The Basis Invariant Flavor Puzzle

Andreas Trautner

based on:

arXiv:2308.00019	JHEP 01 (2024) 024 w/ Miguel P. **Bento** and João P. **Silva**
arXiv:1812.02614	JHEP 05 (2019) 208

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MAX-PLANCK-GESELLSCHAFT

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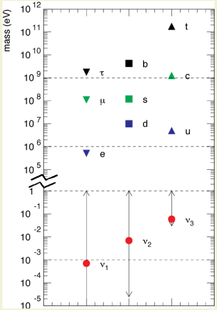
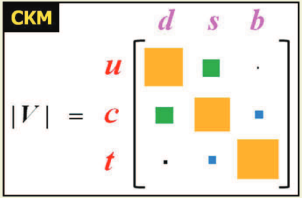
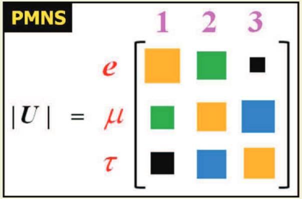
The Standard Model Flavor Puzzle

- **Why** *three* generations of matter Fermions?
- **Why** *hierarchical* masses of Fermions?
- **Why** *small* transition probabilities for $q_i^{\mathrm{up}} \to q_{j\neq i}^{\mathrm{down}}$? $\left(\propto |V_{ij}^{\mathrm{CKM}}|^2\right)$
- **Why** *large* transition probabilities for $\ell_i \to \nu_j$? $\left(\propto |U_{ij}^{\mathrm{PMNS}}|^2\right)$







- **Why** CP violation *only* in combination with *flavor violation*?

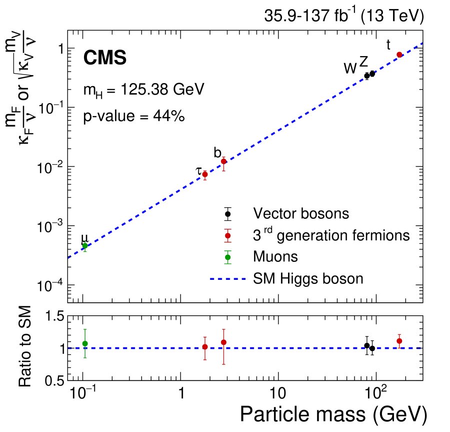
Parametrization independent measure of CP violation:

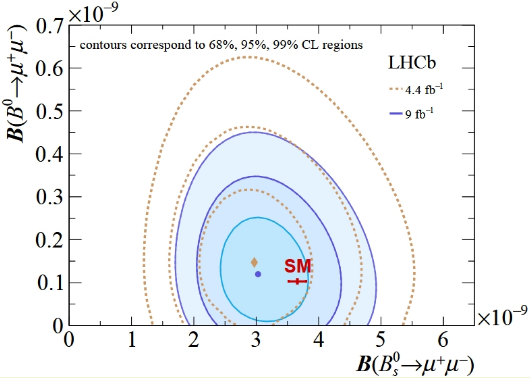
[Greenberg '85, Jarlskog '85]

$$J_{33} \;=\; \det\left[M_u\, M_u^\dagger, M_d\, M_d^\dagger\right] \;\propto\; \boldsymbol{Im}\left[V_{ud}^* V_{cs}^* V_{us} V_{cd}\right] \;=\; 3.08^{+0.15}_{-0.13} \times 10^{-5}\;.$$

The Standard Model Flavor Puzzle

Often underappreciated: Direct confirmation of SM FP at the LHC





And: No hints of New Physics.

see talk by Martinelli

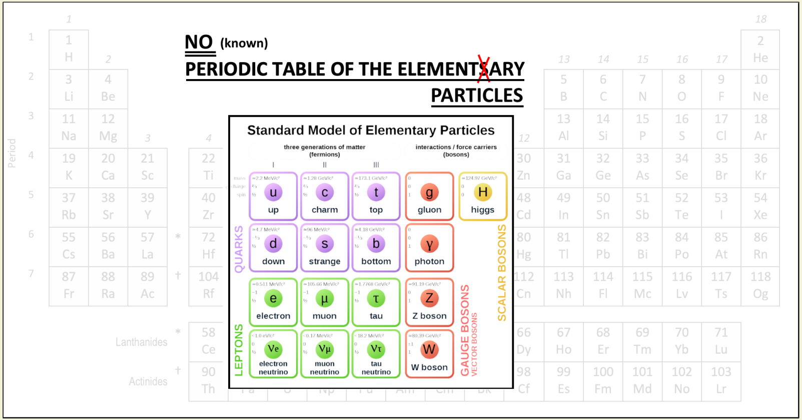
The Standard Model Flavor Puzzle

PERIODIC TABLE OF THE ELEMENTS

Period	1	2	3		4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc		22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y		40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	†	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og

Lanthanides *	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
Actinides †	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr

The Standard Model Flavor Puzzle



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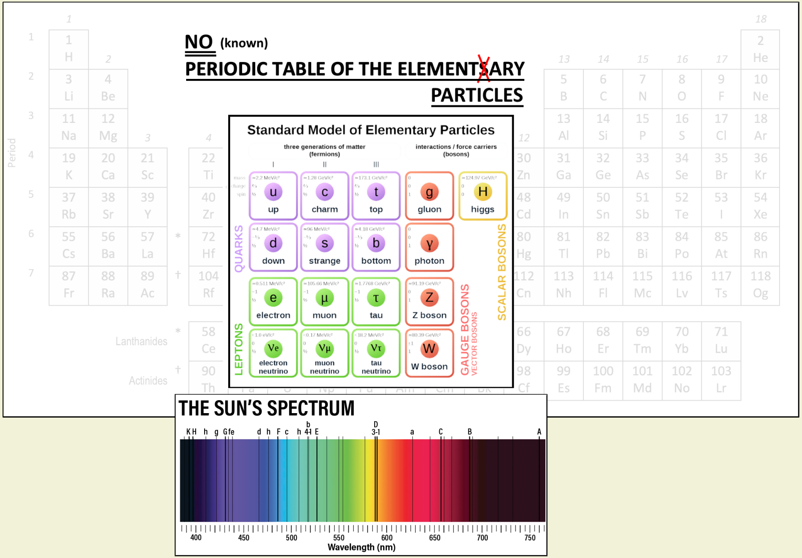


Why use Basis Invariants **(B.I.'s)**?

- Flavor puzzle is *plagued* by ***unphysical*** choice of basis and parametrization.
- Physical observables must be given as function of BIs.
- BI necessary and sufficient conditions for **CPV** in SM. . . . [Greenberg '85; Jarlskog '85]

  . . . and BSM: Multi-scalar 2/3/NHDM, 4th gen., Dirac vs. Majorana ν's, . . .

