Nonlocal field theories and nonlocal gravitational models

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Nonlocal Field Theory

Nonlocality has been introduced within different frameworks

- Efimov (1970) introduced nonlocal interactions in order to obtain a finite theory of quantized fields and confinement.
- Nonlocality of fields emerges from the stochastic nature of space-time (see K. Namsrai, Nonlocal QUF and stochastic quantum mechanics, 1986).

Fundamental or emergent in gravity?

- nonlocal vertexes $e^{\Box/\Lambda^2}$ in strings (E. Witten, Nucl. Phys. B **276** (1986) 291.)
- Emergence of nonlocality at the Planck scale in non-commutative geometries, loop quantum gravity, asymptotic safety and causal sets.
- The trace anomaly induced by quantum corrections due to conformal fields, that is at the basis of the Starobinsky model, induces nonlocal terms
- Emergence of hidden nonlocality in QG and impossibility of probing the spacetime below the Planck-length due to black holes production in scattering processes (Addazi, Mod. Phys. Lett. A **35**, 2050288 (2020))

Renormalizable Stelle model

Quadratic theory is renormalizable

$$S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R + \alpha R_{\mu\nu} R^{\mu\nu} + \alpha R^2 \right] \,,$$

Stelle, PRD **16** (1977) 4 and Gen. Rel. Grav. **9** (1978). N.V. Krasnikov, Theor. Math. Phys. **73** (1987) 1184. Y. Kuzmin, Sov. J. Nucl. Phys. **50** (1989) 1011

However, **Stelle gravity is plagued by ghosts!**

Nonlocal scalar theory

Improve the ultraviolet behavior of the propagator using higher derivatives

$$\mathcal{L}_\phi = -\frac{1}{2}\phi(m^2 + \Box)\phi - V(\phi) \longrightarrow -\frac{1}{2}\phi(m^2 + \Box) f(-\Box/\Lambda^2)\phi - V(\phi)$$

So that the propagator is

$$\mathcal{O}^{-1} \sim \frac{1}{(k^2 - m^2)\, f(k^2/\Lambda^2)}$$

- To improve the ultraviolet behaviour of the propagator one requires $f(k^2/\Lambda^2) \to \infty$ in the ultraviolet $|k^2| \to \infty$
- No ghosts nor tachyons $\Rightarrow$ no new poles of the propagator $\Rightarrow f(k^2/\Lambda^2) \neq 0$ in the finite complex plane k^2.

The Feynman rules can be obtained by path integral formulation in **Euclidean** spacetime:Efimov,Int. J. Theor. Phys. **10**, 1 (1974). Pius, R., Sen, JHEP 2016, 24 (2016). Calcagni, Modesto, arXiv:2402.14785 [hep-th].

Infinite derivatives

As we want to exploit analytic properties of the scattering amplitudes, we assume that the function $f(z)$ is analytic on the $z \in \mathbb{C}$ plane.
We want no ghosts, so that the propagator

$$\mathcal{O}^{-1} \sim \frac{1}{(k^2 - m^2)\, f(k^2/\Lambda^2)}$$

has no extra poles. Indeed

$$f(z) \neq 0 \qquad \forall z \in \mathbb{C}$$

and $f(z)$ can't be polynomial

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \equiv e^{H(-z)}$$

- the theory must contain any power $\Box^n$.
- any truncation of the theory introduces ghosts
- $f(z)$ has an essential singularity at $z = \infty$.

Non local scalar quantum field theory

$$\mathcal{L}_\phi = -\frac{1}{2}\phi\, e^{H(-\sigma\Box)}\left(\Box + m^2\right)\phi - \frac{\lambda}{n!}\phi^n\,,$$

- $e^{H(-z)}$ is the nonlocal form factor.
- $\sigma = \ell^2 = \Lambda^{-2}$ is a new invariant scale of the theory
- $e^{H(-z)} \neq 0 \quad \forall z \in \mathbb{C}$
- $\lim_{x\to\pm\infty} e^{H(-x)} = 0$
- the theory is Lorentz-invariant (in comparison with Horava gravity that contains only spatial higher-derivative terms breaking Lorentz symmetry)

Studied in the framework of Bogolyubov, Medvedev, Polivanov, Shirkov axiomatics

- quantization: Efimov 1968,1973.
- unitarity of S-matrix: Alebastrov, Efimov 1973.
- Bogolyubov causality: Alebastrov, Efimov, 1974.

Field redefinition and non-locality

We introduce the field redefinition

$$\phi(x) = e^{-\frac{1}{2}H(-\sigma\Box)}\varphi(x) = \int e^{-\frac{1}{2}H(\sigma k^2)}\tilde{\varphi}(k)\, e^{ikx} d^4k = \int d^4y\, K(x-y)\,\varphi(y),$$

$$K(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-\frac{1}{2}H(\sigma k^2)} e^{-ik(x-y)}$$

which is invertible in L^2 functional space, because $e^{-\frac{1}{2}H(\sigma k^2)} \neq 0$. One has

$$\mathcal{L}_\varphi = -\frac{1}{2}\varphi\left(\Box + m^2\right)\varphi - \frac{\lambda_n}{n!}(e^{-\frac{1}{2}H(-\sigma\Box)}\varphi)^n\,.$$

and the theory is equivalent to a scalar field with standard kinetic term and nonlocal interactions.

