Hilbert series for covariants and their applications to MFV.

CERN BSM forum - November 2nd 2023

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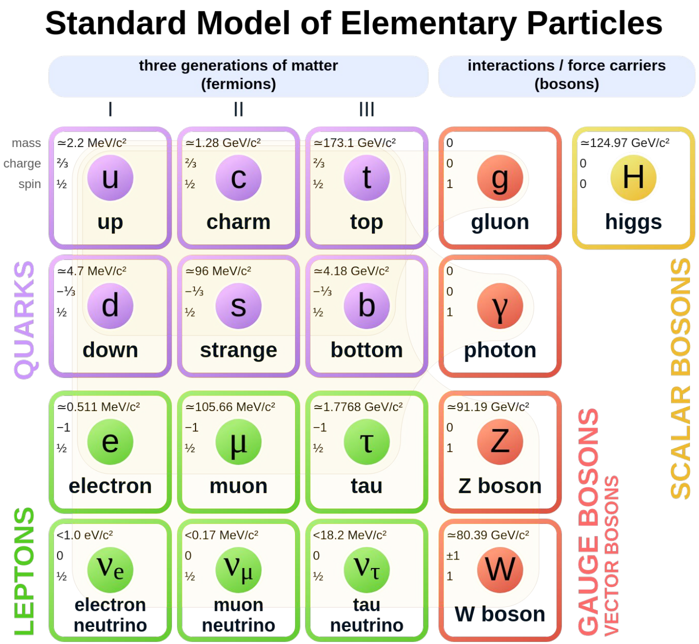
Based on "Hilbert series for covariants and their applications to MFV" (to appear) 2311.XXXX [hep-ph]

in collaboration with B. Grinstein, X. Lu and L. Merlo

The flavor puzzle



▷ Why are there three families?

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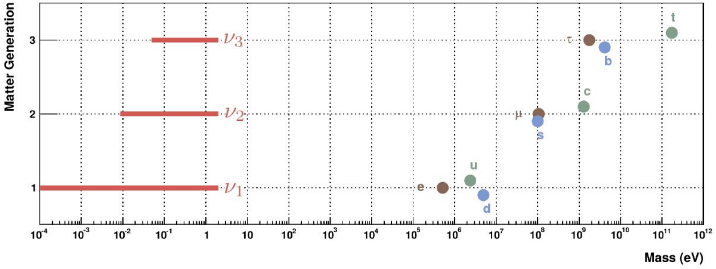
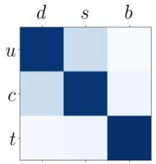
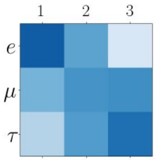


$V_{CKM} \sim$ 

$U_{PMNS} \sim$ 

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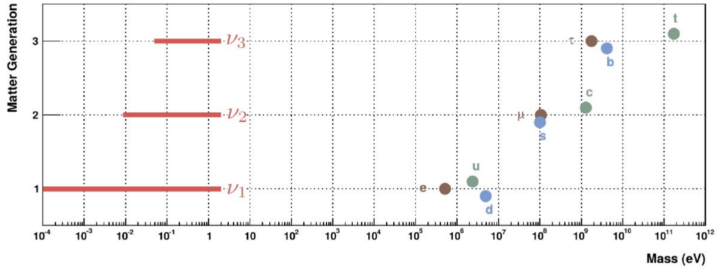
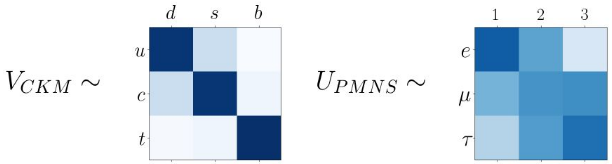
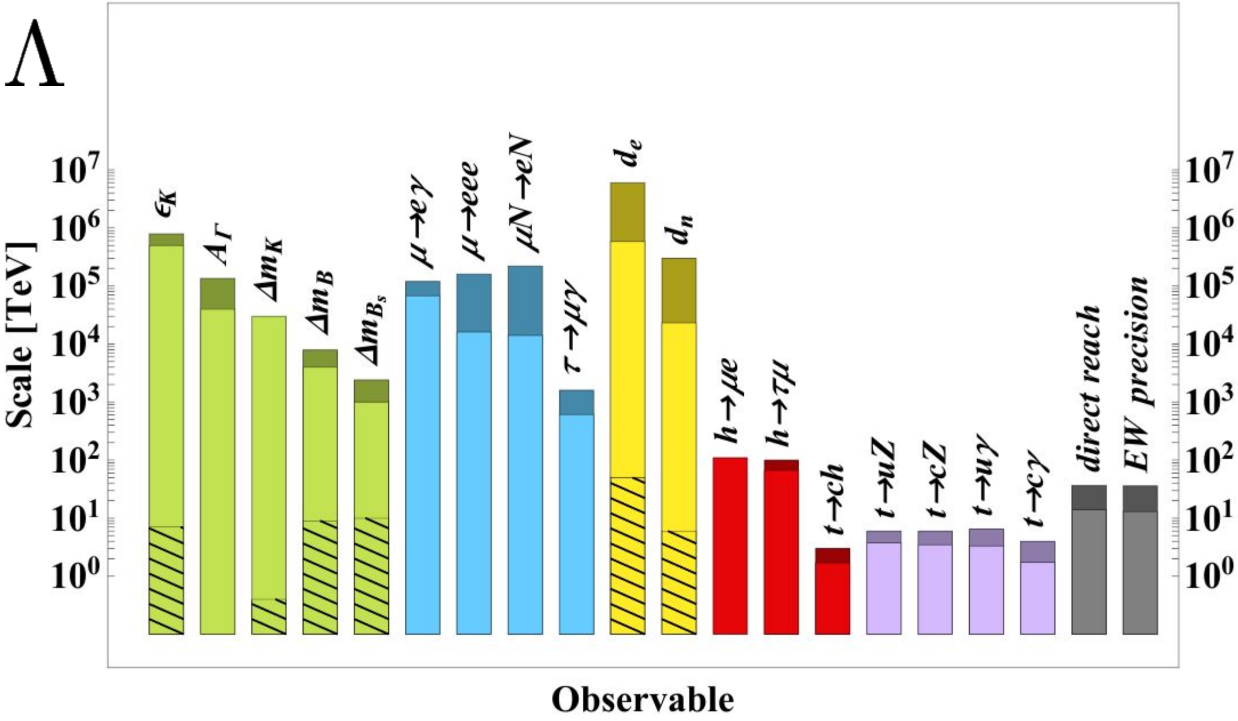




- ▷ Why are there three families?
- ▷ Why do fermions have so different masses?
- ▷ Why is quark mixing so small while lepton mixing is large?

