdot\hat{p}_a}$$

Antenna pattern function

- Total shift of single TOA

$$R_a^{\mathrm{GW}}(t, \Delta t) := \int_t^{t+\Delta t} z_a(t')dt'$$

Observation time window (→ $t+\Delta t$)

- For an isotropic SGWB

Integrated power spectrum (common to all pulsars)

Spatial correlation function Γ_{ab} (specific to pulsar pairs)

$$\langle R_a^{\mathrm{GW}}(t, \Delta t)\, R_b^{\mathrm{GW}}(t, \Delta t)\rangle = \int_0^{\infty} df\, \underbrace{\frac{2}{3}\frac{S_h(f)}{(2\pi f)^2}}_{P^{\mathrm{GWB}}(f)} 4\sin^2(\pi f \Delta t)\, \frac{3}{2}(1 + \delta_{ab}) \int_{S^2} \frac{d\hat{\Omega}}{4\pi} \sum_{A=+,\times} F_a^A(\hat{\Omega})\, F_b^A(\hat{\Omega})$$

Earth Term (1) Pulsar Term (δ_{ab})

SGWB Astrophysical Sources

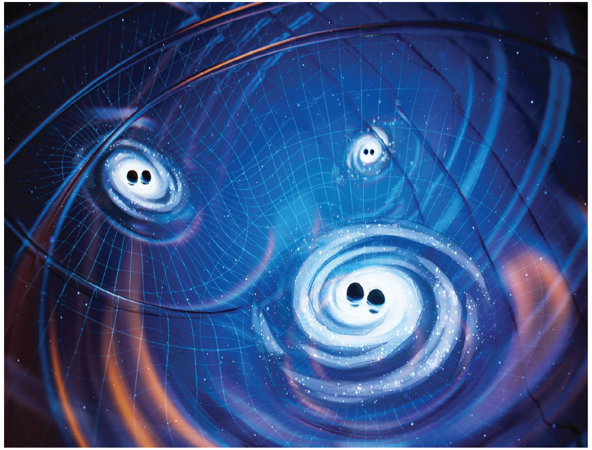


Image by O. Shmahalo for NANOGrav

- **Supermassive Blackhole Binaries (SMBHBs)**
  - Center of *some* galaxies
  - 10^5-10^{10} solar masses
  - Produce GWs during inspiral when <0.01 pc
  - GW frequencies in nHz range (for larger SMBHBs)
  - Inspiral time span on order of 25 million years
  - Could produce a SGWB *if detectable*

- **Characteristic Strain**
  - SMBHBs (Phinney 2001)

$$h_c(f) := \sqrt{2f S_h(f)} = A_{\rm GWB} \left(\frac{f}{f_{\rm yr}}\right)^{\alpha} \qquad \alpha = -2/3$$

  - PTAs capable of detecting SGWB when

$$A_{\rm GWB} \gtrsim 10^{-16}$$ (referenced to f_{yr} = 1/yr)

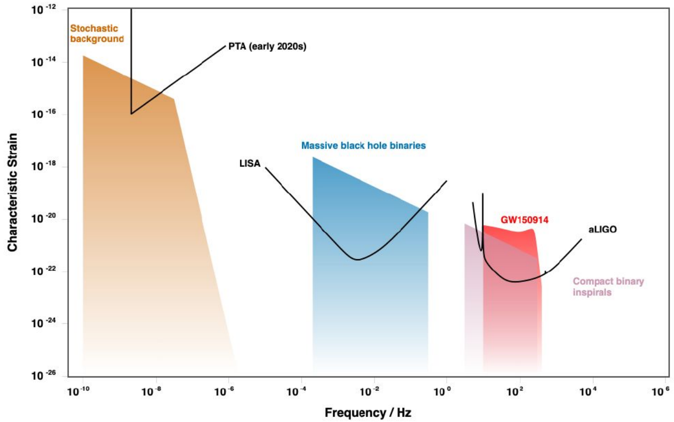


Figure 2.3 of S. Taylor 2021, created using http://gwplotter.com (C. Moore *et al.* 2015)

SGWB Spatial Correlations

- Timing-residual cross-power spectral density for SGWB (Arzoumanian et al. (2016)):

$$P_{ab}^{\mathrm{GWB}}(f) := P^{\mathrm{GWB}}(f)\,\Gamma_{ab} = \left[\frac{A_{\mathrm{GWB}}{}^2}{12\pi^2}\left(\frac{f}{f_{\mathrm{yr}}}\right)^{-\gamma} f_{\mathrm{yr}}^{-3}\right]\Gamma_{ab}$$

$\gamma := 3 - 2\alpha$

$\gamma = 13/3$ for SMBHBs

- Γ_{ab} is the spatial correlation function
  - Function of angle between pulsars a and b: $\hat{p}_a \cdot \hat{p}_b = \cos\Theta_{ab}$
  - Normalized to 1 when $a = b$
  - Referred to as an overlap reduction function (ORF)
  - **For GWs, known as Hellings-Downs (HD) curve** (R. Hellings and G. Downs 1983)

