Black Hole Quadratic Quasi-Normal Modes

Their Frequencies and Amplitudes

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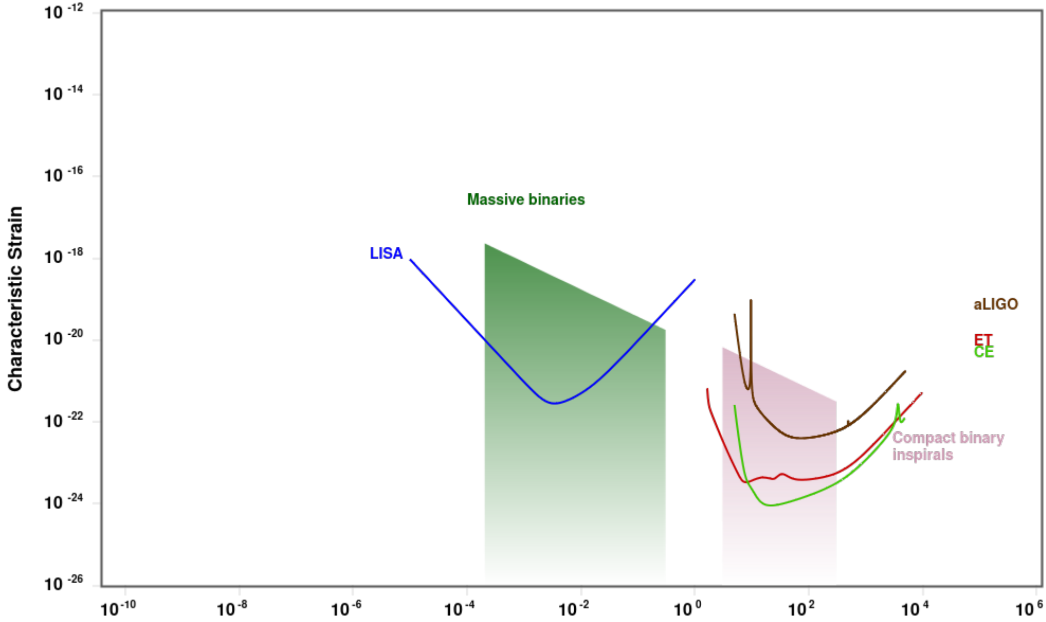
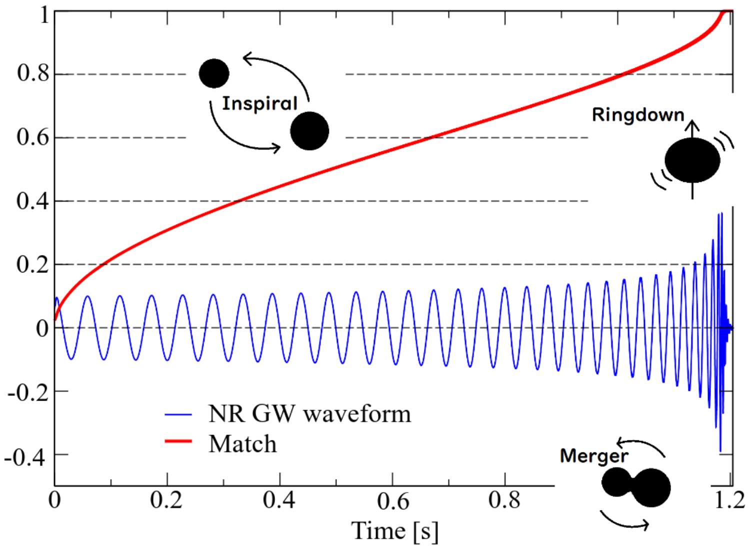

Characteristic Strain
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
Massive binaries
LISA
aLIGO
ET
CE
Compact binary inspirals
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
10^{6}