New Physics Flavor puzzle



$$\mathcal{L} \supset \frac{1}{\Lambda^{n-4}} \mathcal{O}_n$$

Hatched bars: MFV
Darker colors: midterm prospects

[European Strategy for Particle Physics, 19]
Aloni+Dery+Gavela+Nir

Quark flavor symmetry

[Georgi+ Chivukula]

➔ Classical global symmetry of the d=4 Lagrangian for $Y_{u,d} \longrightarrow 0$

$$G_F = U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R}$$

$$Q_L \longrightarrow U_{Q_L} Q_L \,; \qquad d_R \longrightarrow U_{d_R} d_R \,; \qquad u_R \longrightarrow U_{u_R} u_R \,.$$

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- Broken by Yukawas:

$$\mathcal{L}_{\text{Yukawa}} = -\overline{Q}_L Y_u \widetilde{\Phi} u_R - \overline{Q}_L Y_d \Phi d_R + \text{ h.c.}$$

Minimal Flavor Violation

[Georgi+ S. Chivukula]
[Hall, Randall]
[D'Ambrosio+Isidori+Giudice+ Strumia]
[Cirigliano+ Grinstein+Wise]

➔ SM Yukawas are the only source of flavor violation both in SM and BSM

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- ➔ SM Yukawas are promoted to spurions

$$Y_u \longrightarrow U_{Q_L} Y_u U^\dagger_{u_R} \quad Y_d \longrightarrow U_{Q_L} Y_u U^\dagger_{d_R}$$

$$Y_u \sim (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}) \qquad Y_d \sim (\mathbf{3}, \mathbf{1}, \overline{\mathbf{3}})$$

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<u>MFV symmetry principle:</u> All higher dimensional operators built from SM fields and the Yukawa spurions are formally invariant under the flavor group (and CP).

Minimal Flavor Violation

- S nd BSM
- S

$$\frac{C_{pr}}{\Lambda^2}(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{q}_p\gamma^\mu q_r)\,.$$

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$$\frac{C_{pr}}{\Lambda^2}(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{q}_p \gamma^\mu q_r)\,. \longrightarrow \frac{(Y_u Y_u^\dagger)_{pr}}{\Lambda^2}(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar{q}_p \gamma^\mu q_r)\,.$$

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Minimal Flavor Violation: issues

- Example: $$\frac{C_{pr}}{\Lambda^2}(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{q}_p\gamma^\mu q_r)\,.$$

$$C \sim \mathbf{8} \oplus \mathbf{1} \qquad h_u \equiv Y_u Y_u^\dagger \qquad h_d \equiv Y_d Y_d^\dagger$$

$$C = c_0 1 + c_1 h_u + c_2 h_d + c_3 h_u^2 + c_3 h_u h_d + c_4 h_d h_u + c_5 h_d^2 + \ldots \qquad \left(Y_u Y_u^\dagger\right)^n ?$$

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Usually $Y_{u,d}$ are treated as order parameters

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➔ Top Yukawa $y_t \sim 1$

➔ In 2HDM Y_d can also be large

Pure Minimal Flavor Violation

- ➔ Let's take MFV seriously
- ➔ Only symmetry principle, no extra assumptions

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- If not, how many?
- Are there assumption independent correlations among flavor observables?

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HILBERT SERIES

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Number of invariant operators at order n

$$\mathcal{H}_{\text{Inv}}^{U(1),\,(+1,-1)}(q) = 1 + q^2 + q^4 + q^6 + \cdots = \frac{1}{1-q^2}\,, \qquad |q| < 1$$

Hilbert series II (for invariants)

- $G = U(1)$ $\Phi = \left\{\phi_1,\, \phi_1^*,\, \phi_2,\, \phi_2^*\right\}$ $Q = \{+1, -1, +1, -1\}$
- 4 Basic invariants:

$$I_1 \equiv \phi_1 \phi_1^* \,, \qquad I_2 \equiv \phi_2 \phi_2^* \,, \qquad I_3 \equiv \phi_1 \phi_2^* \,, \qquad I_4 \equiv \phi_2 \phi_1^* \,.$$

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- True HS:

$$\boxed{\mathcal{H}_{\mathrm{Inv}} = \frac{1-q^4}{(1-q^2)^4} = \frac{1+q^2}{\left(1-q^2\right)^3}}$$

Hilbert series: primary and sec. invariants

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➔ $$\mathcal{H}_{\text{Inv}} = \frac{1+q^2}{(1-q^2)^3} = \left(1 + q^2 + q^4 + q^6 + \cdots\right)^3 \left(1+q^2\right)$$

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$$\left\{P_1^{k_1}\right\} \otimes \left\{P_2^{k_2}\right\} \otimes \left\{P_3^{k_3}\right\} \otimes (1 \oplus S)\,,$$

3 Primary invariants **1 Secondary invariant**

$$P_1 = \phi_1\phi_1^*\,, \quad P_2 = \phi_2\phi_2^*\,, \quad P_3 = \phi_1\phi_2^* + \phi_2\phi_1^*\,, \quad S = \phi_1\phi_2^* - \phi_2\phi_1^*\,,$$

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- Secondary only arises linearly since:

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Hironaka decomposition:

$$\text{Any Inv. polynomial} = p\big(P_1, P_2, P_3\big) + p_S\big(P_1, P_2, P_3\big)\, S\,,$$

ant

P_1

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How to compute Hilbert series?

$$H(q) = \sum_{n=0}^{\infty} n_{\mathrm{Inv}}(n)\, q^n$$

$$\Phi = \Big\{\phi_1,\, \phi_2,\, \cdots,\, \phi_m\Big\}\,, \qquad R_\Phi = \bigoplus_i R_{\phi_i}\,.$$

$$R_{\Phi^k} = \mathrm{sym}\Big(\underbrace{R_\Phi \otimes R_\Phi \otimes \cdots \otimes R_\Phi}_{k}\Big) = n_{\mathrm{Inv}}(k)\,\mathbf{Inv} \oplus \text{ other irreps}$$

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➔ Character:

$$\chi_{R_\Phi}\big(g(x)\big) = \text{tr}\left(g_{R_\Phi}(x)\right).$$

➔ Character orthogonality:

$$\int \mathrm{d}\mu_G(x)\, \chi^*_{R_1}(x)\chi_{R_2}(x) = \delta_{R_1 R_2}$$

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$$R_1 = \text{Inv and } R_2 = R_{\Phi^k}$$