  [Bernabeau et al. '86], [Branco, Lavoura, Rebelo '86], [Botella, Silva '95], [Davidson, Haber '05], [Yu, Zhou '21],. . .
- BIs and their relations, incl. CP-even BIs, allow to detect symmetries in general.

  [Ivanov, Nishi, Silva, AT '19], [de Meideiros Varzielas, Ivanov '19], [Bento, Boto, Silva, AT '20]
- BI formulation simplifies RGE's, RGE running, and derivation of RGE invariants.

  [Harrison, Krishnan, Scott '10], [Feldmann, Mannel, Schwertfeger '15], [Chiu, Kuo '15], [Bednyakov '18], [Wang, Yu, Zhou '21], . . .

However, no quantitative basis invariant analysis of the flavor puzzle exist.

$\curvearrowright$ This allows an entirely new perspective on the flavor puzzle!

Why hasn't it been done? Technically challenging:

How to construct BI's? **When** to stop?

general answers and technique based on example of 2HDM [AT '18]

Outline

- Motivation

Disclaimer: I will focus entirely on the quark sector here.

for leptons see talk by Davidson

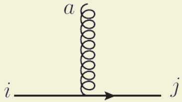
- Standard Model quark sector **flavor covariants**
- Construction of the **complete ring** of quark sector *orthogonal* **basis invariants**
- Determine the basis invariants from experimental data

$\Rightarrow$ An entirely basis invariant picture of the quark flavor puzzle.

- CP transformation of invariants & comments

SM Quark Sector Flavor Invariants – Systematic Construction

Birdtrack diagrams / "Colorflow" / ... $\mathrm{SU}(N)$ tensors

$(t^a)^i_{\ j} \;=$ 

$$\left[t^a, t^b\right] \;=\; \mathrm{i} f^{abc} t^c \,, \qquad \mathrm{Tr}\left(t^a\, t^b\right) \;=\; T_{\boldsymbol{r}}\,\delta^{ab}$$

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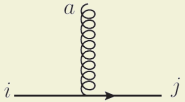

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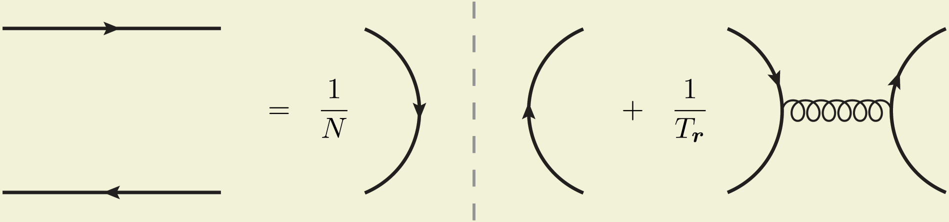
$$\boldsymbol{N} \otimes \overline{\boldsymbol{N}} \;=\; \boldsymbol{1} \;\oplus\; \boldsymbol{adj}$$

$$\delta^i_n \delta^j_m \;=\; \frac{1}{N}\, \delta^i_m \delta^j_n \;+\; \frac{1}{T_{\boldsymbol{r}}}\, (t^a)^i_{\ m}\, (t^a)^j_{\ n}$$



References for introduction to Birdtracks: [Cvitanovic Book ’08, Keppeler and Sjödahl ’13, Keppeler ’17]

Standard Model Quark Sector Flavor **Covariants**

$$-\mathcal{L}_{\text{Yuk.}} = \overline{Q}_{\text{L}}\,\widetilde{H}\,\boldsymbol{Y_u}\,u_{\text{R}} + \overline{Q}_{\text{L}}\,H\,\boldsymbol{Y_d}\,d_{\text{R}} + \text{h.c.}\,,$$

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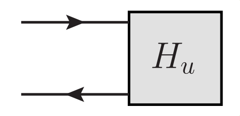
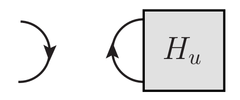
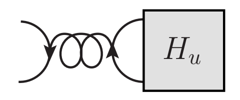


$= \dfrac{1}{N}$



$+ \dfrac{1}{T_{\boldsymbol{r}}}$



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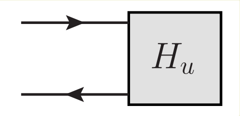

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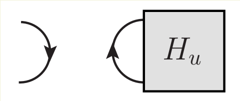

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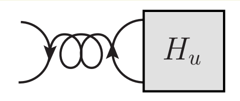

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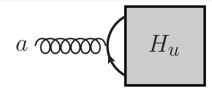
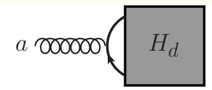
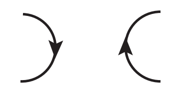


$\boldsymbol{d}^a = \mathrm{Tr}\left[t^a H_d\right] =$


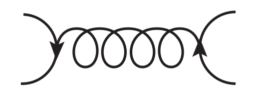
Orthogonal Covariant Projection Operators

What does orthogonal mean here?

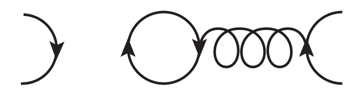
Orthogonality on the level of **projection operators**!



$P_{(\mathbf{1})}$



$P_{(\mathbf{8})}$



$P_{(\mathbf{1})} \cdot P_{(\mathbf{8})} = 0 \quad (\propto \mathrm{Tr}\, t^a)$.

Projection operators: $P_i^2 = P_i$, $\quad \mathrm{Tr}\, P_i = \dim(\boldsymbol{r}_i)$,

Orthogonality: $P_i \cdot P_j = 0$.

Using orthogonal **singlet** projectors, we find invariants that are orthogonal to each other!

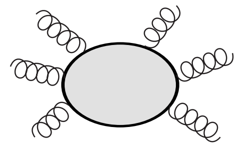
What is necessary to construct **Basis Invariants**

$$\mathbf{8}_u \otimes \mathbf{8}_u \otimes \ldots \mathbf{8}_d \otimes \mathbf{8}_d \otimes \cdots = \mathbf{8}_u^{\otimes k} \otimes \mathbf{8}_d^{\otimes \ell} = \sum_{\oplus} \boldsymbol{r}_i$$

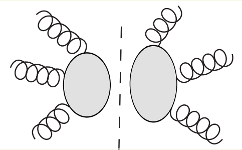
Singlet projection operators:

$$\mathbf{8}_u^{\otimes k} \otimes \mathbf{8}_d^{\otimes \ell} \supset \mathbf{1}_{(1)} \oplus \mathbf{1}_{(2)} \oplus \ldots$$

Singlet projection operators are characterized by ***factorization***. For example:



$$\mathbf{8}^{\otimes 3} \to \mathbf{8}^{\otimes 3}$$



$$\Leftrightarrow \quad \mathbf{8}^{\otimes 3} \supset \mathbf{1}$$

How many ***independent*** singlets exist? (here: in contractions $\mathbf{8}_u^{\otimes k} \otimes \mathbf{8}_d^{\otimes \ell}$ **for all** k, ℓ)

Jargon of invariant theory

- *Algebraic* **(in-)dependence:**

  Invariants $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots$ are ***algebraically dependent*** if and only if

$$\exists \quad \mathrm{Polynomial}\,(\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \ldots) = 0\ .$$

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  A maximal set of algebraically independent invariants.

of primary invariants = # of physical parameters.