Inspire-Merger-Ringdown

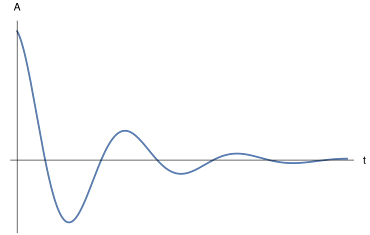


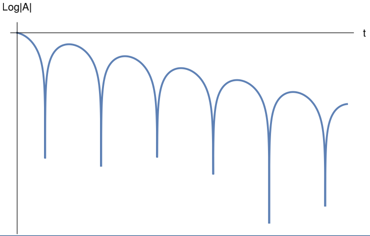
picture from Isoyama, Sturani, Nakano '20

Quasi-Normal Modes

at Linear Order

ℓ	n	Frequency
2	0	$0.37367 + 0.08896i$
	1	$0.34671 + 0.27391i$
	2	$0.30105 + 0.47828i$
	3	$0.25150 + 0.70515i$
3	0	$0.59944 + 0.09270i$
	1	$0.58264 + 0.28130i$
	2	$0.55168 + 0.47909i$
	3	$0.51196 + 0.69034i$
4	0	$0.80918 + 0.09416i$
	1	$0.79663 + 0.28433i$
	2	$0.77271 + 0.47991i$
	3	$0.73984 + 0.68392i$

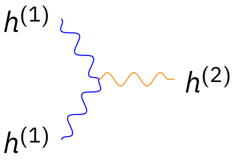


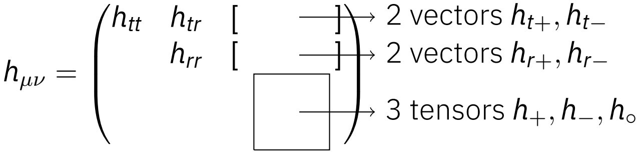


Black Hole Perturbation Theory

Metric Ansatz and Einstein Equations

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu}$$





Einstein Equations in vacuum

$$G^{(1)}_{\mu\nu}\left[\epsilon\, h^{(1)}\right] = 0$$

In Regge-Wheeler gauge

$$h_{t+} = h_{r+} = h_{+} = h_{-} = 0$$

$$G^{(1)}_{\mu\nu}\left[\epsilon^2\, h^{(2)}\right] = -G^{(2)}_{\mu\nu}\left[\epsilon\, h^{(1)}, \epsilon\, h^{(1)}\right] \equiv \epsilon^2 S_{\mu\nu}\left[h^{(1)}, h^{(1)}\right]$$

Can take eigenstates of angular momentum, frequency, and parity.

Selection Rules

for Quadratic Order Modes

Before doing any computation, let's exploit symmetry.

Couple two linear modes of given

frequencies $\omega_{1,2}$, angular momenta $(\ell_{1,2}, m_{1,2})$, parity $P_{1,2} = 0, 1$

- $\cos\omega_1 t\, e^{-\gamma_1 t} \cos\omega_2 t\, e^{-\gamma_2 t} \propto (\cos(\omega_1 + \omega_2)t + \cos(\omega_1 - \omega_2)t)e^{-(\gamma_1+\gamma_2)t}$
- $\ell = |\ell_1 - \ell_2|, \ldots, \ell_1 + \ell_2; \quad m_1 + m_2 = m$
- $(-1)^{\ell_1+P_1}(-1)^{\ell_2+P_2} = (-1)^{\ell+P}$

But what are the **amplitudes** of the quadratic modes?

Black Hole PT

Master Scalars, Linear Order

The two physical d.o.f. of the graviton are captured by **master scalars**

$$\psi_+ = \frac{2r}{\lambda_1^2}\left[r^{-2}\tilde{h}_\circ + \frac{2}{\Lambda(r)}\left(f^2\tilde{h}_{rr} - rf(r^{-2}\tilde{h}_\circ)'\right)\right]$$
$$\psi_- = \frac{2r}{\mu^2}\left[\partial_r\tilde{h}_{t-} + \frac{M}{r^2 f(r)}(\tilde{h}_{t-} - \tilde{h}_{r-}) - \partial_t\tilde{h}_{r-} - \frac{2}{r}\tilde{h}_{t-}\right]$$

which obey the Regge-Wheeler and Zerilli equations

$$\frac{\mathrm{d}\psi_\pm}{\mathrm{d}r_*^2} + \omega^2\psi_\pm - V_\pm(r)\psi_\pm = 0, \quad r_* = r + \ln(1 - 2M/r)$$