- Functional form of HD curve:

$$\Gamma_{ab} = \left[\frac{1}{2} - \frac{1}{4}x_{ab} + \frac{3}{2}x_{ab}\ln(x_{ab})\right](1 + \delta_{ab})$$

$$x_{ab} = \frac{1 - \cos\Theta_{ab}}{2}$$

Pulsar Term (not included in figure)

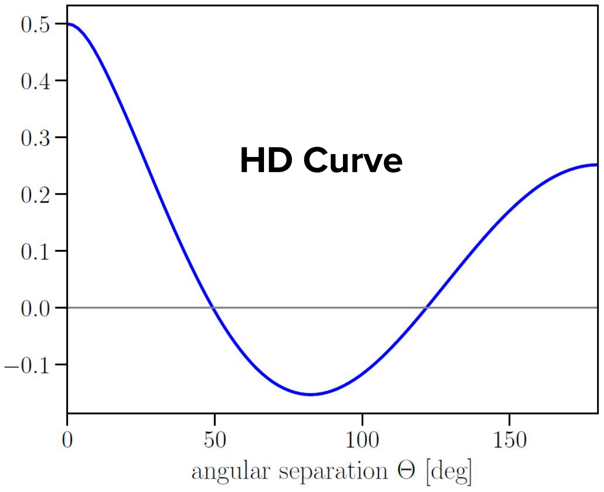


Figure adapted from W. Qin, K. Boddy, M. Kamionkowski, & L.Dai (2019)

PTA Evidence for SGWB (June 2023)

PTA Collaboration	No. of Pulsars	Max Obs. Time (yrs)	A_{GWB} ($x10^{-15}$)*	HD Bayes Factor	False Alarm Probability
NANOGrav	68 - 1	~16	2.4 +0.7/-0.6	**~200**	10^{-3} to $5x10^{-5}$
EPTA+InPTA	25	~11	2.0 +0.3/-0.2	**~60**	$\sim 10^{-3}$
PPTA	32 - 2	~18	2.5 +0.7/-0.7	1.5	0.02

Evidence of a SGWB!

* Reference frequency of 1/yr at γ=13/3; median posterior values with 90% CI (68% CI for PPTA)

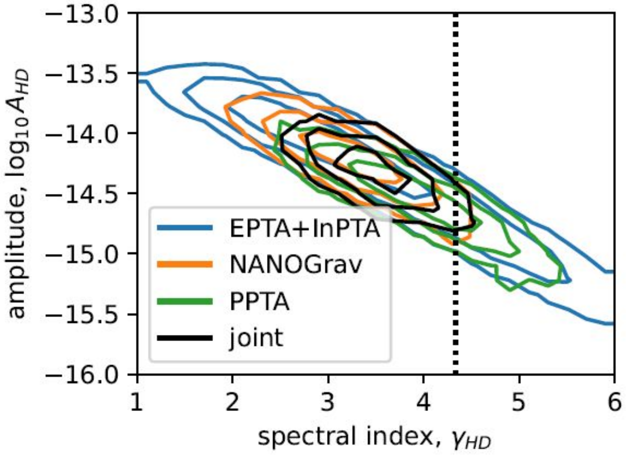


Figure 1. from G. Agazie *et al.* 9/1/2023

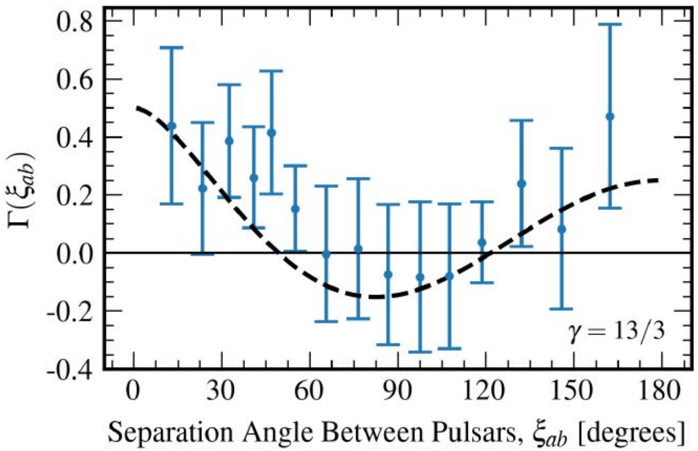


Figure 1.c. from G. Agazie *et al.* 2023 (NANOGrav 15-yr GWB paper)

Extending to Alternate Theories of Gravity

- Beyond-GR theories can have GWs with primarily quadrupolar spatial correlations *that are different from HD correlations*

But can a Bayes factor methodology or frequentist calculation that look like:

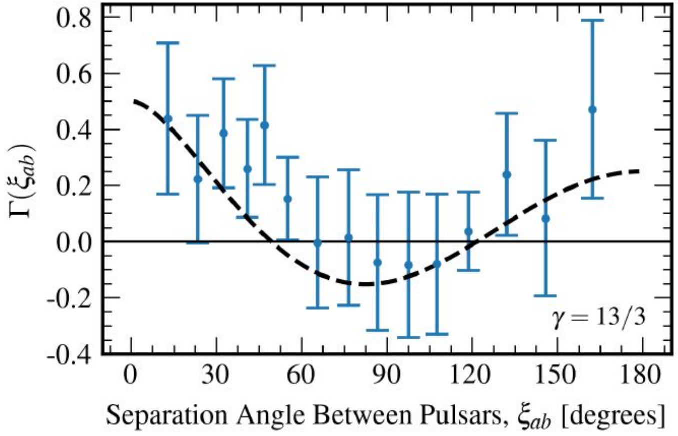
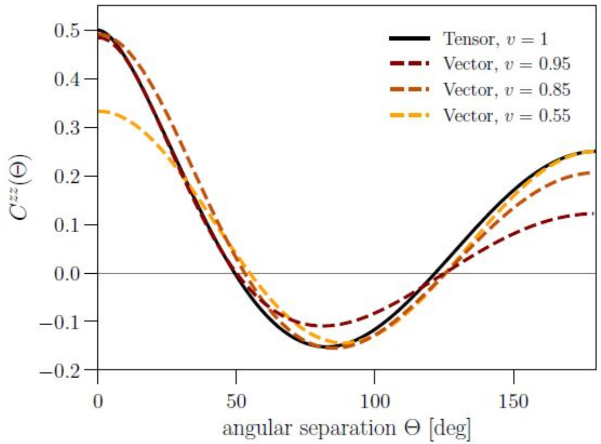


Figure 1.c. from G. Agazie *et al.* 2023 (NANOGrav 15-yr GWB paper)

constrain spatial correlations from beyond-GR theories, e.g.:



Partial Figure 5 from W. Qin, K. Boddy, & M. Kamionkowski (2021)

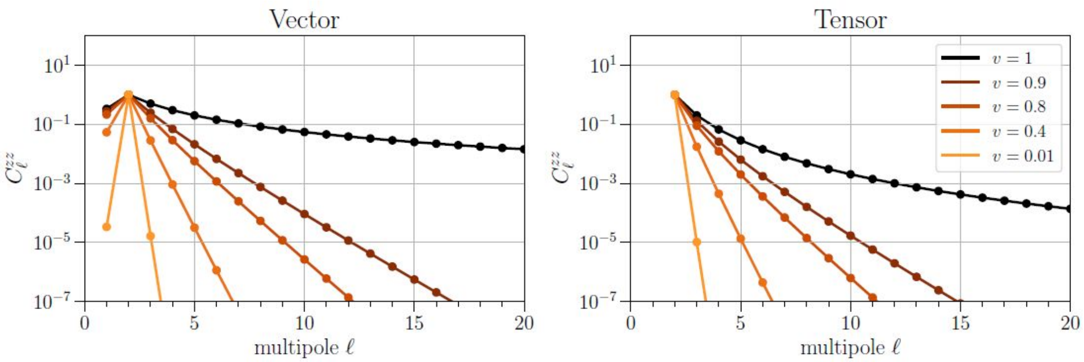
Harmonic Analysis Overview

- GW timing residual in spherical harmonic basis: $R_{\ell m}^{\mathrm{GW}}(t) = \int_{\mathcal{S}^2} d\hat{p}\, R^{\mathrm{GW}}(t,\hat{p})\, Y_{\ell m}^{*}(\hat{p})$
- Isotropic, stationary background: $\langle R_{\ell m}^{\mathrm{GW}}(t_{ai}) R_{\ell' m'}^{\mathrm{GW}*}(t_{bj}) \rangle = C_\ell(|t_{ai} - t_{bj}|)\, \delta_{\ell\ell'}\, \delta_{mm'}$
- Angular power spectrum: $C_\ell(|t_{ai} - t_{bj}|) = 24\pi\, |F_\ell|^2 \int_0^\infty df\ P^{\mathrm{GWB}}(f)\, \cos\left(2\pi f (t_{ai} - t_{bj})\right)$

Detector Response Function

- Angular power spectrum characterizes spatial correlations of GWs and is unique to beyond-GR theories such as:
  - Alternate polarizations
  - Subluminal propagation speeds
  - Massive gravity

Quadrupole normalized to 1



W. Qin, K. Boddy, M. Kamionkowski, & L. Dai (2018),
D. Mihaylov et al. (2019),
W. Qin, K. Boddy, & M. Kamionkowski (2021)

Partial Figure 1 from W. Qin, K. Boddy, & M. Kamionkowski (2021)

Harmonic Analysis Overview

- Angular power spectrum from two-point angular correlation function

$$\langle\, R^{\rm GW}(t_{ai},\hat{p}_a)R^{\rm GW}(t_{bj},\hat{p}_b)\,\rangle = \sum_\ell \frac{2\ell+1}{4\pi}\, C_\ell\, P_\ell\,(\cos\Theta_{ab})$$