The non-interacting nonlocal theory is equivalent to the local free theory.

nonlocality $\leftrightarrow$ interactions

Examples of form factors

Asymptotically polinomial form factor

Y.V. Kuz'min, Yad. Fiz. 50 (1989) 1630; E.T. Tomboulis,arXiv:hep-th/9702146; L. Modesto, Phys. Rev. D 86 (2012) 044005.

$$H(z) = \left[\gamma_E + \Gamma\left(0, p_\gamma(z)\right) + \log\left(p_\gamma(z)\right)\right]$$

where $\gamma_E = 0.577216$ is the Euler's constant,and $\Gamma(a,z) = \int_z^{+\infty} t^{a-1} e^{-t} dt$

$$\frac{-i}{(k^2 - m^2 + i\epsilon)e^{H(\sigma\, k^2)}} \sim k^{-2(\gamma+1)} \quad \text{for} \quad k^2 \to +\infty$$

Exponential form factor

$$H(k^2\sigma) = P_n(k^2\sigma) \Longrightarrow \frac{-i}{(k^2 - m^2 + i\epsilon)e^{H(\sigma\, k^2)}} \sim e^{-(\sigma\, k^2)^n}$$

Power counting

Interaction $\lambda\phi^n$ in $4-D$

- **Asymptotically polynomial form factor**
  $\omega_d = 4\,L - 2(\gamma+1)P$. Using $L = P - V + 1$ and $nV = N + 2P$

$$\omega_d = \frac{2(\gamma+1)n}{n-2} - \frac{2(\gamma+1)\mathrm{N}}{n-2} + L\left(4 - \frac{2(\gamma+1)n}{n-2}\right)$$

  - the theory is renormalizable for $\gamma = \frac{n-4}{n}$
  - the theory is super-renormalizable for $\gamma > \frac{n-4}{n}$
- **Exponential form factor**: exponential convergence of diagrams

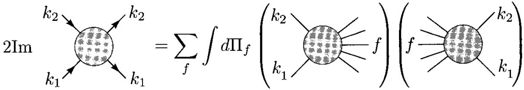
Non-renormalizable interactions can be rendered renormalizable in the nonlocal framework (new physics in the SM?)

Unitarity of the Euclidean theory

Proved in the axiomatic definition of S-matrix Alebastrov, Efimov, Comm. Math. Phys. **31** (1973).

$$\mathcal{M}_{ba} - \mathcal{M}^*_{ab} = i \sum_c \mathcal{M}^*_{cb} \mathcal{M}_{ca} (2\pi)^4 \delta^{(4)}(p_c - p_a)\,,$$

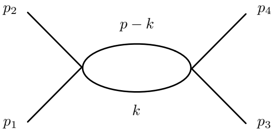
$\mathcal{M}_{ba} = \langle b|\mathcal{M}|a\rangle$ sum of all the connected amputated diagrams for the process $a \to b$.

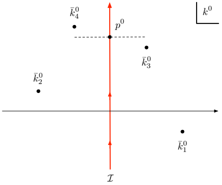


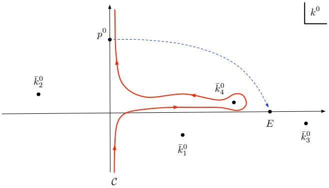
It can be proved by means of Cutkosky rules

- The imaginary part of complex amplitudes is given by Cutkosky rules
- Anomalous thresholds do not contribute
- The Minkoskian theory is not unitary

F. Briscese, L. Modesto, Phys. Rev. D **99** (2019) 104043. R. Pius, A. Sen, JHEP 11 (2018) **094**; JHEP **10** (2016) 024. P. Chin, E.T. Tomboulis, JHEP **06** (2018) 014. Efimov, Sov. J. Nuc. Phys. 4, 2 (1967).







The amplitude is calculated assuming the external energy p^0 purely imaginary and integrating for k^0 from $-i\infty$ to $+i\infty$ in the imaginary axis of the complex k^0 plane

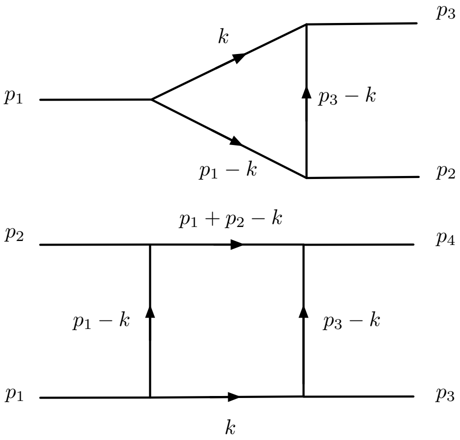
$$\mathcal{M}(\sigma, p, m, \epsilon) = -\frac{i\lambda^2}{2} \int_{\mathcal{I}\times\mathbb{R}^3} \frac{d^4k}{(2\pi)^4} \frac{e^{-H\left[\sigma\left(k^2-m^2\right)\right]}}{k^2 - m^2 + i\epsilon} \frac{e^{-H\left[\sigma\left((k-p)^2-m^2\right)\right]}}{(k-p)^2 - m^2 + i\epsilon}, \qquad (1)$$