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- ***Generating set*** of invariants $\equiv$ all ***primary* + *secondary*** invariants.

  $\Rightarrow$ *All* invariants can be written as a polynomial in the ***generating set*** of invariants.

  $$\mathcal{I} = \mathrm{Polynomial}\,(\mathcal{I}_1, \mathcal{I}_2, \ldots)\ .$$

Number and structure of invariants

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- **How to find their structure in terms of covariants?**

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[Noether 1916; Getzler & Kapranov '94]

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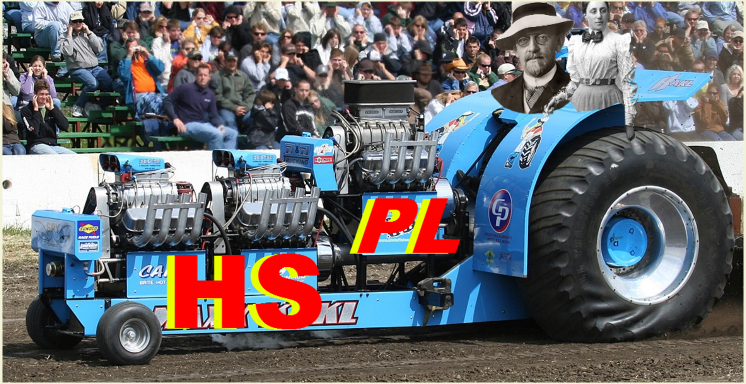

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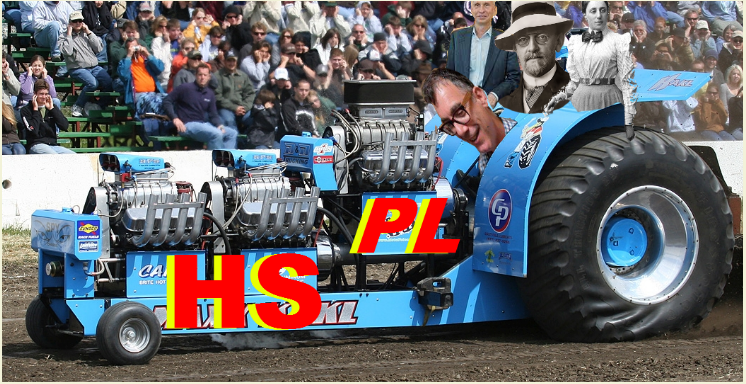

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↷ HS/PL **output**: [Jenkins & Manohar '09]

- # of primary invariants and their sub-structure (covariant content):

linear	$(\boldsymbol{u})$	$(\boldsymbol{d})$		
quadratic	$\boldsymbol{u}^2$	$\boldsymbol{d}^2$	$\boldsymbol{ud}$	
cubic	$\boldsymbol{u}^3$	$\boldsymbol{d}^3$	$\boldsymbol{u}^2\boldsymbol{d}$	$\boldsymbol{ud}^2$
quartic	$\boldsymbol{u}^2\boldsymbol{d}^2$			

(10 primary invariants $\widehat{=}$ 10 physical parameters).

- 1 secondary invariant of structure: $\boldsymbol{u}^3\boldsymbol{d}^3$. (Jarlskog invariant)
- Relation (***Syzygy***) of order $\boldsymbol{u}^6\boldsymbol{d}^6$ between primaries and the secondary.

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For this we use ***orthogonal projection operators***. (here in adjoint space of $\mathrm{SU}(3)_{Q_\mathrm{L}}$)

[AT '18]

Those can be constructed via ***birdtrack*** diagrams

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$\delta^{ab} =$  .

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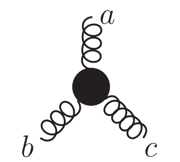
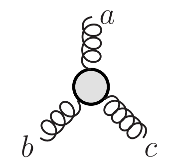
- $\mathbf{8}^{\otimes 3} \to \mathbf{1}$



$= \mathrm{i}\, f^{abc}$ and



$= d^{abc}$.

$$f^{abc} = \frac{1}{\mathrm{i}\, T_r} \mathrm{Tr}\left(\left[t^a, t^b\right] t^c\right) \qquad d^{abc} = \frac{1}{T_r} \mathrm{Tr}\left(\left\{t^a, t^b\right\} t^c\right)$$

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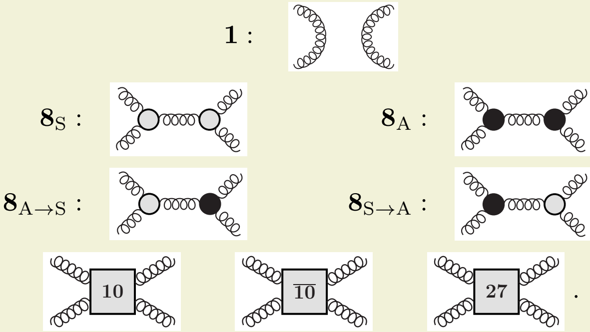

Can understand the different contraction channels from

$$\mathbf{8}^{\otimes 2} = \mathbf{1} \oplus \mathbf{8}_\mathrm{S} \oplus \mathbf{8}_\mathrm{A} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}\,.$$

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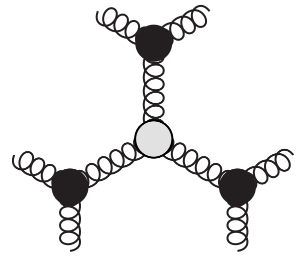
- $\mathbf{8}^{\otimes 4} \to \mathbf{1}$
- $\mathbf{8}^{\otimes 6} \to \mathbf{1}$

many operators exist in $\mathbf{8}^{\otimes 6} \to \mathbf{1}$, we only need one:



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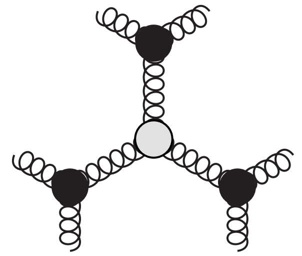
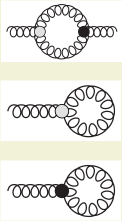




$= 0$

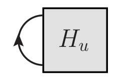
$= 0$

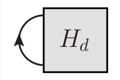
$= 0$

All of these operators are **orthogonal** to each other.
We now use them to construct the orthogonal invariants.