Average over all distinct pulsar pairs with $\hat{p}_a \cdot \hat{p}_b = \cos\Theta_{ab}$

- *Normalized* Legendre coefficients for an isotropic SGWB

$$c_\ell = \frac{2\ell+1}{4\pi}\frac{C_\ell}{P^{\rm GWB}} = \frac{3}{2}(2\ell+1)\,\frac{(\ell-2)!}{(\ell+2)!} \qquad \ell \geq 2$$

$|F_\ell|^2$

$c_0 = c_1 = 0$

$c_2 = 0.3125$

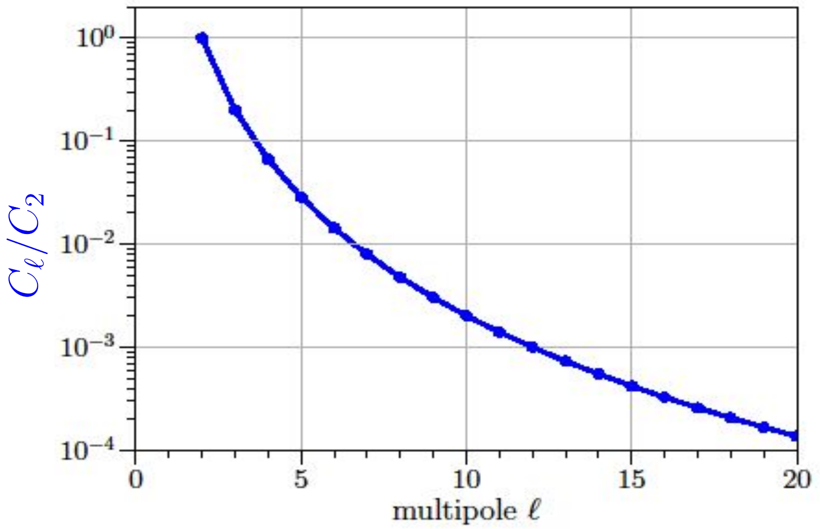
$c_3 = 0.0875$

$\vdots$

Dominant Quadrupole

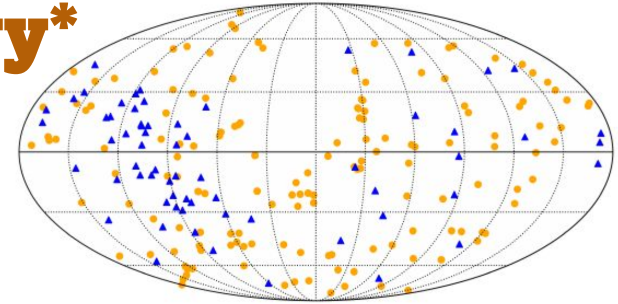
See also: Mingarelli et al. (2013), J. Gair et al. (2014), E. Roebber et al. (2017)

Isotropic SGWB Angular Power Spectrum



Adapted from W. Qin, K. Boddy & M. Kamionkowski (2021)

Harmonic Analysis Methodology*



Blue = Actual IPTA pulsars
Yellow = Mock dataset pulsars

- Generate mock PTA datasets with varying number of pulsars and observation times
  - Inject white noise, pulsar intrinsic red noise, and SGWB signal
- Create Bayesian analysis likelihood model

$$P_{ab}(f) = \frac{A^2_{\mathrm{GWB}}}{12\pi^2}(f)^{-13/3}(1+\delta_{ab})\sum_{\ell=2}^{8} c_\ell\, P_\ell(\cos\Theta_{ab}) + \frac{A^2_{\mathrm{RN,a}}}{12\pi^2}(f)^{\gamma_{RN,a}}\,\delta_{ab}$$

- Perform MCMC sampling with parameters
- Evaluate posterior distributions for multipole evidence
- Reconstruct spatial correlation function from best-fit posteriors

Parameter Name	No. of MCMC Parameters	MCMC Prior
$\log_{10}A_{\mathrm{GWB}}$	1	U(-18,-14)
c_2 through c_8	7	U(0,1)
$\log_{10}A_{\mathrm{RN,a}}$	No. of Pulsars	U(-20,-11)
$\gamma_{\mathrm{RN,a}}$	N0. of Pulsars	U(0,7)

* **JN**, K. Boddy, T. Smith, and C. Mingarelli, *Harmonic Analysis for Pulsar Timing Arrays*, 2023, ArXiv: 2306.06168