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Molien formula to compute HS

→

$$\mathcal{H}_{\text{Inv}}^{G,R_\Phi}(q) = \sum_{k=0}^{\infty} \int \mathrm{d}\mu_G(x)\, \chi_{R_{\Phi^k}}(x)\, q^k = \int \mathrm{d}\mu_G(x) \frac{1}{\det\left[1 - q g_{R_\Phi}(x)\right]}$$

→

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Applications of Hilbert Series

- ➔ Supersymmetric gauge theories , general supersymmetric EFTs [Benvenuti et al, 07] [Feng et al, 07] [Gray et al, 08] [Delgado et al, 23]
- ➔ SMEFT, SMEFT with gravity [Lehman et al, 15] [Henning, et al, 15] [Lehman et al, 16] [Henning, et al, 17] [Marinissen et al, 20] [Kondo, et al, 23] [Ruhdorfer et al, 19]
- ➔ QCD Chiral Lagrangian, Higgs EFT, NRQED and NRQCD [Graf et al, 21] [Graf et al, 22] [Sun, et al, 22] [Kobach, et al, 17] [Kobach, et al, 18]
- ➔ EFTs for axion-like particles [Grojean et al, 23]
- ➔ Primary observables at colliders [Chang, et al, 22]
- ➔ **Flavor invariants** [Jenkins+Manohar, 09] [Hanany et al, 10]

Hilbert Series for flavor invariants

[Jenkins+Manohar, 09]
[Hanany et al, 10]
[Broer, 94]

$$\mathcal{L}_{\text{Yukawa}} = -\overline{Q}_L Y_u \widetilde{\Phi} u_R - \overline{Q}_L Y_d \Phi d_R + \text{ h.c.}$$

$h_u \equiv Y_u Y_u^\dagger \quad h_d \equiv Y_d Y_d^\dagger$

- ➔ Group: $G_F = U(3)_{Q_L} \times U(3)_{u_R} \times U(3)_{d_R}$
- ➔ Building blocks: $Y_u \sim (\mathbf{3}, \overline{\mathbf{3}}, \mathbf{1}) \qquad Y_d \sim (\mathbf{3}, \mathbf{1}, \overline{\mathbf{3}})$
- ➔ Hilbert series: $\mathcal{H}_{\text{Inv}}(q) = \dfrac{1+q^{12}}{(1-q^2)^2 (1-q^4)^3 (1-q^6)^4 (1-q^8)}$

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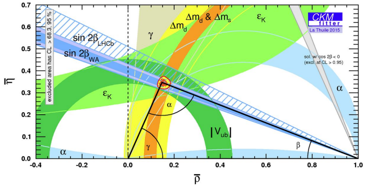

- ➔ Properties
  - 10 prim. inv. = 10 phys. param.
  - Polynomial invariants form a ring
  - Positive coefs. in numerator
  - Palindromic numerator
  - Hironaka decomposition

10 Primary invariants

$$P_{2,0} = \text{Tr}\left[h_u\right], \qquad P_{0,2} = \text{Tr}\left[h_d\right],$$
$$P_{4,0} = \text{Tr}\left[h_u^2\right], \qquad P_{0,4} = \text{Tr}\left[h_d^2\right],$$
$$P_{2,2} = \text{Tr}\left[h_u h_d\right],$$
$$P_{6,0} = \text{Tr}\left[h_u^3\right], \qquad P_{0,6} = \text{Tr}\left[h_d^3\right],$$
$$P_{4,2} = \text{Tr}\left[h_u^2 h_d\right], \qquad P_{2,4} = \text{Tr}\left[h_u h_d^2\right],$$
$$P_{4,4} = \text{Tr}\left[h_u^2 h_d^2\right],$$

1 Secondary invariant

$$S = \text{Im Tr}\left[h_u h_d h_u^2 h_d^2\right]$$
$$= -\frac{i}{2} \det\left[Y_u Y_u^\dagger, Y_d Y_d^\dagger\right]$$
$$\equiv \text{Jarlskog determinant}$$

$$J^2 = \text{poly}(P_1, \ldots, P_{10})$$

Extension: Hilbert series for covariants

➔ Hilbert Series can also count rep-R covariants

$$R_{\Phi^k} = n_R(k)\, R \oplus \text{ other irreps}\,.$$

$$n_{\text{Inv}}(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_{\text{Inv}}(x)\, \chi_{R_{\Phi^k}}(x)$$

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$$n_{\rm Inv}(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_{\rm Inv}(x)\, \chi_{R_{\Phi^k}}(x)$$

$$\chi^*_{\rm Inv}(x) = 1 \longrightarrow \chi^*_R(x)$$

$$n_R(k) = \int \mathrm{d}\mu_G(x)\, \chi^*_R(x)\, \chi_{R_{\Phi^k}}(x)\,.$$

$$\mathcal{H}_R^{G,\, R_\Phi}(q) \equiv \sum_{k=0}^{\infty} n_R(k)\, q^k = \int \mathrm{d}\mu_G(x)\, \chi^*_R(x)\, \frac{1}{\det\left[1 - q\, g_{R_\Phi}(x)\right]}\,.$$

Hilbert series for covariants: example

- Group: $G = U(1)$
- Building blocks: $\Phi = \{\phi_1,\, \phi_1^*,\, \phi_2,\, \phi_2^*\}$ $\quad Q = \{+1, -1, +1, -1\}$
- Goal representation: $Q = +2$
- Hilbert series:

$$\mathcal{H}_{\text{Inv}} = \frac{1+q^2}{\left(1-q^2\right)^3}$$

$$\mathcal{H}_{+2}^{U(1),\,2\times(+1,-1)}(q) = \oint_{|z|=1} \frac{\mathrm{d}z}{2\pi i}\, \frac{1}{z}\, z^{-2}\, \frac{1}{(1-qz)^2(1-qz^{-1})^2}$$

$$= \left[\frac{\mathrm{d}}{\mathrm{d}z}\, \frac{1}{z(1-qz)^2}\right]\Bigg|_{z=q} = \frac{3q^2-q^4}{\left(1-q^2\right)^3}\,.$$

Hilbert series for covariants: Properties

➔ Rep-R covariants form a module over the ring of invariants $\mathcal{M}_R^{G,\,R_\Phi}$

$$r_i \in \mathbb{r}_{\rm Inv}\,, \quad v_i \in \mathcal{M}_R^{G,\,R_\Phi} \qquad \Longrightarrow \qquad \sum_i r_i\, v_i \in \mathcal{M}_R^{G,\,R_\Phi}\,.$$

$$\mathcal{H}_{\rm Inv} = \frac{1+q^2}{(1-q^2)^3}$$

$$\mathcal{H}_{+2}(q) = \frac{3q^2 - q^4}{(1-q^2)^3}\,.$$

➔ Negative coefficients arise in the numerator => redundancies

➔ The denominator corresponds to the primary invariants

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➔ Negative coefficients arise in the numerator => redundancies

➔ The denominator corresponds to the primary invariants

➔ Generating set: Every covariant is a linear combination of them

➔ Linear independence

➔ Basis is not guaranteed to exist. If it does, the module is free.