Orthogonal Invariants

The 10 *algebraically independent* and orthogonal invariants are given by: $I_{\text{#u's,#d's}}$

$I_{10} \propto$ 

and $I_{01} \propto$ 

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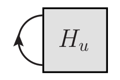
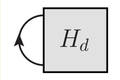
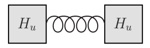
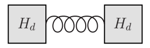
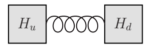
$I_{10} \propto$ 

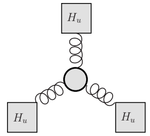
and $I_{01} \propto$ 

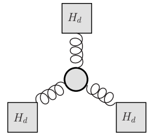
.

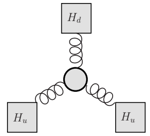
$I_{20} \propto$ 

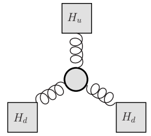
$I_{02} \propto$ 

$I_{11} \propto$ 

$I_{30} \propto$ 

$I_{03} \propto$ 

$I_{21} \propto$ 

$I_{12} \propto$ 

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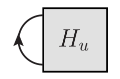
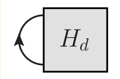
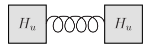
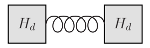
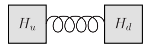
$I_{10} \propto$ 

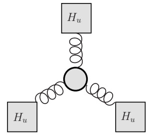
and $I_{01} \propto$ 

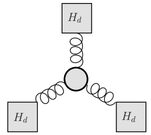
.

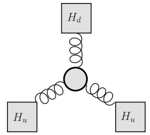
$I_{20} \propto$ 

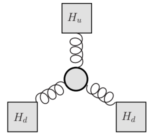
$I_{02} \propto$ 

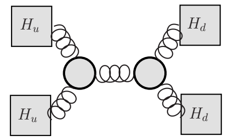
$I_{11} \propto$ 

$I_{30} \propto$ 

$I_{03} \propto$ 

$I_{21} \propto$ 

$I_{12} \propto$ 

$I_{22} \propto$ 

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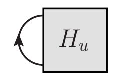
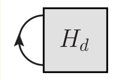
$I_{10} \propto$ 

and $I_{01} \propto$ 

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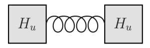
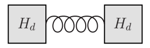
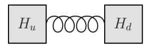
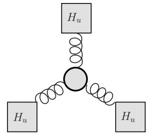
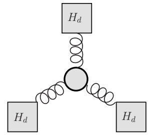
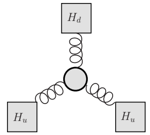
$I_{20} \propto$ 

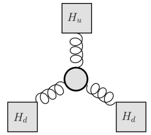
$I_{02} \propto$ 

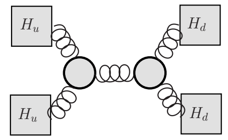
$I_{11} \propto$ 

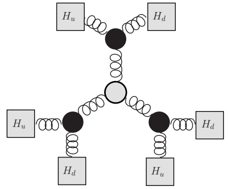
$I_{30} \propto$ 

$I_{03} \propto$ 

$I_{21} \propto$ 

$I_{12} \propto$ 

$I_{22} \propto$ 

Secondary invariant: $J_{33} \propto$ 

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The 10 algebraically independent and orthogonal invariants are given by:

$$I_{10} := \mathrm{Tr}\,\widetilde{H}_u \qquad \text{and} \qquad I_{01} := \mathrm{Tr}\,\widetilde{H}_d\;.$$

$$I_{20} := \mathrm{Tr}(H_u^2)\,,\quad I_{02} := \mathrm{Tr}(H_d^2)\,,\quad I_{11} := \mathrm{Tr}(H_u H_d)\,,$$

$$I_{30} := \mathrm{Tr}(H_u^3)\,,\quad I_{03} := \mathrm{Tr}(H_d^3)\,,\quad I_{21} := \mathrm{Tr}(H_u^2 H_d)\,,\quad I_{12} := \mathrm{Tr}(H_u H_d^2)\,,$$

$$I_{22} := 3\,\mathrm{Tr}(H_u^2 H_d^2) - \mathrm{Tr}(H_u^2)\,\mathrm{Tr}(H_d^2)\,.$$

Secondary invariant: exactly the Jarlskog invariant,

$$J_{33} := \mathrm{Tr}(H_u^2 H_d^2 H_u H_d) - \mathrm{Tr}(H_d^2 H_u^2 H_d H_u) \equiv \frac{1}{3}\,\mathrm{Tr}\,[H_u, H_d]^3\;.$$

Note: Here $\widetilde{H}_u \equiv Y_u Y_u^\dagger$, $\widetilde{H}_d \equiv Y_d Y_d^\dagger$, and $H_{u,d} \equiv \widetilde{H}_{u,d} - \mathbb{1}\,\mathrm{Tr}\frac{\widetilde{H}_{u,d}}{3}$. **“Traces of traceless matrices”**

The Syzygy

With our orthogonal invariants, the syzygy is given by

$$
\begin{aligned}
(J_{33})^2 = &- \frac{4}{27} I_{22}^3 + \frac{1}{9} I_{22}^2 I_{11}^2 + \frac{1}{9} I_{22}^2 I_{02} I_{20} + \frac{2}{3} I_{22} I_{30} I_{03} I_{11} - \frac{2}{3} I_{22} I_{21} I_{12} I_{11} - \frac{1}{9} I_{22} I_{11}^2 I_{20} I_{02} \\
&+ \frac{2}{3} I_{22} I_{21}^2 I_{02} + \frac{2}{3} I_{22} I_{12}^2 I_{20} - \frac{2}{3} I_{22} I_{30} I_{12} I_{02} - \frac{2}{3} I_{22} I_{03} I_{21} I_{20} \\
&- \frac{1}{3} I_{30}^2 I_{03}^2 + I_{21}^2 I_{12}^2 + 2 I_{30} I_{03} I_{21} I_{12} - \frac{4}{9} I_{30} I_{03} I_{11}^3 \\
&+ \frac{1}{18} I_{30}^2 I_{02}^3 + \frac{1}{18} I_{03}^2 I_{20}^3 - \frac{4}{3} I_{30} I_{12}^2 - \frac{4}{3} I_{03} I_{21}^2 \\
&- \frac{1}{3} I_{30} I_{21} I_{11} I_{02}^2 - \frac{1}{3} I_{03} I_{12} I_{11} I_{20}^2 + \frac{2}{3} I_{30} I_{12} I_{11}^2 I_{02} + \frac{2}{3} I_{03} I_{21} I_{11}^2 I_{20} \\
&- \frac{2}{3} I_{21} I_{12} I_{20} I_{02} I_{11} - \frac{1}{108} I_{20}^3 I_{02}^3 + \frac{1}{36} I_{20}^2 I_{02}^2 I_{11}^2 + \frac{1}{6} I_{21}^2 I_{20} I_{02}^2 + \frac{1}{6} I_{12}^2 I_{02} I_{20}^2 \ .
\end{aligned}
$$

This is the **shortest relation ever** expressed for the SM quark flavor ring and has 27 terms.

