Nilpotent Fierz-Pauli

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Einstein (1915)

$$R_{\mu\nu} = 0$$

Nonlinear equations . . No superposition

$$\lambda\, g^{(1)}_{\mu\nu}(x) + \xi\, g^{(2)}_{\mu\nu}(x)$$ Not a solution!

It is very useful, but difficult, to find exact solutions!

They are few and far between!

Very important in order to stablish the presence of Horizons, singularities, etc

Fierz-Pauli (1939)

Free Spin 2 (and higher) in flat space

Those are linear equations!

It happens to be the first order in the perturbative expansion of Einstein equations

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Fierz-Pauli and its fiends: Transverse and unimodular

$$\mathcal{L} = \mathcal{L}^{\text{I}} + \beta \mathcal{L}^{\text{II}} + a \mathcal{L}^{\text{III}} + b \mathcal{L}^{\text{IV}},$$

$$\mathcal{L}^{\text{I}} = \frac{1}{4} \partial_\mu h^{\nu\rho} \partial^\mu h_{\nu\rho}, \qquad \mathcal{L}^{\text{II}} = -\frac{1}{2} \partial_\mu h^{\mu\rho} \partial_\nu h^\nu_\rho,$$

$$\mathcal{L}^{\text{III}} = \frac{1}{2} \partial^\mu h \partial^\rho h_{\mu\rho}, \qquad \mathcal{L}^{\text{IV}} = -\frac{1}{4} \partial_\mu h \partial^\mu h.$$

Transverse/UG

in unimodular variables

(III,IV)= $\left(\frac{2}{n}, \frac{n+2}{n^2}\right)$

UG in unrestricted

variables

$$\gamma_{\mu\nu} \equiv |g|^{-\frac{1}{n}} g_{\mu\nu}$$

E.A., Blas, Garriga and Verdaguer, 2006

Both transverse and unimodular are invariant under transverse (volume preserving) gauge transformations

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \qquad \partial_\mu \xi^\mu = 0$$

Unimodular is also invariant under local conformal tramsformations

$$\delta h_{\mu\nu} = \frac{2}{n} \phi \eta_{\mu\nu}$$

The action we are going to be interested at is the transverse UG one

$$L_T = \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial_\mu h^{\mu\rho}\partial_\nu h^\nu_\rho$$

(This is UG in unimodular variables) $\gamma_{\mu\nu}$

$$\{h_{\mu\lambda}h^{\lambda\nu} = 0\} \Longleftrightarrow \{\exists l_\mu \quad l^2 = 0 \qquad h_{\mu\nu} = l_\mu l_\nu\}$$

Classical implications

Study rank one traceless deformations of an arbitrary background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + l_\mu l_\nu \qquad l^2 = 0.$$

Why we want to do that?

In general the inverse metric is an infinite power series in Newton's constant

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h_\lambda^\nu + \ldots$$

Nilpotency however implies $$g^{\alpha\beta} = \bar{g}^{\alpha\beta} - l^\alpha l^\beta$$

This is an exact result!

The Ricci tensor is the sum of the background one, a linear piece and a nonlinear one

$$R_{\mu\nu} = \overline{R}_{\mu\nu} + X_{\alpha\beta}[l] + A_{\alpha\beta}[l]$$

This exact, no approximations are involved

No more terms in the expansion!!!

The linear piece is essentially the Fierz-Pauli equation as deduced from the transverse lagrangian

$$X_{\alpha\beta}[l] \equiv \frac{1}{2}\left[-\overline{\Box}(l_\alpha l_\beta) + \overline{\nabla}_\mu \overline{\nabla}_\alpha (l^\mu l_\beta) + \overline{\nabla}_\mu \overline{\nabla}_\beta (l^\mu l_\alpha)\right]$$

$$X \equiv \eta^{\alpha\beta} X_{\alpha\beta} = \overline{\nabla}_\alpha \overline{\nabla}_\beta \left(l^\alpha l^\beta\right) = \overline{\nabla}_\mu \dot{l}^\mu + \dot{\theta} + \theta^2$$

$$l^\alpha l^\beta X_{\alpha\beta} = -\dot{l}_\alpha \dot{l}^\alpha$$

$$g^{\alpha\beta} X_{\alpha\beta} = \overline{\nabla}_\mu \dot{l}^\mu + \dot{\theta} + \theta^2 + \dot{l}_\alpha \dot{l}^\alpha$$