Harmonic Analysis Results

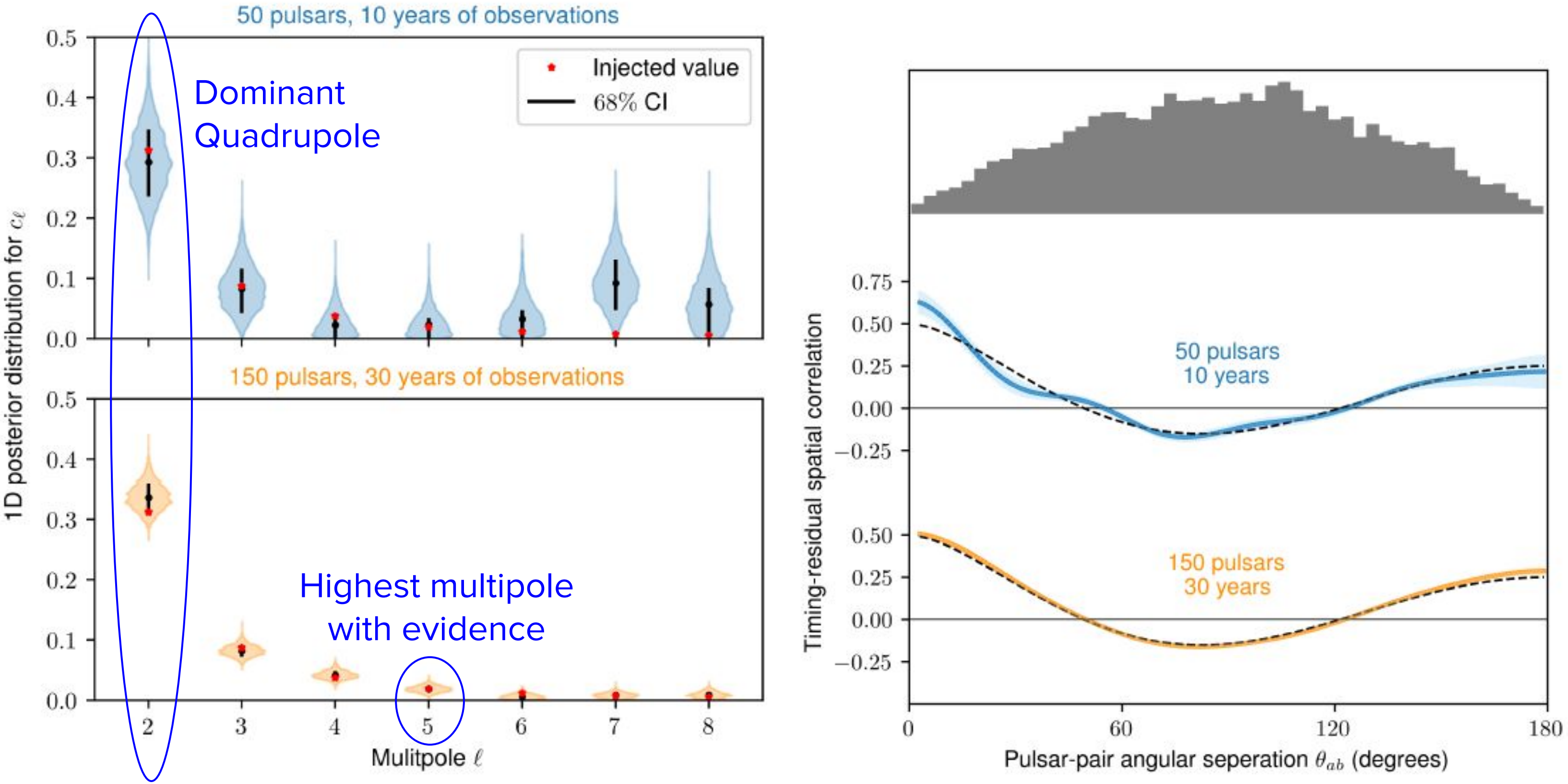


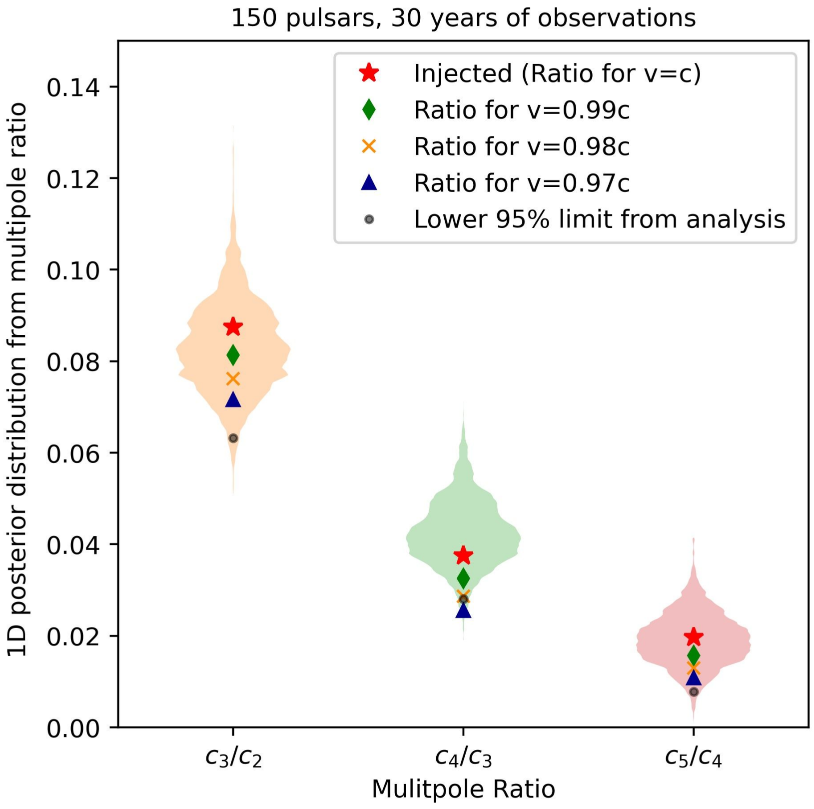
Figure 6 from JN, et al. (2023)