(this should be compared to result of [Jenkins&Manohar'09] with 241 terms using non-orthogonal invariants).

SM Quark Sector Flavor Invariants – Quantitative Analysis

Measuring the Invariants

In order to evaluate the invariants, one can use *any* parametrization. We use PDG:

$$\widetilde{H}_u = \mathrm{diag}(\, y_u^2\,,\, y_c^2\,,\, y_t^2\,)$$

and
$$\widetilde{H}_d = V_{\mathrm{CKM}}\, \mathrm{diag}(\, y_d^2\,,\, y_s^2\,,\, y_b^2\,)\, V_{\mathrm{CKM}}^\dagger\,,$$

1. **Explore the *possible* parameter space**: scan $\mathcal{O}(10^7)$ uniform random points
   - $s_{12}, s_{13}, s_{23} \in [-1, 1]$ and $\delta \in [-\pi, \pi]$ together with:

A) Linear measure: $y_{u,c} \in [0, 1]y_t$, $y_{d,s} \in [0, 1]y_b$.

B) Log measure: $(m_{u,c}/\mathrm{MeV}) \in 10^{[-1,\log(m_t/\mathrm{MeV})]}$, $(m_{d,s}/\mathrm{MeV}) \in 10^{[-1,\log(m_b/\mathrm{MeV})]}$.

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2. **"Measure" the parameter point realized in Nature.**

We use PDG data and errors and evaluate at the EW scale $\mu = M_Z$. see e.g. [Huang, Zhou '21]

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For convenience of the presentation we normalize the invariants as

$$\hat{I}_{ij} := \frac{I_{ij}}{\left(y_t^2\right)^i \left(y_b^2\right)^j} \,.$$

Experimental values of the invariants

Invariant	best fit and error	Normalized invariant	best fit and error
I_{10}	$0.9340(83)$	$\hat{I}_{10}$	$1.00001358(^{+85}_{-88})$
I_{01}	$2.660(49) \times 10^{-4}$	$\hat{I}_{01}$	$1.000351(^{+63}_{-71})$
I_{20}	$0.582(10)$	$\hat{I}_{20}$	$0.66665761(^{+59}_{-57})$
I_{02}	$4.71(17) \times 10^{-8}$	$\hat{I}_{02}$	$0.666432(^{+47}_{-42})$
I_{11}	$1.651(45) \times 10^{-4}$	$\hat{I}_{11}$	$0.664783(^{+91}_{-87})$
I_{30}	$0.1811(48)$	$\hat{I}_{30}$	$0.22221769(^{+29}_{-28})$
I_{03}	$4.18(23) \times 10^{-12}$	$\hat{I}_{03}$	$0.222105(^{+24}_{-21})$
I_{21}	$5.14(^{+18}_{-19}) \times 10^{-5}$	$\hat{I}_{21}$	$0.221593(^{+30}_{-29})$
I_{12}	$1.463(^{+65}_{-68}) \times 10^{-8}$	$\hat{I}_{12}$	$0.221555(^{+38}_{-36})$
I_{22}	$1.366(^{+73}_{-76}) \times 10^{-8}$	$\hat{I}_{22}$	$0.221554(^{+38}_{-36})$
J_{33}	$4.47(^{+1.23}_{-1.58}) \times 10^{-24}$	$\hat{J}_{33}$	$2.92(^{+0.74}_{-0.93}) \times 10^{-13}$
J	$3.08(^{+0.16}_{-0.19}) \times 10^{-5}$		

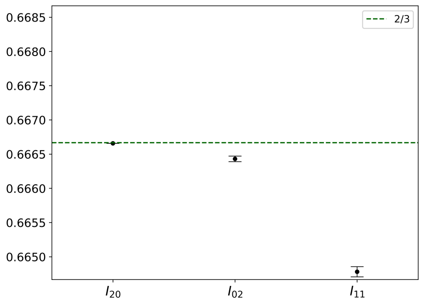
Table: Experimental values of the quark sector basis invariants evaluated using PDG data. Uncertainties are 1σ. Left: orthogonal invariants at face value. Right: the same invariants normalized to the largest Yukawa couplings.

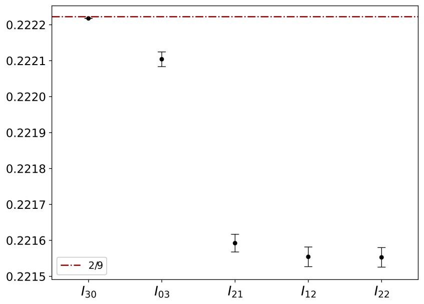
Experimental values of the Invariants

$$\hat{I}_{10} \approx \hat{I}_{01} \approx 1, \qquad \hat{I}_{11} \approx \hat{I}_{20} \approx \hat{I}_{02} \approx \frac{2}{3},$$

$$\hat{I}_{30} \approx \hat{I}_{03} \approx \hat{I}_{21} \approx \hat{I}_{12} \approx \hat{I}_{22} \approx \frac{2}{9}.$$