$$\dot{l}^\alpha \equiv l^\lambda \overline{\nabla}_\lambda l^\alpha$$

$$\theta \equiv \overline{\nabla}_\lambda l^\lambda$$

The nonlinear piece is more complicated

$$A_{\mu\nu} \equiv \tfrac{1}{2}\Big\{\theta\left(\dot{l}_\mu l_\nu + l_\mu \dot{l}_\nu\right) + \ddot{l}_\mu l_\nu + \dot{l}_\mu \dot{l}_\nu + l_\mu \ddot{l}_\nu - \dot{l}^\rho \bar{\nabla}_\rho \left(l_\mu l_\nu\right) - \\ - l_\mu l_\nu \bar{\nabla}^\rho l_\sigma \bar{\nabla}^\sigma l_\rho + l_\mu l_\nu \bar{\nabla}^\sigma l^\rho \bar{\nabla}_\sigma l_\rho - l_\mu l_\nu \dot{l}_\rho \dot{l}^\rho \Big\}$$

$$A \equiv \eta^{\alpha\beta} A_{\alpha\beta} = g^{\alpha\beta} A_{\alpha\beta} = -\frac{1}{2}\dot{l}_\alpha \dot{l}^\alpha$$

There are no more terms.

The whole series has reduced to two terms

G\”urses-G\”ursey, Xanthopulos’ work (1978)

When X=0 then both the deformed and the original Ricci tensors are the same

$$R_{\alpha\beta} = \overline{R}_{\alpha\beta}$$

What happens with the A tensor?

In fact the vanishing of the first condition implies the vanishing of the second one

$$\{X_{\alpha\beta} = 0\} \Longrightarrow \{A_{\alpha\beta} = 0\}$$

A linear condition implies a nonlinear one !

Examples

Flat space as seed

Null vector $$l_\mu \equiv \sqrt{\frac{r_s}{r}}\left(1, -\frac{x^j}{r}\right)$$ It obeys X=0

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu - \frac{r_s}{r}(dt + dr)^2$$

In this way we reach the Kerr-Schild family of spacetimes

(Schwarzschild, Kerr, Reissner-Nordstrom, Kerr-Newman, etc)

Ricci flat spacetime as seed

Kasner spacetime

$$ds^2 = dt^2 - \sum_{i=1}^{3} t^{2p_i} dx_i^2$$

Null vector $l_\mu = (1, t^{p_1}, 0, 0)$ X=0

$$ds^2 = 2dt^2 + 2t^{p_1} dt dx - t^{2p_2} dy^2 - t^{2p_3} dx^2$$

The deformed spacetime is again Ricci flat

Gravitational waves

Plane fronted GW $$ds^2 \equiv \overline{g}_{\mu\nu} dx^\mu dx^\nu = du dv - \sum_{a,b=1}^{2} g_{ab} dx^a dx^b$$

Those are exact solutions of the vacuum Einstein equations

Null vector $$l = f(v) du$$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = f^2(v) du^2 + \overline{g}_{\mu\nu} dx^\mu dx^\nu$$

When $l = \sqrt{v} du$ The deformed spacetime is Ricci flat

de Sitter as seed

Constant curvature spacetime $$\mathrm{d}s^2_{(\mathrm{CCS})} = \bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - e^{2Ht}\delta_{ij}\mathrm{d}x^i\mathrm{d}y^j$$

Null vector $$l_\mu = (1, -e^{Ht}, 0, 0)$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - e^{2Ht}\delta_{ij}\mathrm{d}x^i\mathrm{d}y^j + \left(dt - e^{Ht}dx\right)^2 =$$
$$= 2dt^2 - 2e^{Ht}dtdx - e^{2Ht}\left(dy^2 + dz^2\right)$$

Ricci-flat!!