Hilbert series for covariants: example

- $G = U(1) \quad \Phi = \{\phi_1,\, \phi_1^*,\, \phi_2,\, \phi_2^*\} \quad Q = \{+1, -1, +1, -1\} \quad Q = +2$
- HS: $\mathcal{H}_{+2}(q) = \frac{3q^2 - q^4}{(1-q^2)^3}\,. \quad \mathcal{H}_{\rm Inv} = \frac{1+q^2}{(1-q^2)^3}$ $\quad P_1 = \phi_1\phi_1^*, \quad P_2 = \phi_2\phi_2^*, \quad P_3 = \phi_1\phi_2^* + \phi_2\phi_1^*,\; S = \phi_1\phi_2^* - \phi_2\phi_1^*$
- Generating set:

$$v_1 = \phi_1\phi_1\,, \qquad v_2 = \phi_2\phi_2\,, \qquad v_3 = \phi_1\phi_2$$

- Not linearly independent, there is a redundancy $O(q^4)$

$$P_3\, v_3 = P_2\, v_1 + P_1\, v_2$$

Hilbert series for covariants: Rank

- Rank: “Maximal number of linearly independent vectors”
- Computation:

$$\operatorname{rank}\left({}_{\mathfrak{r}_{\mathrm{Inv}}}\mathcal{M}_R^{G,\,R_\Phi}\right) = \left.\frac{\mathcal{H}_R^{G,\,R_\Phi}(q)}{\mathcal{H}_{\mathrm{Inv}}^{G,\,R_\Phi}(q)}\right|_{q=1}$$

Hilbert series for covariants: Rank

➔ Rank: “Maximal number of linearly independent vectors”

➔ Computation:

$$\mathrm{rank}\left({}_{r_{\mathrm{Inv}}}\mathcal{M}_R^{G,\,R_\Phi} \right) = \left. \frac{\mathcal{H}_R^{G,\,R_\Phi}(q)}{\mathcal{H}_{\mathrm{Inv}}^{G,\,R_\Phi}(q)} \right|_{q=1}$$

➔ Bound on the rank:

$$\mathrm{rank}\left({}_{r_{\mathrm{Inv}}}\mathcal{M}_R^{G,\,R_\Phi} \right) \leq \dim(R)\,.$$

➔ Rank saturation:

$$\mathrm{rank}\left({}_{\sigma_{\mathrm{Inv}}}\,\mathcal{M}_R^{G,R_\Phi} \right) = \dim(R)$$

Hilbert series for covariants: Rank

- Rank: “Maximal number of linearly independent vectors”
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$$\operatorname{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_R^{G,\,R_\Phi}\right) = \left.\frac{\mathcal{H}_R^{G,\,R_\Phi}(q)}{\mathcal{H}_{\text{Inv}}^{G,\,R_\Phi}(q)}\right|_{q=1}$$

- Bound on the rank: $\operatorname{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_R^{G,\,R_\Phi}\right) \leq \dim(R)\,.$
- Rank saturation: $\operatorname{rank}\left({}_{\sigma_{\text{Inv}}}\mathcal{M}_R^{G,R_\Phi}\right) = \dim(R)$ → **One can build the most general rep-R covariant!**
- Theorem by **[Brion, 93]**

$$\operatorname{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_R^{G,\,R_\Phi}\right) = \dim\left(R^H\right)$$

Hilbert series for covariants: Applications

- ➔ OPE (Operator Product Expansion)
- ➔ Counting form factors
- ➔ Spurion analysis ➔ e.g. **Minimal Flavor Violation**

Pure Minimal Flavor Violation

- ➔ Let's take MFV seriously
- ➔ Only symmetry principle, no extra assumptions

$$C = c_0 1 + c_1 h_u + c_2 h_d + c_3 h_u^2 + c_3 h_u h_d + c_4 h_d h_u + c_5 h_d^2 + \ldots$$

- ➔ Are there really infinite textures? $\left(Y_u Y_u^\dagger\right)^n$?
- ➔ If not, how many?
- ➔ Are there assumption independent correlations among flavor observables?

HILBERT SERIES

$5: \psi^2 H^3 + \text{h.c.}$		$SU(3)_{Q_L,u_R,d_R}$
Q_{eH}	$(H^\dagger H)(\bar{l}_p e_r H)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
Q_{uH}	$(H^\dagger H)(\bar{q}_p u_r \widetilde{H})$	$(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$
Q_{dH}	$(H^\dagger H)(\bar{q}_p d_r H)$	$(\mathbf{3},\mathbf{1},\bar{\mathbf{3}})$

$6: \psi^2 XH + \text{h.c.}$		$SU(3)_{Q_L,u_R,d_R}$
Q_{eW}	$(\bar{l}_p \sigma^{\mu\nu} e_r)\tau^I H W^I_{\mu\nu}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
Q_{eB}	$(\bar{l}_p \sigma^{\mu\nu} e_r) H B_{\mu\nu}$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
Q_{uG}	$(\bar{q}_p \sigma^{\mu\nu} T^A u_r)\widetilde{H}\, G^A_{\mu\nu}$	$(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$
Q_{uW}	$(\bar{q}_p \sigma^{\mu\nu} u_r)\tau^I \widetilde{H}\, W^I_{\mu\nu}$	$(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$
Q_{uB}	$(\bar{q}_p \sigma^{\mu\nu} u_r)\widetilde{H}\, B_{\mu\nu}$	$(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$
Q_{dG}	$(\bar{q}_p \sigma^{\mu\nu} T^A d_r) H\, G^A_{\mu\nu}$	$(\mathbf{3},\mathbf{1},\bar{\mathbf{3}})$
Q_{dW}	$(\bar{q}_p \sigma^{\mu\nu} d_r)\tau^I H\, W^I_{\mu\nu}$	$(\mathbf{3},\mathbf{1},\bar{\mathbf{3}})$
Q_{dB}	$(\bar{q}_p \sigma^{\mu\nu} d_r) H\, B_{\mu\nu}$	$(\mathbf{3},\mathbf{1},\bar{\mathbf{3}})$