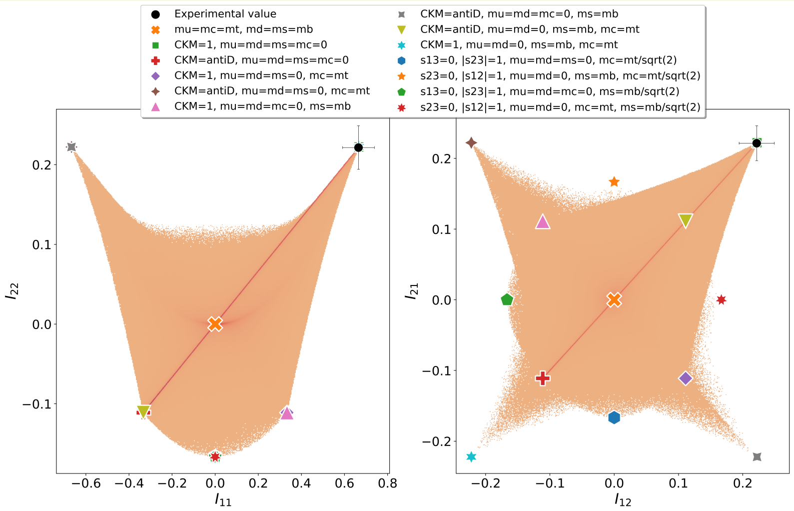
$$\left(\hat{I}_{ij} := \frac{I_{ij}}{(y_t^2)^i \, (y_b^2)^j} \,. \right)$$





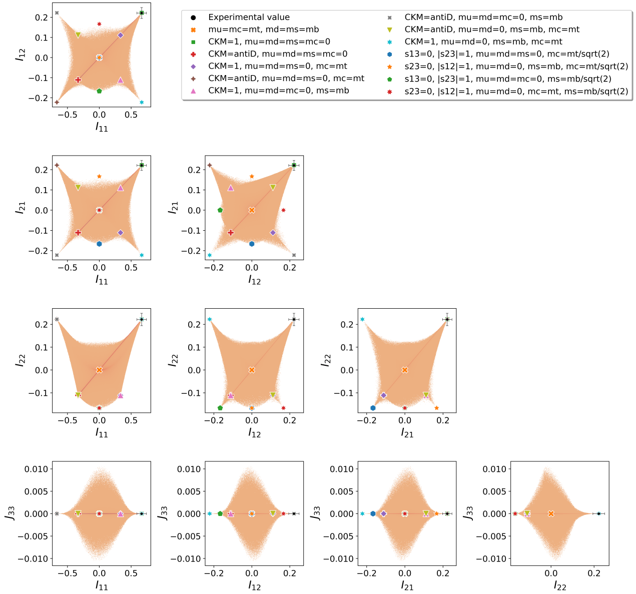
- Deviations from maximal possible values are significant.
- Deviations from each other, e.g. $\hat{I}_{21} - \hat{I}_{12} \neq 0$ and $\hat{I}_{12} - \hat{I}_{22} \neq 0$, are significant.

Parameter space and experimental values



Error bars: $1\sigma \times 1000$

Parameter space and experimental values



Results and empirics

- Observed primary invariants are *very close to* maximal – with small but significant deviations.
- Small deviations from max. correspond to 1./2. gen. masses and mixings.
- Explaining the value of the invariants and their misalignment from maximal point amounts to solving the flavor puzzle in the language of invariants.

Results and empirics

- Observed primary invariants are *very close to* maximal – with small but significant deviations.
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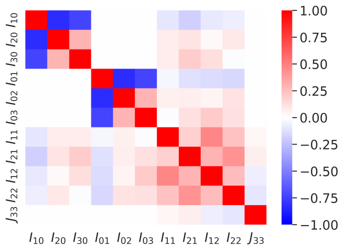
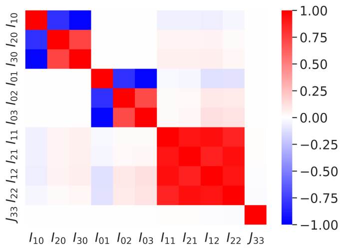
- Explaining the value of the invariants and their misalignment from maximal point amounts to solving the flavor puzzle in the language of invariants.
- The invariants are ***strongly correlated*** (for the observed hierarchical parameters).

linear scan:



log scan:



This is **not** true for anarchical parameters, or points with increased symmetry.

RGE running of invariants

$$\mathcal{D} := 16\pi^2 \mu \frac{\mathrm{d}}{\mathrm{d}\mu} ,$$

$$a_\Delta := -8\, g_s^2 - \frac{9}{4} g^2 - \frac{17}{12} g'^2 ,$$

$$a_\Gamma := -8\, g_s^2 - \frac{9}{4} g^2 - \frac{5}{12} g'^2 ,$$

$$a_\Pi := -\frac{9}{4} g^2 - \frac{15}{4} g'^2 ,$$

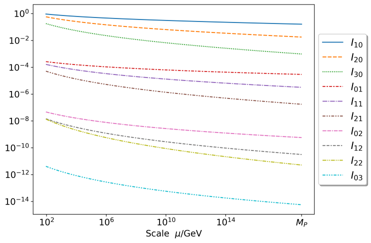
$$t_{udl} := 3\,\mathrm{Tr}\tilde{H}_u + 3\,\mathrm{Tr}\tilde{H}_d + \mathrm{Tr}\tilde{H}_\ell .$$

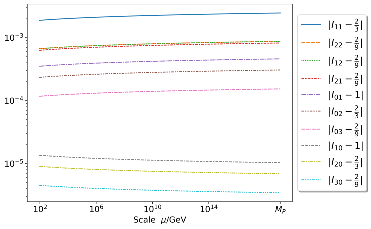
$$\mathcal{D}\tilde{H}_u = 2\,(a_\Delta + t_{udl})\ \tilde{H}_u + 3\,\tilde{H}_u^2 - \frac{3}{2}\left(\tilde{H}_d\tilde{H}_u + \tilde{H}_u\tilde{H}_d\right) ,$$

$$\mathcal{D}\tilde{H}_d = 2\,(a_\Gamma + t_{udl})\ \tilde{H}_d + 3\,\tilde{H}_d^2 - \frac{3}{2}\left(\tilde{H}_d\tilde{H}_u + \tilde{H}_u\tilde{H}_d\right) ,$$

$$\mathcal{D}\tilde{H}_\ell = 2\,(a_\Pi + t_{udl})\ \tilde{H}_\ell + 3\,\tilde{H}_\ell^2 ,$$

$$\mathcal{D}g_s = -7\, g_s^3 , \qquad \mathcal{D}g = -\frac{19}{6} g^3 , \qquad \mathcal{D}g' = \frac{41}{6} g'^3 .$$





CP transformation of covariants and invariants

CP is trafo under $\mathrm{Out}\,(\mathrm{SU}(N)) = \mathbb{Z}_2$.
Covariants:

$$\begin{aligned} \boldsymbol{u}^a &\mapsto -R^{ab}\,\boldsymbol{u}^b\,, \\ \boldsymbol{d}^a &\mapsto -R^{ab}\,\boldsymbol{d}^b\,, \end{aligned}$$

e.g. in Gell-Mann basis for the generators:
$R = \mathrm{diag}(-1,+1,-1,-1,+1,-1,+1,-1)$.