These equations are solved numerically with QNM b.c. $\psi \sim \mathcal{A}e^{\pm i\omega r_*}$

Knowing $\psi_\pm$, we can reconstruct the metric $h_{\mu\nu} \longleftrightarrow \psi_\pm$

Black Hole PT

Master Scalars, Quadratic Order

Putting together the master scalars $\psi_{\pm}^{(2)}$ using $h_{\mu\nu}^{(2)}$, they now obey

$$\frac{\mathrm{d}\psi_{\pm}^{(2)}}{\mathrm{d}r_*^2} + \omega^2\psi_{\pm}^{(2)} - V_{\pm}(r)\psi_{\pm}^{(2)} = S[\psi_{\pm}^{(1)}, \psi_{\pm}^{(1)}] \leftarrow \text{ Source term}$$

[Hui et Al. '22; Spiers,Pound,Wardell '23]

But $\psi_{\pm}^{(2)}$ diverge at large r as $\psi_{\pm}^{(2)} \sim r^2\, e^{i\omega r}$, so for them

- QNM b.c. cannot be imposed
- Cannot extract a finite amplitude $\mathcal{A}$

We expect these divergences to be due to a *poor choice* of master scalars.

[Ioka, Nakano '07; Brizuela et Al. '09]

Resolution

The Good Master Scalars

We perform a master scalar redefinition where $\Delta(r)$ is a function to be determined

$$\Psi = \psi^{(2)} + \Delta(r)\psi^{(1)}\psi^{(1)}, \quad \Delta(r) = c_2 r^2 + c_1 r$$

The RWZ equation for Ψ gets a modified source $\mathfrak{S}$

$$\frac{\mathrm{d}\Psi_\pm}{\mathrm{d}r_*^2} + \omega^2\Psi_\pm - V_\pm(r)\Psi_\pm = \mathfrak{S} = S[\psi_\pm^{(1)}, \psi_\pm^{(1)}] + \left(\frac{\mathrm{d}}{\mathrm{d}r_*^2} + \omega^2 - V_\pm(r)\right)\left(\Delta(r)\psi^{(1)}\psi^{(1)}\right)$$

and we design $\Delta(r)$ so that $\Psi_\pm \sim \mathcal{A}_\pm^{(2)} e^{i\omega r}$ at large r.

Now we can impose QNM b.c. and extract $\mathcal{A}^{(2)}$ by numerically integrating.

Physical Waveform

We can now reconstruct the metric $h^{(2)}_{\mu\nu}$.

Here is one component when Even $\times$ Odd $\to$ Odd

$$h^{(2)}_{r-} \overset{r\to\infty}{\sim} \left(-\frac{i\omega}{2} r M \mathcal{A}^{(2)}_{-} e^{i\omega r_*} + i r\omega \frac{\Delta_{+-;-}(r)\psi^{(1)}_{+}(r)\psi^{(1)}_{-}(r)}{2f(r)} \right) + \frac{2r^2}{\mu_\ell^2} S_{r-}$$