$7: \psi^2 H^2 D$		$SU(3)_{Q_L,u_R,d_R}$
$Q^{(1)}_{Hl}$	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{l}_p \gamma^\mu l_r)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$Q^{(3)}_{Hl}$	$(H^\dagger i\overleftrightarrow{D}^I_\mu H)(\bar{l}_p \tau^I \gamma^\mu l_r)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
Q_{He}	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{e}_p \gamma^\mu e_r)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$Q^{(1)}_{Hq}$	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{q}_p \gamma^\mu q_r)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1})$
$Q^{(3)}_{Hq}$	$(H^\dagger i\overleftrightarrow{D}^I_\mu H)(\bar{q}_p \tau^I \gamma^\mu q_r)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1})$
Q_{Hu}	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{u}_p \gamma^\mu u_r)$	$(\mathbf{1},\mathbf{1}\oplus\mathbf{8},\mathbf{1})$
Q_{Hd}	$(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{d}_p \gamma^\mu d_r)$	$(\mathbf{1},\mathbf{1},\mathbf{1}\oplus\mathbf{8})$
Q_{Hud} + h.c.	$i(\widetilde{H}^\dagger D_\mu H)(\bar{u}_p \gamma^\mu d_r)$	$(\mathbf{1},\mathbf{3},\bar{\mathbf{3}})$

$8: (\bar{L}L)(\bar{L}L)$		$SU(3)_{Q_L,u_R,d_R}$
Q_{ll}	$(\bar{l}_p \gamma_\mu l_r)(\bar{l}_s \gamma^\mu l_t)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
$Q^{(1)}_{qq}$	$(\bar{q}_p \gamma_\mu q_r)(\bar{q}_s \gamma^\mu q_t)$	$(\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27},\mathbf{1},\mathbf{1})$
$Q^{(3)}_{qq}$	$(\bar{q}_p \gamma_\mu \tau^I q_r)(\bar{q}_s \gamma^\mu \tau^I q_t)$	$(\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27},\mathbf{1},\mathbf{1})$
$Q^{(1)}_{lq}$	$(\bar{l}_p \gamma_\mu l_r)(\bar{q}_s \gamma^\mu q_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1})$
$Q^{(3)}_{lq}$	$(\bar{l}_p \gamma_\mu \tau^I l_r)(\bar{q}_s \gamma^\mu \tau^I q_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1})$

$8: (\bar{R}R)(\bar{R}R)$		$SU(3)_{Q_L,u_R,d_R}$
Q_{ee}	$(\bar{e}_p \gamma_\mu e_r)(\bar{e}_s \gamma^\mu e_t)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
Q_{uu}	$(\bar{u}_p \gamma_\mu u_r)(\bar{u}_s \gamma^\mu u_t)$	$(\mathbf{1},\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27},\mathbf{1})$
Q_{dd}	$(\bar{d}_p \gamma_\mu d_r)(\bar{d}_s \gamma^\mu d_t)$	$(\mathbf{1},\mathbf{1},\mathbf{1}\oplus\mathbf{8}\oplus\mathbf{10}\oplus\overline{\mathbf{10}}\oplus\mathbf{27})$
Q_{eu}	$(\bar{e}_p \gamma_\mu e_r)(\bar{u}_s \gamma^\mu u_t)$	$(\mathbf{1},\mathbf{1}\oplus\mathbf{8},\mathbf{1})$
Q_{ed}	$(\bar{e}_p \gamma_\mu e_r)(\bar{d}_s \gamma^\mu d_t)$	$(\mathbf{1},\mathbf{1},\mathbf{1}\oplus\mathbf{8})$
$Q^{(1)}_{ud}$	$(\bar{u}_p \gamma_\mu u_r)(\bar{d}_s \gamma^\mu d_t)$	$(\mathbf{1},\mathbf{1}\oplus\mathbf{8},\mathbf{1}\oplus\mathbf{8})$
$Q^{(8)}_{ud}$	$(\bar{u}_p \gamma_\mu T^A u_r)(\bar{d}_s \gamma^\mu T^A d_t)$	$(\mathbf{1},\mathbf{1}\oplus\mathbf{8},\mathbf{1}\oplus\mathbf{8})$

$8: (\bar{L}L)(\bar{R}R)$		$SU(3)_{Q_L,u_R,d_R}$
Q_{le}	$(\bar{l}_p \gamma_\mu l_r)(\bar{e}_s \gamma^\mu e_t)$	$(\mathbf{1},\mathbf{1},\mathbf{1})$
Q_{lu}	$(\bar{l}_p \gamma_\mu l_r)(\bar{u}_s \gamma^\mu u_t)$	$(\mathbf{1},\mathbf{1}\oplus\mathbf{8},\mathbf{1})$
Q_{ld}	$(\bar{l}_p \gamma_\mu l_r)(\bar{d}_s \gamma^\mu d_t)$	$(\mathbf{1},\mathbf{1},\mathbf{1}\oplus\mathbf{8})$
Q_{qe}	$(\bar{q}_p \gamma_\mu q_r)(\bar{e}_s \gamma^\mu e_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1})$
$Q^{(1)}_{qu}$	$(\bar{q}_p \gamma_\mu q_r)(\bar{u}_s \gamma^\mu u_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1}\oplus\mathbf{8},\mathbf{1})$
$Q^{(8)}_{qu}$	$(\bar{q}_p \gamma_\mu T^A q_r)(\bar{u}_s \gamma^\mu T^A u_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1}\oplus\mathbf{8},\mathbf{1})$
$Q^{(1)}_{qd}$	$(\bar{q}_p \gamma_\mu q_r)(\bar{d}_s \gamma^\mu d_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1}\oplus\mathbf{8})$
$Q^{(8)}_{qd}$	$(\bar{q}_p \gamma_\mu T^A q_r)(\bar{d}_s \gamma^\mu T^A d_t)$	$(\mathbf{1}\oplus\mathbf{8},\mathbf{1},\mathbf{1}\oplus\mathbf{8})$

$8: (\bar{L}R)(\bar{L}R) + \text{h.c.}$		$SU(3)_{Q_L,u_R,d_R}$
$Q^{(1)}_{quqd}$	$(\bar{q}^j_p u_r)\epsilon_{jk}(\bar{q}^k_s d_t)$	$(\bar{\mathbf{3}}\oplus\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}})$
$Q^{(8)}_{quqd}$	$(\bar{q}^j_p T^A u_r)\epsilon_{jk}(\bar{q}^k_s T^A d_t)$	$(\bar{\mathbf{3}}\oplus\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}})$
$Q^{(1)}_{lequ}$	$(\bar{l}^j_p e_r)\epsilon_{jk}(\bar{q}^k_s u_t)$	$(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$
$Q^{(3)}_{lequ}$	$(\bar{l}^j_p \sigma_{\mu\nu} e_r)\epsilon_{jk}(\bar{q}^k_s \sigma^{\mu\nu} u_t)$	$(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})$