$\mathrm{SU}(3)$ tensors (projection ops.):

$$\begin{aligned} f^{abc} &\mapsto R^{aa'}\,R^{bb'}\,R^{cc'}\,f^{a'b'c'} = f^{abc}\,, \\ d^{abc} &\mapsto R^{aa'}\,R^{bb'}\,R^{cc'}\,d^{a'b'c'} = -d^{abc}\,. \end{aligned}$$

CP trafo of invariants is easily read-off from birdtrack projection operator:

Invariants are **CP even** / **CP odd** iff their projection operator contains and **even** / **odd # of** f **tensors**.

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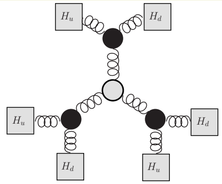
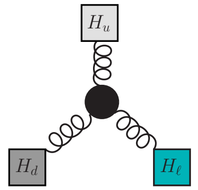
$\Rightarrow$ Only CP-odd in SM: $J_{33} \propto$



BSM: CPV at order 3 ?
$\mathrm{i}f^{abc}\,\mathrm{Tr}[t^a\,H_u]\,\mathrm{Tr}[t^b\,H_d]\,\mathrm{Tr}[t^c\,\boldsymbol{H_\ell}]$



CP trafo of invariants is easily read-off from birdtrack projection operator:

> Invariants are **CP even** / **CP odd** iff their projection operator contains and **even** / **odd # of f tensors**.

Comments

- $I_{01}, I_{02}, I_{03}, I_{10}, I_{20}, I_{30}$ correspond to masses.
- CP-even $I_{11}, I_{21}, I_{12}, I_{22}$ correspond to mixings.
- CPV requires interplay of 8 CP-even primary invariants (all besides the “trivial” invariants I_{10}, I_{01}).
- Non-trivial $\hat{I}_{ij}$'s being close to maximal forces the Jarlskog invariant to be ***small***.
- Any reduction of # of parameters corresponds to relation between invariants.
- **All** flavor observables can be expressed as

$$\mathcal{O}_{\text{flavor}} = \text{Polynomial}_1(I_{ij}) + J_{33} \times \text{Polynomial}_2(I_{ij})\,.$$

  This is guaranteed since our primary and secondary invariants form a “Hironaka decomposition” of the ring.

- Our invariants provide easy targets for fits of any BSM model to SM flavor structure.
- Our procedure is *completely general*, can be applied to all BSM scenarios.

Outlook

- Ambiguity in choice of I_{22} needs to be clarified. Contributions to different contraction channels could be very relevant to decipher flavor puzzle.
- Relative alignments of 8-plet covariants are in 1:1 relation with invariant relations.
  see other examples [Merle, Zwicky '12], [Bento, Boto, Silva, AT '20]
- Maximization and strong correlation of invariants could point to possible **information theoretic** argument to set parameters! $\rightarrow$ should be done.
  see e.g. [Bousso, Harnik, Kribs, Perez '07], [Beane, Kaplan, Klco, Savage '19], [Carena, Low, Wagner, Xiao '23]
- Extension to lepton sector with **orthogonal** invariants $\rightarrow$ should be done.
  for HS/PL and non-orthogonal invariants see [Hanany, Jenkins, Manhoar, Torri '10], [Wang, Yu, Zhou '21], [Yu, Zhou '21].
- Using orthogonal BIs in $\mathrm{SU}(3)_{Q_\mathrm{L}}$ fundamental space $\rightarrow$ should be done.
- RGE's directly in terms of invariants $\rightarrow$ should be done.
- Investigation of $u \leftrightarrow d$ custodial flavor symmetry $\rightarrow$ should be done.
- General relation of BI's to observables $\rightarrow$ should be done.

Conclusion

- We have for the first time obtained a quantitative analysis of the flavor puzzle exclusively in terms of basis invariants.
- This uncovers an entirely new angle on the flavor puzzle.
- The (quark) flavor puzzle in invariants amounts to explaining:
  - **Why** are the invariants very close to maximal?
  - **What** explains their tiny deviations from the maximal values?
  - **Why** are the (*orthogonal, a priori independent*) invariants so strongly correlated?
- **Any** explanation of the flavor structure will have to answer these questions.

This is just the beginning of an entirely new exploration of the flavor puzzle.

Thank You!

Backup slides

General Procedure / Algorithm

for the construction of basis invariants.

Three steps:

1. Construction of *basis covariant* objects: “building blocks”.
   - Determine CP transformation behavior of the building blocks.
2. Derive Hilbert series & Plethystic logarithm.
   - $\Rightarrow$ # and order of primary invariants.
   - $\Rightarrow$ # and structure of generating set of invariants.
   - $\Rightarrow$ interrelations between invariants ($\equiv$ syzygies).
3. Construct all invariants and interrelations explicitly.

Application here:
Characterize SM flavor sector invariants.

Hilbert Series and Plethystic Logarithm

Covariant building blocks as **input** for the ring:

$$\mathbf{8}_u \;\widehat{=}\; u\,, \quad \mathbf{8}_d \;\widehat{=}\; d.$$

From input, compute Hilbert series (HS) and Plethystic logarithm (PL):
introduced in math: [Getzler, Kapranov '94], physics [Benvenuti, Feng, Hanany, He '06]

$$\mathfrak{H}(u,d) = \int_{\mathrm{SU}(3)} d\mu_{\mathrm{SU}(3)}\, \mathrm{PE}\left[z_1, z_2; u; \mathbf{8}\right] \mathrm{PE}\left[z_1, z_2; d; \mathbf{8}\right]\,,$$

$$\mathrm{PL}\left[\mathfrak{H}\left(u,d\right)\right] := \sum_{k=1}^{\infty} \frac{\mu(k)\, \ln \mathfrak{H}\left(u^k, d^k\right)}{k}\,.$$

$$\mathfrak{H}(u,d) = \frac{1+u^3 d^3}{(1-u^2)(1-d^2)(1-ud)(1-u^3)(1-d^3)(1-ud^2)(1-u^2d)(1-u^2d^2)}\,.$$

$$\mathrm{PL}\left[\mathfrak{H}(u,d)\right] = u^2 + ud + d^2 + u^3 + d^3 + u^2 d + ud^2 + u^2 d^2 + u^3 d^3 - u^6 d^6\,.$$

Möbius function $\mu(n) = \begin{cases} \overset{+}{(-)}1, & \text{if } n \text{ is square free with even(odd) \# number of prime factors,} \\ 0, & \text{else.} \end{cases}$