In this case Fierz-Pauli does not vanish, but it generates a cosmological constant

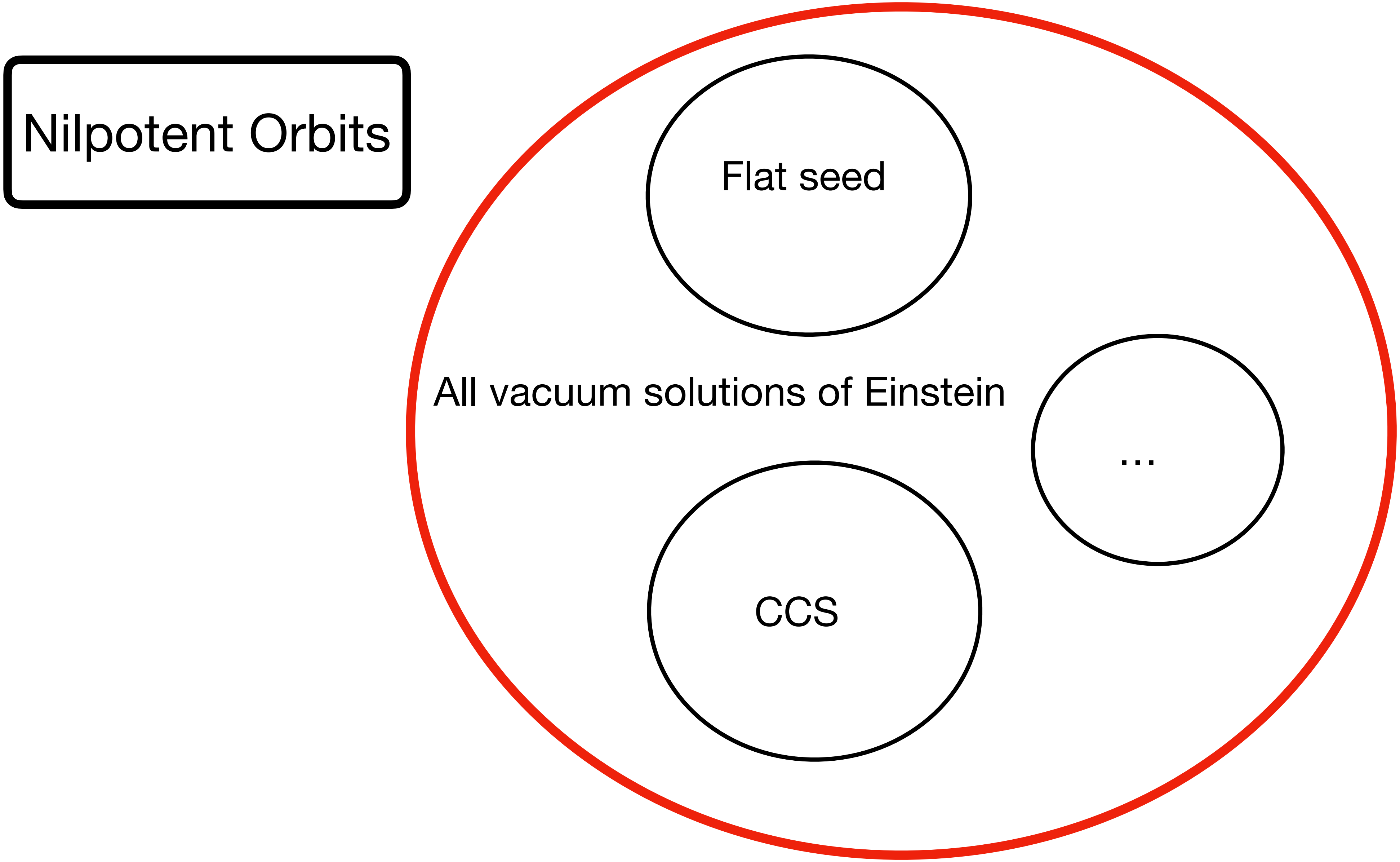
$$X_{\mu\nu}[l] + A_{\mu\nu}[l] = 3H^2\bar{g}_{\mu\nu}$$

$$l_\mu = \left(1, \frac{1}{\sqrt{3}}e^{Ht}, \frac{1}{\sqrt{3}}e^{Ht}, \frac{1}{\sqrt{3}}e^{Ht}\right)$$

$$X_{\mu\nu} = H^2 \begin{pmatrix} 0 & -\sqrt{3}e^{Ht} & -\sqrt{3}e^{Ht} & -\sqrt{3}e^{Ht} \\ -\sqrt{3}e^{Ht} & -4e^{2Ht} & -e^{2Ht} & -e^{2Ht} \\ -\sqrt{3}e^{Ht} & -e^{2Ht} & -4e^{2Ht} & -e^{2Ht} \\ -\sqrt{3}e^{Ht} & -e^{2Ht} & -e^{2Ht} & -4e^{2Ht} \end{pmatrix}$$