$8: (\bar{L}R)(\bar{R}L) + \text{h.c.}$		$SU(3)_{Q_L,u_R,d_R}$
Q_{ledq}	$(\bar{l}^j_p e_r)(\bar{d}_s q_{tj})$	$(\bar{\mathbf{3}},\mathbf{1},\mathbf{3})$

Hilbert series for all d=6 MFV covariants

$$\mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})} = \frac{1+q^{12}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} = \frac{2\left(q^2+2q^4+2q^6+2q^8+q^{10}\right)}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})} = \frac{q^2\left(1+2q^2+3q^4+4q^6+4q^8+2q^{10}+q^{12}-q^{16}\right)}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{3},\overline{\mathbf{3}},\mathbf{1})} = \frac{q\left(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12}\right)}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{3},\overline{\mathbf{3}})} = \frac{q^2\left(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12}\right)}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{27},\mathbf{1},\mathbf{1})} = \frac{3q^4+8q^6+17q^8+20q^{10}+19q^{12}+8q^{14}-q^{16}-8q^{18}-7q^{20}-4q^{22}-q^{24}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{10},\mathbf{1},\mathbf{1})} = \frac{q^4(1+6q^2+7q^4+8q^6+4q^8-3q^{12}-2q^{14}-q^{16})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{10},\mathbf{1})} = \frac{q^6(2+3q^2+6q^4+7q^6+6q^8+2q^{10}-3q^{14}-2q^{16}-q^{18})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{10},\mathbf{1})} = \frac{q^6(2+3q^2+6q^4+7q^6+6q^8+2q^{10}-3q^{14}-2q^{16}-q^{18})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{27},\mathbf{1})} = \frac{q^4(1+2q^2+6q^4+10q^6+17q^8+18q^{10}+16q^{12}+6q^{14}-2q^{16}-8q^{18}-7q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{8},\mathbf{8},\mathbf{1})} = \frac{q^2(1+6q^2+17q^4+30q^6+39q^8+38q^{10}+24q^{12}+6q^{14}-7q^{16}-12q^{18}-9q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{8})} = \frac{q^4(2+8q^2+19q^4+32q^6+40q^8+36q^{10}+21q^{12}+4q^{14}-9q^{16}-12q^{18}-8q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\overline{\mathbf{3}},\mathbf{3},\overline{\mathbf{3}})} = \frac{q^2(1+4q^2+9q^4+14q^6+15q^8+12q^{10}+5q^{12}-3q^{16}-2q^{18}-q^{20})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{6},\overline{\mathbf{3}},\overline{\mathbf{3}})} = \frac{q^2(1+4q^2+12q^4+22q^6+32q^8+32q^{10}+24q^{12}+8q^{14}-4q^{16}-10q^{18}-8q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{10},\mathbf{1},\mathbf{1})} = \mathcal{H}_{(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})}\,,\quad \mathcal{H}_{(\mathbf{1},\mathbf{10},\mathbf{1})} = \mathcal{H}_{(\mathbf{1},\overline{\mathbf{10}},\mathbf{1})} = \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{10})} = \mathcal{H}_{(\mathbf{1},\mathbf{1},\overline{\mathbf{10}})}\,,$$
$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})} = \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{8})} \quad \text{and} \quad \mathcal{H}_{(\mathbf{1},\mathbf{27},\mathbf{1})} = \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{27})}$$

Hilbert series (8,1,1)

Ex. $\frac{C_{pr}}{\Lambda^2}(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{q}_p\gamma^\mu q_r)\,.$

$$\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} = \frac{2\left(q^2+2q^4+2q^6+2q^8+q^{10}\right)}{\left(1-q^2\right)^2\left(1-q^4\right)^3\left(1-q^6\right)^4\left(1-q^8\right)}$$

$$\begin{array}{lll}
\mathcal{O}(q^2): & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^2,a} = h_u\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^2,b} = h_d\,, \\
\mathcal{O}(q^4): & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,a} = h_u^2\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,b} = h_d^2\,, \\
 & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,c} = h_u h_d\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,d} = h_d h_u\,, \\
\mathcal{O}(q^6): & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,a} = h_u^2 h_d\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,c} = h_d^2 h_u \\
 & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,b} = h_u h_d^2\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,d} = h_d h_u^2\,, \\
\mathcal{O}(q^8): & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,a} = h_u^2 h_d^2\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,b} = h_d^2 h_u^2\,, \\
 & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,c} = h_u^2 h_d h_u\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,d} = h_d^2 h_u h_d\,, \\
\mathcal{O}(q^{10}): & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^{10},a} = h_u^2 h_d h_u h_d\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^{10},b} = h_d^2 h_u h_d h_u\,.
\end{array}$$

→ Reproduced with traditional methods

Cayley-Hamilton Theorem:

$$\mathbf{A}^3 = (\mathrm{tr}\,\mathbf{A})\mathbf{A}^2 - \frac{1}{2}\left((\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\left(\mathbf{A}^2\right)\right)\mathbf{A} + \det(\mathbf{A})I_3$$

$$\mathcal{H}_{\rm Inv}(q) = \frac{1+q^{12}}{\left(1-q^2\right)^2\left(1-q^4\right)^3\left(1-q^6\right)^4\left(1-q^8\right)}$$

[Mercolli+Smith, 09]

Hilbert series (8,1,1)

Ex. $\frac{C_{pr}}{\Lambda^2}(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{q}_p\gamma^\mu q_r)\,.$

$$\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} = \frac{2\,(q^2+2q^4+2q^6+2q^8+q^{10})}{(1-q^2)^2\,(1-q^4)^3\,(1-q^6)^4\,(1-q^8)}$$

$$\begin{aligned}
\mathcal{O}(q^2):\quad & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^2,a} = h_u\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^2,b} = h_d\,,\\
\mathcal{O}(q^4):\quad & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,a} = h_u^2\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,b} = h_d^2\,,\\
& V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,c} = h_u h_d\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^4,d} = h_d h_u\,,\\
\mathcal{O}(q^6):\quad & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,a} = h_u^2 h_d\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,c} = h_d^2 h_u\\
& V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,b} = h_u h_d^2\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^6,d} = h_d h_u^2\,,\\
\mathcal{O}(q^8):\quad & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,a} = h_u^2 h_d^2\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,b} = h_d^2 h_u^2\,,\\
& V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,c} = h_u^2 h_d h_u\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^8,d} = h_d^2 h_u h_d\,,\\
\mathcal{O}(q^{10}):\quad & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^{10},a} = h_u^2 h_d h_u h_d\,, & V^{(\mathbf{8},\mathbf{1},\mathbf{1})}_{q^{10},b} = h_d^2 h_u h_d h_u\,.
\end{aligned}$$