Harmonic Analysis Results

- Angular power spectrum can place constraints on beyond-GR theories
- Example:
  - Mock data with 150 pulsars, 30 years of observation, optimistic noise quality
  - GW single-phase subluminal propagation speed v ≤ c
  - Detector response function is a function of v (W. Qin, K. Boddy, & M. Kamionkowski (2021))

$$\frac{c_{\ell+1}}{c_\ell} = \frac{2\ell+3}{2\ell+1}\left(\frac{C_{\ell+1}}{C_\ell}\right) = \frac{2\ell+3}{2\ell+1}\left(\frac{|F_{\ell+1}(v)|^2}{|F_\ell(v)|^2}\right)$$

  - v > 0.98c at 95% confidence from multipole ratio c_4 / c_3 (most constraining)



Summary and Future Work

- PTA collaborations have found evidence for an isotropic SGWB
- Current PTA methodology does not answer questions such as
  - "How close is observed quadrupole correlation to its theoretical value?"
  - "Is the ratio of octupole to quadrupole correlations consistent with GR?"
- Harmonic analysis approach
  - Methodology provided in **J. Nay, K. Boddy, T. Smith, and C. Mingarelli, *Harmonic Analysis for Pulsar Timing Arrays*, 2023, ArXiv: 2306.06168**
  - Provides best-fit angular power spectrum of PTA timing data
  - Strength of multipoles quantified and/or bounded
  - Place constraints on beyond-GR theories
- Currently applying harmonic analysis methodology to NANOGrav 15-yr dataset (NANOGrav collaboration project)

Questions?

GW Energy Density Spectrum

- GW energy density

$$\rho_{\rm GW} = T^{00} = \frac{1}{32\pi}\left\langle \dot{h}_{ij}^{\rm TT}\,\dot{h}_{ij}^{\rm TT}\right\rangle$$