$$A_{\mu\nu} = H^2 \begin{pmatrix} 3 & \sqrt{3}e^{Ht} & \sqrt{3}e^{Ht} & \sqrt{3}e^{Ht} \\ \sqrt{3}e^{Ht} & e^{2Ht} & e^{2Ht} & e^{2Ht} \\ \sqrt{3}e^{Ht} & e^{2Ht} & e^{2Ht} & e^{2Ht} \\ \sqrt{3}e^{Ht} & e^{2Ht} & e^{2Ht} & e^{2Ht} \end{pmatrix}$$

Conspiracy between X and A in such a way that the deformed spacetime is Ricci flat


Nilpotent Orbits
Flat seed
All vacuum solutions of Einstein
...
CCS

We have traded in some sense a nonlinear equation

(Ricci flatness) by a linear one (Fierz-Pauli)

Is there superposition of solutions?

The sum of two nilpotent matrices needs not be nilpotent

$$l_\mu^{(1)} l_\nu^{(1)}$$

$$h_{\mu\nu} = l_\mu^{(3)} l_\nu^{(3)}$$

$$l_\mu^{(2)} l_\nu^{(2)}$$

The condition for that to happen is

$$\det\, h_{\mu\nu} = 0$$

$$\eta^{\mu\nu} h_{\mu\nu} = 0$$

$$h_{\mu\nu} \equiv l_\mu^{(1)} l_\nu^{(1)} + l_\mu^{(2)} l_\nu^{(2)}$$

$$l^\lambda_{(1)} l_\lambda^{(2)} = 0$$

$$l_\lambda^{(1)} = l_\lambda^{(2)}$$

Only scaling survives.

Quantum implications to follow!

fluctuations around nilpotent Fierz-Pauli

BACKUP

$$R_{\nu\sigma} = \frac{1}{2}\left(-\partial_\mu\partial^\mu(l_\nu l_\sigma) + \partial_\nu\partial_\mu(l^\mu l_\sigma) + \partial_\mu\partial_\sigma(l^\mu l_\nu) + \kappa\partial_\mu\left(l^\mu l^\delta\partial_\delta(l_\nu l_\sigma)\right)\right) -$$

$$+\frac{1}{4}\Big\{ -\partial^\mu(l_\lambda l_\sigma)\partial^\lambda(l_\nu l_\mu) - \partial^\mu(l_\lambda l_\sigma)\partial_\nu(l^\lambda l_\mu) + \partial^\mu(l_\lambda l_\sigma)\partial_\mu(l^\lambda l_\nu) +$$

$$+\partial_\lambda(l^\mu l_\sigma)\partial^\lambda(l_\nu l_\mu) - \partial_\lambda(l^\mu l_\sigma)\partial_\nu(l^\lambda l_\mu) - \partial_\lambda(l^\mu l_\sigma)\partial_\mu(l^\lambda l_\nu) - \partial_\lambda(l^\mu l_\sigma)l^\lambda l^\delta\partial_\delta(l_\nu l_\mu) +$$

$$+\partial_\sigma(l^\mu l_\lambda)\partial^\lambda(l_\nu l_\mu) - \partial_\sigma(l^\mu l_\lambda)\partial_\mu(l^\lambda l_\nu) -$$

$$-l^\mu l^\delta\partial_\delta(l_\lambda l_\sigma)\partial_\mu(l^\lambda l_\nu)\Big\}$$

$$R \equiv g^{\nu\sigma}R_{\nu\sigma} = \left(\eta^{\nu\sigma} - l^\nu l^\sigma\right)R_{\nu\sigma}$$

$$l^\nu l^\sigma R_{\nu\sigma} = 0 \qquad (2$$

Ergo

$$R = \eta^{\nu\sigma} R_{\nu\sigma} = \partial_\nu \partial_\mu (l^\mu l^\nu) + \frac{1}{4}\Big\{ \partial_\mu (l_\lambda l_\sigma) \partial^\lambda (l^\sigma l^\mu) - \partial^\mu (l_\lambda l_\mu) \partial^\sigma (l^\lambda l^\mu) - \partial_\lambda (l^\mu l_\sigma) \partial_\sigma (l^\lambda l_\mu) -$$