➔ Reproduced with traditional methods

Cayley-Hamilton Theorem:

$$\mathbf{A}^3 = (\operatorname{tr}\mathbf{A})\mathbf{A}^2 - \frac{1}{2}\left((\operatorname{tr}\mathbf{A})^2 - \operatorname{tr}\left(\mathbf{A}^2\right)\right)\mathbf{A} + \det(\mathbf{A})I_3$$

➔ No factor $(1+q^{12})$ in the numerator:

$$Jh_u = \sum c_i V_i$$

➔ Generating set is **not** linearly independent

$$\mathcal{H}_{\text{Inv}}(q) = \frac{1+q^{12}}{(1-q^2)^2\,(1-q^4)^3\,(1-q^6)^4\,(1-q^8)}$$

[Mercolli+Smith, 09]

Hilbert series (1,8,1)

Ex. $\frac{C_{pr}}{\Lambda^2}(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{u}_p\gamma^\mu u_r)\,.$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})}(q) = \frac{q^2\left(1+2q^2+3q^4+4q^6+4q^8+2q^{10}+q^{12}-q^{16}\right)}{\left(1-q^2\right)^2\left(1-q^4\right)^3\left(1-q^6\right)^4\left(1-q^8\right)}\,.$$

- ➔ Can be understood from $H_{(8,1,1)}(q)$ and $H_{(1,1,1)}(q)$

$$V_{(\mathbf{1},\mathbf{8},\mathbf{1})} \sim Y_u^\dagger\, V_{(\mathbf{8},\mathbf{1},\mathbf{1})}\, Y_u \qquad \text{or} \qquad V_{(\mathbf{1},\mathbf{8},\mathbf{1})} \sim Y_u^\dagger\, V_{(\mathbf{1},\mathbf{1},\mathbf{1})}\, Y_u\,.$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})}\Big|_{\text{naive}} = q^2\,\left[\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} + \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right] = \frac{q^2\left(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12}\right)}{\left(1-q^2\right)^2\left(1-q^4\right)^3\left(1-q^6\right)^4\left(1-q^8\right)}$$

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$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})}\Big|_{\text{naive}} = q^2\,\left[\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} + \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right] = \frac{q^2\left(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12}\right)\ ?}{(1-q^2)^2\,(1-q^4)^3\,(1-q^6)^4\,(1-q^8)}$$

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Ex. $\frac{C_{pr}}{\Lambda^2}(H^\dagger i\overleftrightarrow{D}_\mu H)(\bar{u}_p\gamma^\mu u_r)\,.$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})}(q) = \frac{q^2\left(1+2q^2+\boxed{3q^4}+4q^6+4q^8+2q^{10}+q^{12}\boxed{-q^{16}}\right)}{(1-q^2)^2\,(1-q^4)^3\,(1-q^6)^4\,(1-q^8)}\,.$$

➔ Can be understood from $H_{(8,1,1)}(q)$ and $H_{(1,1,1)}(q)$

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$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})}\Big|_{\text{naive}} = q^2\,\left[\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})}+\mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right] = \frac{q^2\left(1+2q^2+\boxed{4q^4}+4q^6+4q^8+2q^{10}+q^{12}\right)}{(1-q^2)^2\,(1-q^4)^3\,(1-q^6)^4\,(1-q^8)}\ \boxed{?}$$

➔ But there are 2 redundancies:

$$\mathcal{O}(q^6):\quad Y_u^\dagger h_u^2 Y_u = (Y_u^\dagger Y_u)^3 \longrightarrow \text{Cayley-Hamilton} \qquad \longrightarrow -\frac{q^6}{D(q)}$$

$$\mathcal{O}(q^{18}):\quad Y_u^\dagger J h_u^2 Y_u = J\left(Y_u^\dagger Y_u\right)^3 \longrightarrow \text{Cayley-Hamilton} \qquad \longrightarrow -\frac{q^{18}}{D(q)}$$

$$\downarrow$$

$$Jh_u = \sum c_i V_i$$

Hilbert series for $(\mathbf{3},\overline{\mathbf{3}},\mathbf{1})$, $(\mathbf{3},\mathbf{1},\overline{\mathbf{3}})$ and $(\mathbf{1},\mathbf{3},\overline{\mathbf{3}})$

$$V_{(\mathbf{3},\overline{\mathbf{3}},\mathbf{1})} \sim \left(V_{(\mathbf{8},\mathbf{1},\mathbf{1})} + V_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right) Y_u \qquad V_{(\mathbf{3},\mathbf{1},\overline{\mathbf{3}})} \sim \left(V_{(\mathbf{8},\mathbf{1},\mathbf{1})} + V_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right) Y_d$$

$$V_{(\mathbf{1},\mathbf{3},\overline{\mathbf{3}})} \sim Y_u^\dagger \left(V_{(\mathbf{8},\mathbf{1},\mathbf{1})} + V_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right) Y_d$$

$$\mathcal{H}_{(\mathbf{3},\overline{\mathbf{3}},\mathbf{1})} = \mathcal{H}_{(\mathbf{3},\mathbf{1},\overline{\mathbf{3}})} = \frac{q\left(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12}\right)}{\left(1-q^2\right)^2\left(1-q^4\right)^3\left(1-q^6\right)^4\left(1-q^8\right)} = q\left[\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} + \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right]$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{3},\overline{\mathbf{3}})} = \frac{q^2\left(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12}\right)}{\left(1-q^2\right)^2\left(1-q^4\right)^3\left(1-q^6\right)^4\left(1-q^8\right)} = q^2\left[\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} + \mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})}\right] \qquad (3.24)$$

Hilbert series for all d=6 MFV covariants

$$\mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})} = \frac{1+q^{12}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} = \frac{2(q^2+2q^4+2q^6+2q^8+q^{10})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})} = \frac{q^2(1+2q^2+3q^4+4q^6+4q^8+2q^{10}+q^{12}-q^{16})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})} = \frac{q(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{3},\bar{\mathbf{3}})} = \frac{q^2(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{27},\mathbf{1},\mathbf{1})} = \frac{3q^4+8q^6+17q^8+20q^{10}+19q^{12}+8q^{14}-q^{16}-8q^{18}-7q^{20}-4q^{22}-q^{24}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{10},\mathbf{1},\mathbf{1})} = \frac{q^4(1+6q^2+7q^4+8q^6+4q^8-3q^{12}-2q^{14}-q^{16})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{10},\mathbf{1})} = \frac{q^6(2+3q^2+6q^4+7q^6+6q^8+2q^{10}-3q^{14}-2q^{16}-q^{18})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