- GW energy density spectrum

$$\Omega_{\rm GW}(f) := \frac{1}{\rho_c}\frac{d\rho_{\rm GW}}{d\log f}$$

$$= \frac{2\pi^2}{3{\rm H}_0^2}\, f^3\, S_h(f)$$

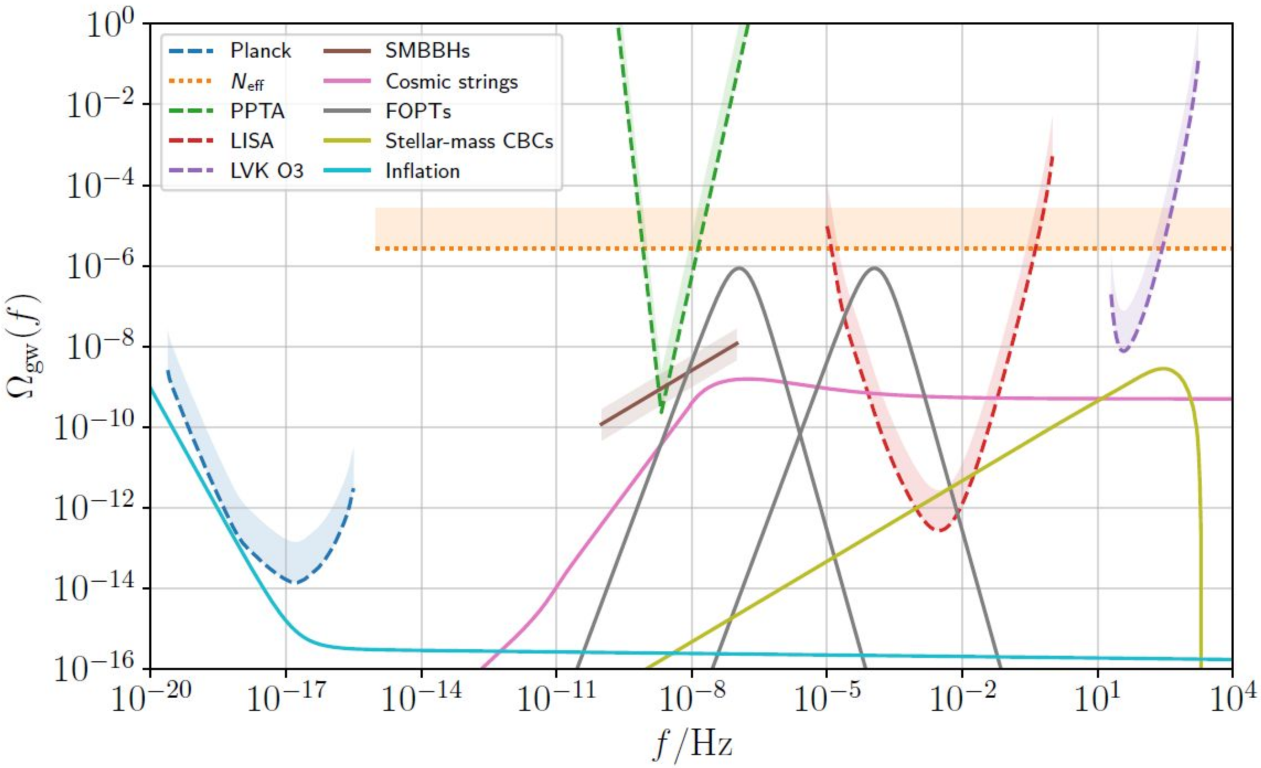


Figure 1 of A. Renzini *et al.* 2022

Generate Synthetic Timing Data

General Attributes	Number of pulsars (N_{psrs})	50, 100, 150
	Pulsar spatial distribution	Full-sky uniform, Galactic-plane restricted
	Pulsar observation time (T_{obs}) (14 day cadence)	10 years, 20 years, 30 years
Injected Random Noise	Instrument/Measurement Uncertainty White Noise	Moderate quality: $\mathcal{O}$(100 ns) High quality: $\mathcal{O}$(10 ns)
	Pulsar Intrinsic Red Noise (different for each pulsar)	$A_{RN} < A_{GW}$ $1 < \gamma_{RN} < 5$
Injected Signal	Isotropic Stochastic Gravitational Wave Background	$A_{GW} = 2 \times 10^{-15}$ $\gamma_{GW} = 13/3 = 4.333$

Injected Signals

- Injected white noise covariance matrix

$$N_{ab} = (EFAC_a)^2 \left((TOAerr_a)^2 + (EQUAD_a)^2\right) \delta_{ab}$$

- Injected pulsar intrinsic red noise power-spectrum

$$S_{ab}^{\mathrm{RN}} \propto (A_{RN,a})^2 \, (f)^{-\gamma_{RN,a}} \, \delta_{ab}$$

- Injected isotropic SGWB power spectrum

$$S_{ab}^{\mathrm{GW}} \propto (2 \times 10^{-15})^2 \, (f)^{-13/3} \, \Gamma_{ab}^{\mathrm{HD}}$$

- Reminder: Wiener-Khinchin Theorem to convert red process power spectra into covariance matrices

Future Research

- Vary spatial correlations and frequency spectra
  - Monopole and dipole signals for noise, new physics, etc.
- Explore use of different MCMC techniques for large parameter spaces
  - Recent developments in Hamiltonian MCMC for PTA analyses
- Develop techniques to separate auto-correlation amplitude from cross-correlation amplitude
  - Reduce correlation between GW amplitude and quadrupole coefficient
- Improve modeling of pulsar intrinsic red noise
  - Pulsar “drop-out” analyses shown to reduce bias in recovered GW amplitude

Research Overview

- **Research Goal:** Evaluate robustness of harmonic analysis for PTAs to determine *spatial correlation function*
- Bayesian statistics only
- Synthetic pulsars for control of input parameters
- Use PTA collaborations' analysis techniques and software pipeline
    - TEMPO2 (G. Hobbs, R. Edwards & R. Manchester 2006)
    - ENTERPRISE (J. Ellis, M. Vallisneri, S. Taylor & P. Baker 2020)
    - ENTERPRISE Extensions (S. Taylor, P. Baker, J. Hazboun, J. Simon & S. Vigeland 2021)
    - PTMCMCSampler (J. Ellis & R. van Haasteren 2017)

Method of Calculating Bayes Factors

- CURN likelihood function PSD (**no cross-correlations between pulsars**)

$$S_{ab}(f) = \frac{A^2_{\rm GWB}}{12\pi^2}\left(\frac{f}{f_{\rm yr}}\right)^{-\gamma} f_{\rm yr}^{-3}\, \delta_{ab}$$

- HD likelihood function PSD (**pulsar cross-correlations given by HD curve**)

$$S_{ab}(f) = \frac{A^2_{\rm GWB}}{12\pi^2}\left(\frac{f}{f_{\rm yr}}\right)^{-\gamma} f_{\rm yr}^{-3}\, \Gamma^{\rm HD}_{ab}$$