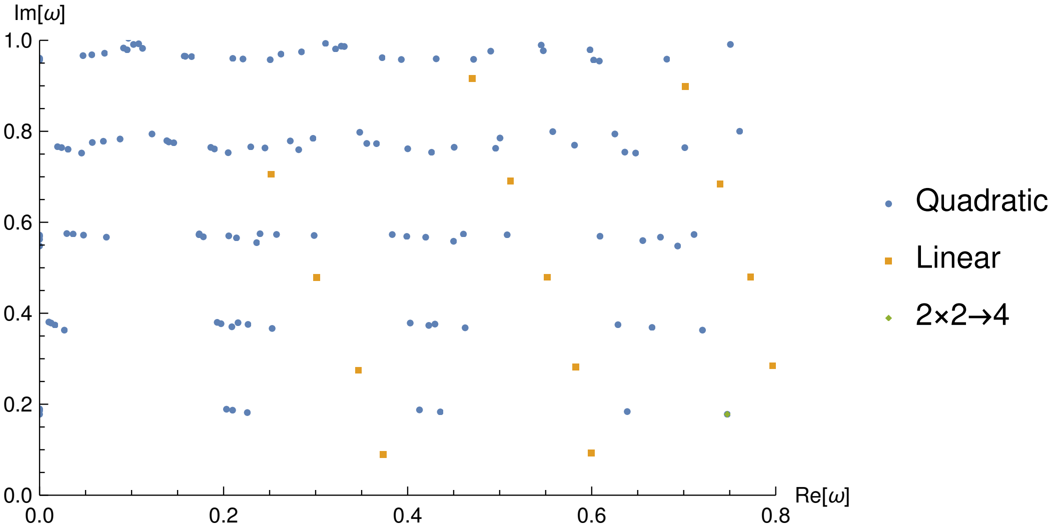
To extract the physical waveform, we go to *asymptotically flat gauge*.
For the $+, \times$ polarizations

$$\mathfrak{h}^{(2)}_{+} - i\mathfrak{h}^{(2)}_{\times} = \frac{M}{r} \sum \mathcal{R}_{\ell m \mathcal{N}} e^{-i\omega_{\ell\mathcal{N}}(r_* - t)} \; {}_{-2}Y^{\ell m}(\theta, \phi)$$

and $\mathcal{R}_{\ell m \mathcal{N}}$ is computed.

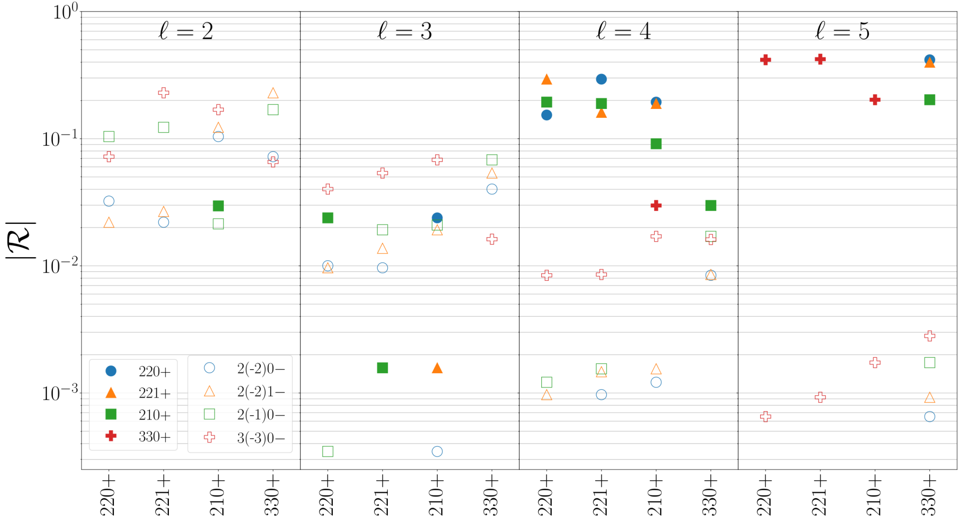
Conclusions

Quadratic Frequencies



Conclusions

Quadratic Amplitudes



$\mathcal{A}^{(1)} \sim 10\%$

$\mathcal{A}^{(2)} \sim 1\%$

(We assumed reflection symmetry in this plot)

Conclusions

- We confirmed $\mathcal{R}$ of $2,2 \times 2,2 \to 4,4$ and $2,2 \times 3,3 \to 5,5$ against existing NR simulations
- Trusting GR, we offer a more detailed model of ringdown for the same number of free parameters (the linear amplitudes)
- Detection prospects of Quadratic QNMs (see 2403.09767): ground detectors should see $\mathcal{O}(10)/y$, LISA $\mathcal{O}(10^2)/y$

Thank you