CKM in PDG parametrization

$V_{\mathrm{CKM}} := V_{u,\mathrm{L}}^{\dagger} V_{d,\mathrm{L}}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix. In PDG parametrization

$$V_{\mathrm{CKM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

Explicit expressions for Invariants in physical basis

In "physical parameters" of SM the normalized invariants can be approximated using the (empirically observed) parametric hierarchies $y_t \gg y_{c,u}$, $y_b \gg y_{s,d}$ and $\lambda \ll 1$,

$$\hat{I}_{20} = \frac{2}{3} - 2\frac{y_c^2 + y_u^2}{y_t^2} + \text{h.o.}\,,$$

$$\hat{I}_{30} = \frac{2}{9} - \frac{y_c^2 + y_u^2}{y_t^2} + \text{h.o.}\,,$$

$$\hat{I}_{11} = \frac{2}{3} - A^2\lambda^4 - \frac{y_c^2 + y_u^2}{y_t^2} - \frac{y_s^2 + y_d^2}{y_b^2} + \text{h.o.}\,,$$

$$3\,\hat{I}_{21} = \frac{2}{3} - A^2\lambda^4 - 2\frac{y_c^2 + y_u^2}{y_t^2} - \frac{y_s^2 + y_d^2}{y_b^2} + \text{h.o.}\,,$$

$$3\,\hat{I}_{12} = \frac{2}{3} - A^2\lambda^4 - \frac{y_c^2 + y_u^2}{y_t^2} - 2\frac{y_s^2 + y_d^2}{y_b^2} + \text{h.o.}\,,$$

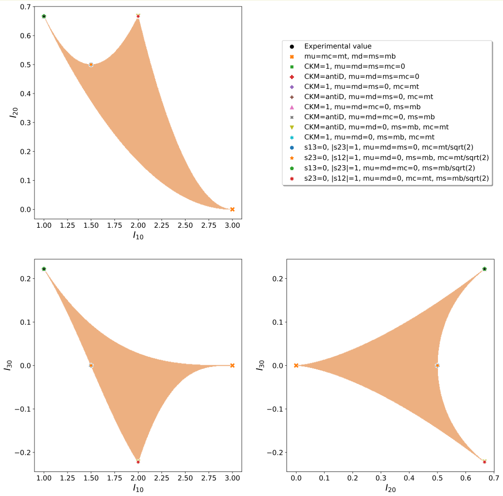
$$3\,\hat{I}_{22} = \frac{2}{3} - A^2\lambda^4 - 2\frac{y_c^2 + y_u^2}{y_t^2} - 2\frac{y_s^2 + y_d^2}{y_b^2} + \text{h.o.}\,.$$

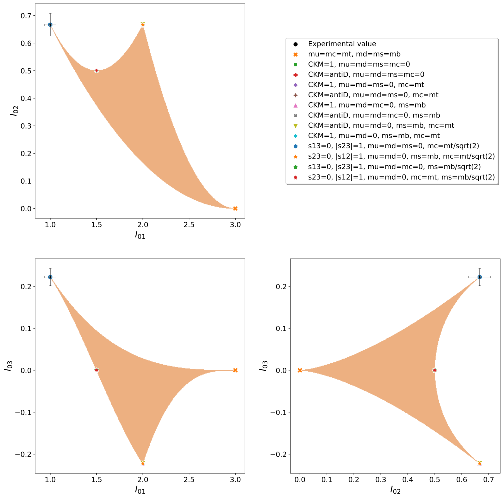
$$\hat{I}_{02} = \frac{2}{3} - 2\frac{y_s^2 + y_d^2}{y_b^2} + \text{h.o.}\,,$$

$$\hat{I}_{03} = \frac{2}{9} - \frac{y_s^2 + y_d^2}{y_b^2} + \text{h.o.}\,,$$

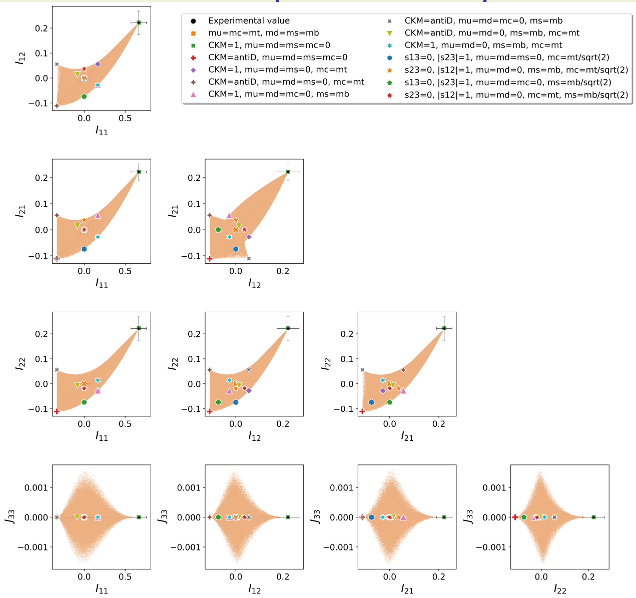
h.o. here refers to higher order corrections in λ or higher powers of the Yukawa coupling ratios. This shows that the values $2/3$ and $2/9$'ths become exact in the limit of zero mixing and zero 1st and 2nd-generation fermion masses.

Correlation of “mass” invariants $I_{10}, I_{20}, I_{30}, I_{01}, I_{02}, I_{03}$





Parameter space and experimental values

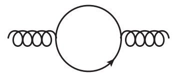


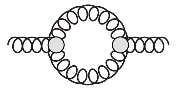
Arguably even "more basis invariant" alternative choice of normalization:

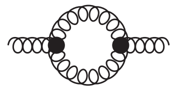
$$\hat{I}_{ij}^{\text{alt}} := \frac{I_{ij}}{I_{10}^{i}\, I_{01}^{j}} \, .$$

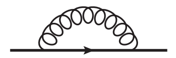
Birdtrack Identities

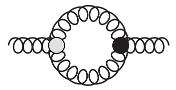
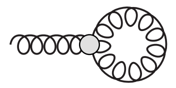
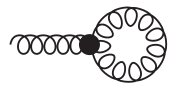
We mostly use the conventions of [Keppeler '17] with the following identities

 $= T_{\boldsymbol{r}}$  with $T_{\boldsymbol{r}}\delta^{ab} = \mathrm{Tr}[t^a t^b]\;,$

 $= C_D$  with $C_D = \dfrac{N^2-4}{N}\;,$

 $= C_A$  with $C_A = 2T_{\boldsymbol{r}}N\;.$

 $= C_F$  with $C_F = T_{\boldsymbol{r}}\dfrac{N^2-1}{N}\;,$

 $=$  $=$  $= 0$