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$$\mathcal{H}_{(\mathbf{1},\mathbf{27},\mathbf{1})} = \frac{q^4(1+2q^2+6q^4+10q^6+17q^8+18q^{10}+16q^{12}+6q^{14}-2q^{16}-8q^{18}-7q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{8},\mathbf{8},\mathbf{1})} = \frac{q^2(1+6q^2+17q^4+30q^6+39q^8+38q^{10}+24q^{12}+6q^{14}-7q^{16}-12q^{18}-9q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{8})} = \frac{q^4(2+8q^2+19q^4+32q^6+40q^8+36q^{10}+21q^{12}+4q^{14}-9q^{16}-12q^{18}-8q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\bar{\mathbf{3}},\mathbf{3},\bar{\mathbf{3}})} = \frac{q^2(1+4q^2+9q^4+14q^6+15q^8+12q^{10}+5q^{12}-3q^{16}-2q^{18}-q^{20})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{6},\bar{\mathbf{3}},\bar{\mathbf{3}})} = \frac{q^2(1+4q^2+12q^4+22q^6+32q^8+32q^{10}+24q^{12}+8q^{14}-4q^{16}-10q^{18}-8q^{20}-4q^{22}-q^{24})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

Hilbert series for all d=6 MFV covariants

$$\mathcal{H}_{(\mathbf{1},\mathbf{1},\mathbf{1})} = \frac{1+q^{12}}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{8},\mathbf{1},\mathbf{1})} = \frac{2(q^2+2q^4+2q^6+2q^8+q^{10})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{8},\mathbf{1})} = \frac{q^2(1+2q^2+3q^4+4q^6+4q^8+2q^{10}+q^{12}-q^{16})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{3},\bar{\mathbf{3}},\mathbf{1})} = \frac{q(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

$$\mathcal{H}_{(\mathbf{1},\mathbf{3},\bar{\mathbf{3}})} = \frac{q^2(1+2q^2+4q^4+4q^6+4q^8+2q^{10}+q^{12})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

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$$\mathcal{H}_{(1,\mathbf{10},1)} = \frac{q^6(2+3q^2+6q^4+7q^6+6q^8+2q^{10}-3q^{14}-2q^{16}-q^{18})}{(1-q^2)^2(1-q^4)^3(1-q^6)^4(1-q^8)}$$

- ➔ Finitely generated (as for any reductive G) [Hochster+Roberts, 74]
- ➔ Denominator → primary invariants
- ➔ Numerator with negative coef. → not free module
  - ◆ Positive terms → generating set
  - ◆ Negative terms → redundancies (no basis)
  - ◆ No common factor ($1+q^{12}$)
- ➔ **Rank saturates for all MFV representations**

$$\text{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_R^{G_F,Y_u,Y_d}\right) = \dim(R)$$

Rank saturation for MFV

- ➔ Rank saturates for all MFV representations

$$\text{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_R^{G_F,Y_u,Y_d}\right) = \dim(R)$$

- ➔ Out of Y_u and Y_d We can build as many rep-R covariants as dimension of the representation

Rank saturation for MFV

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➔ Out of Y_u and Y_d We can build as many rep-R covariants as dimension of the representation

➔ Ex. $(\mathbf{27},\mathbf{1},\mathbf{1})$ covariants

$$C_{pqrs}\left(\bar{q}_p\gamma_\mu q_r\right)\left(\bar{q}_s\gamma^\mu q_t\right)$$

$$\text{rank}\left({}_{r_{\text{Inv}}}\mathcal{M}_{(\mathbf{27},\mathbf{1},\mathbf{1})}^{G_F,Y_u,Y_d}\right) = 27 \implies \exists\{V_i^{(27,1,1)}\}_{i=1}^{27} \text{ independent covariants}$$

$$\text{Any } C_{pqrs} \sim \sum_{i=1}^{27} a_i V_i^{(27,1,1)}$$

Rank saturation for MFV

➔ Rank saturates for all MFV representations

The MFV symmetry principle does not restrict the EFT

$$\text{MFV SMEFT} \equiv \text{SMEFT}.$$

Note: It is not obvious. This does not hold for smaller number of building blocks (e.g. only Y_u).

Quo vadis MFV?

- ➔ Still is a good guiding principle organizing different contributions
- ➔ **<u>"Physics lies in the extra assumptions"</u>**
  - ◆ $Y_{u,d}$ as order parameters
  - ◆ Only Y_d as order parameter
  - ◆ Only Y_u as order parameter

} Expanding a order k, the Hilbert series tells you how many structures there are.

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  - ◆ One operator at a time: ratios of different observables $\mathcal{O}_1/\mathcal{O}_2$ may be able to distinguish among the covariants of the generating set. Currently exploring the pheno.
- ➔ No assumption. In terms of finding an origin of flavor it may be useful to use these generating sets as a parametrization of any flavor operator.

Conclusions

- ➔ Hilbert series are really useful tools to count not only invariants but also covariants.
- ➔ The set of rep-R covariants form a module over the ring of invariants (finitely generated…)
- ➔ Rank saturation
- ➔ Application to MFV: we computed all HS for d=6 MFV SMEFT
- ➔ The rank of all of the reps saturates → $\boxed{\text{MFV SMEFT} \equiv \text{SMEFT}.}$
- ➔ Physics lies on the extra assumptions (not the MFV symmetry principle).
- ➔ Outlook: alternative MFV EFTs, other spurion analysis, OPEs, form factors, amplitudes…

Thank you

Back up slides

SMEFT

- ➔ field content + symmetries ⇒ Lagrangian

$$\mathcal{L}_{\mathrm{SMEFT}} = \mathcal{L}_{\mathrm{SM}} + \sum c_i \mathcal{O}_i$$

- ➔ At dimension d=6

[Buchmuller+Wyler, 86]
[Grzadkowski et al, 10]
[Alonso et al, 13]

$$\text{For } n_g = 1, \quad \exists \quad \mathbf{59} \text{ ops} \quad \longrightarrow \quad \text{For } n_g = 3, \quad \exists \quad \mathbf{2499} \text{ ops}$$

Simplifying flavor assumption?