$$-\partial_\lambda (l^\mu l_\sigma) \partial_\mu (l^\lambda l^\sigma) + \partial_\sigma (l^\mu l_\lambda) \partial^\lambda (l^\sigma l_\mu) - \partial_\sigma (l^\mu l_\lambda) \partial_\mu (l^\lambda l_\sigma) \Big\} \qquad (2$$

$$\boxed{X_{\mu\nu} = \frac{1}{2}\left(-\overline{\Box}\left(l_\mu l_\nu\right) + \overline{\nabla}^\lambda \overline{\nabla}_\mu \left(l_\lambda l_\nu\right) + \overline{\nabla}^\lambda \overline{\nabla}_\nu \left(l_\lambda l_\mu\right)\right) = 0}$$

$$\dot{l}^\alpha \equiv l^\lambda \partial_\lambda l^\alpha = \phi\, l^\alpha$$

$$\overline{\nabla}_\lambda l_\mu \overline{\nabla}^\lambda l^\mu = -\dot{\theta} - \dot{\phi} - \phi^2 - \theta\phi - \overline{R}_{\alpha\beta} l^\alpha l^\beta$$

$$\overline{\nabla}_\mu l^\nu \overline{\nabla}_\nu l^\mu = \dot{\phi} - \dot{\theta} + \phi\theta - \overline{R}_{\alpha\beta} l^\alpha l^\beta$$

A full expansion of the Ricci tensor

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + R^{(3)}_{\mu\nu}$$

Power counting with nilpotent fluctuations is not trivial

$$\begin{aligned}
R^{(1)}_{\mu\nu} &= \frac{\kappa}{2}\left[\bar{\nabla}_\lambda \bar{\nabla}_\mu (h_\nu{}^\lambda) + \bar{\nabla}_\lambda \bar{\nabla}_\nu (h_\mu{}^\lambda) - \bar{\Box}(h_{\mu\nu})\right] \\
R^{(2)}_{\mu\nu} &= \frac{\kappa^2}{2}\Big[\bar{\nabla}_\sigma \left[h^{\sigma\rho}\bar{\nabla}_\rho (h_{\mu\nu})\right] - \bar{\nabla}_\sigma (h_\mu{}^\rho)\bar{\nabla}_\rho (h_\nu{}^\sigma) + \\
&\quad + \bar{\nabla}^\rho (h_\mu{}^\sigma)\bar{\nabla}_\rho (h_{\nu\sigma}) - \tfrac{1}{2}\bar{\nabla}_\mu (h_{\rho\sigma})\bar{\nabla}_\nu (h^{\rho\sigma})\Big] \\
R^{(3)}_{\mu\nu} &= -\frac{\kappa^3}{2} h^{\rho\sigma}\bar{\nabla}_\rho (h_{\mu\lambda})\bar{\nabla}_\sigma (h_\nu{}^\lambda)
\end{aligned}$$

$$\{R^{(1)}_{\mu\nu} = 0\} \Longrightarrow \{R^{(2)}_{\mu\nu} + R^{(3)}_{\mu\nu} = 0\}$$

Nilpotency not only makes the expansion finite, but also

makes it linear at the level of the EM

$$S_{\mathrm{EH}}^{(2)} = -\frac{1}{2}\int d^4x \left[\frac{1}{4}h_{\alpha\beta}\bar{\Box}h^{\alpha\beta} - \frac{1}{2}h^{\mu\lambda}\partial_\mu\partial_\nu h^\nu_\lambda\right]$$

$$-\left[\frac{3}{4}h^{\rho\sigma}\overline{\nabla}_\rho h_{\alpha\beta}\overline{\nabla}_\sigma h^{\alpha\beta} - h^{\rho\sigma}h^{\alpha\beta}\overline{\nabla}_\rho\overline{\nabla}_\alpha h_{\sigma\beta} + h^{\rho\sigma}h^{\alpha\beta}\overline{\nabla}_\rho\overline{\nabla}_\sigma h_{\alpha\beta}\right] \tag{24}$$

This is UG at lowest order in unimodular variables

$$\boxed{R = \overline{R} + \overline{\nabla}_\mu \dot{l}^\mu + \dot{\theta} + \theta^2 + \frac{1}{2}\dot{l}^2}$$

Hilbert’s lagrangian vanishes on shell