- Combine HD and CURN likelihood models into “product space” Bayesian analysis
- Add model switching parameter ‘n’ to MCMC parameters, uniform prior on [0,1]

  n < 0.5 = sample CURN model subspace, n > 0.5 = sample HD model subspace

- Perform Bayesian analysis

  Bayes Factor = (length of MCMC chain with n > 0.5) / (length of MCMC chain with n < 0.5)

Calculation of False Alarm Probability

- Randomly scramble pulsar sky positions
  - Use HD match statistic to create approximately orthogonal skies
  - Purpose is to determine probability that cross-correlations in timing data aren't a result of "lucky" positions
- Calculate $HD_{scrambled}$ vs CURN Bayes factors for thousands of sky scrambles
- Create "background" distribution of Bayes factors from scrambled skies
- Compare actual HD vs CURN Bayes factor to background and calculate false alarm probability

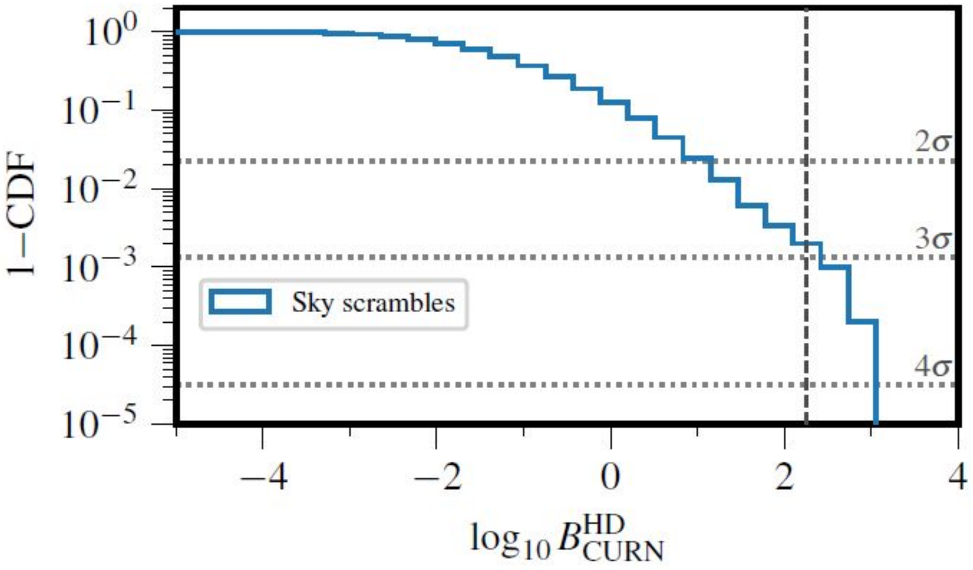


Figure 14 from G. Agazie *et al.* 2023 (NANOGrav 15-yr GWB paper)

Summary of PTA Collaborations Results

- NANOGrav, EPTA, and PPTA have consistent, decisive evidence for a low-frequency power-law spectrum common to all pulsars (BF $\sim 10^{12}$)
  - Amplitude and power law broadly consistent with (large) population of inspiraling SMBHBs
- **NANOGrav and EPTA have evidence for spatial correlations in PTA timing data that are primarily quadrupolar! (as of June 2023)**
  - PPTA does not yet have evidence for quadrupolar correlations
  - Some PTA collaborations also use *Frequentist* methods, which give results consistent with *Bayesian* methods
- No *known* systematics create quadrupolar correlations in PTA timing data
  - Clock errors cause monopolar correlations
  - Solar system barycenter errors cause dipolar correlations
- No physics within GR and the SM create quadrupolar correlations in PTA timing data, *except gravitational waves!*

June 29, 2023 PTA Collaboration Publications

Collaboration	Publication	ArXive
NANOGrav	The NANOGrav 15 yr Data Set: Evidence for a Gravitational-wave Background	2306.16213
EPTA	The second data release from the EPTA III. Search for gravitational wave signals	2306.16214
PPTA	Search for an Isotropic Gravitational-wave Background with the Parkes Pulsar Timing Array	2306.16215
CPTA	Searching for the nano-Hertz SGWB with the Chinese Pulsar Timing Array Data Release I	2306.16216
NANOGrav	The NANOGrav 15 yr Data Set: Observations and Timing of 68 Millisecond Pulsars	2306.16217
NANOGrav	The NANOGrav 15 yr Data Set: Detector Characterization and Noise Budget	2306.16218
NANOGrav	The NANOGrav 15 yr Data Set: Search for Signals from New Physics	2306.16219
NANOGrav	The NANOGrav 15 yr Data Set: Constraints on SMBHBs from the Gravitational-wave Background	2306.16220
NANOGrav	The NANOGrav 15-year Data Set: Search for Anisotropy in the Gravitational-Wave Background	2306.16221
NANOGrav	The NANOGrav 15 yr Data Set: Bayesian Limits on Gravitational Waves from Individual SMBHBs	2306.16222
NANOGrav	The NANOGrav 15-year Gravitational-Wave Background Analysis Pipeline	2306.16223
EPTA	+5 more publications	
PPTA	+2 more publications	