Cutkosky rules

$$\mathcal{M} - \mathcal{M}^* = -\frac{\lambda^2}{2} \int_{(\mathbb{R}^4)} \frac{i\, d^4k}{(2\pi)^4} (-2\pi i)^2\, \delta(k^2 - m^2)\delta((p-k)^2 - m^2)$$

Anomalous thresholds

F. Briscese, L. Modesto, Phys. Rev. D **99** (2019) 104043.



Källén-Lehmann representation

FB, G. Calcagni, G. Nardelli, L. Modesto, arXiv:2305.16422v2

$$\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \int_0^\infty dS\,\Delta_F\,(x,s)\;\rho(s). \quad \text{F.T.} = \int_0^\infty dS\,\frac{i\rho(s)}{k^2 - s^2 + i\epsilon} \sim k^{-2}$$

Spectral density

$$\rho(s) = e^{-H(s-m^2)} \sum_\lambda \delta(s - m_\lambda^2)\,|\langle\Omega|\tilde{\varphi}(0)|\lambda_0\rangle|^2 \geq 0, \quad \varphi(x) = e^{\frac{1}{2}H(-\sigma\Box)}\phi(x)$$

Compare with the nonlocal propagator

From Stelle theory to nonlocal gravity

The occurrence of ghosts in Stelle gravity can be avoided introducing nonlocal form factors

$$\mathcal{L} = R + R\,\gamma_0(-\sigma\Box)\,R + R_{\mu\nu}\,\gamma_2(-\sigma\Box)\,R^{\mu\nu})$$

Where

$$\gamma_0\left(\sigma\Box\right) = -\frac{\left(e^{H_0(\sigma\Box)} - 1\right) + 2\left(e^{H_2(\sigma\Box)} - 1\right)}{6\Box}, \quad \gamma_2\left(\sigma\Box\right) = \frac{\left(e^{H_2(\sigma\Box)} - 1\right)}{\Box}$$

Graviton Propagator

Quantization around Minkowski $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_D \, h_{\mu\nu}$, gives the quadratic part of the Lagrangian

$$\mathcal{L}_{\rm g} = \mathcal{L}_{\rm g} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\rho\sigma} h^{\rho\sigma} + \xi^{-1} \partial^{\nu} h_{\mu\nu} w(-\sigma\Box) \partial_{\rho} h^{\rho\mu} + O(h^3)$$

where we have added a gauge fixing term. The propagator is

$$\mathcal{O}^{-1} = \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2 \, w(\sigma k^2)} + \left(\frac{P^{(2)}}{k^2 e^{H_2(\sigma k^2)}} - \frac{P^{(0)}}{2k^2 e^{H_0(\sigma k^2)}} \right)$$

Power counting

Consider the asymptotic polynomial form factor $e^{H_0(\sigma k^2)} \sim e^{H_2(\sigma k^2)} \sim k^{2(\gamma+1)}$, the propagator scales as

$$e^{H(-\sigma\Box)} \sim \Box^{\gamma+1} \Rightarrow \mathcal{O}^{-1} \sim k^{-2} e^{-H(\sigma k^2)} \sim \frac{1}{k^{2(\gamma+2)}}$$

Interactions coming from the terms

$$\mathcal{R}\,\gamma(-\Box)\,\mathcal{R} = \mathcal{R}\,\frac{e^{H(-\sigma\Box)}-1}{\Box}\,\mathcal{R} \sim \Box h^n \frac{(-\sigma\Box)^{\gamma+1}}{\Box}\Box h^m \sim h^n \Box^{\gamma+2} h^m + \ldots$$

The contribution at each vertex scales as $k^{2(\gamma+2)}$ and the superficial degree of divergence is

$$\delta = 4L + (V - I) \times 2(\gamma + 2)$$

and using $I = V + L - 1$ one has

$$\delta = 4 - 2\gamma(L - 1)$$

If $\gamma \geqslant 2$ one has only one loop divergences

Classical solutions

$$S_{\rm g} = -\frac{2}{\kappa_D^2}\int d^4x\sqrt{-g}\,[R + G_{\mu\nu}\gamma(\Box)R^{\mu\nu}]\,, \quad \text{with} \quad \gamma(\Box) \equiv \left(e^{H(-\sigma\Box)} - 1\right)/\Box$$

This non-local theory has all the vacuum solutions of GR ($R_{\mu\nu} = 0$) including BH and GW
(other authors are interested in non-singular solutions)

The equations of motion for the action read

$$E_{\mu\nu} \equiv (1 + \Box\,\gamma\,(\Box))\,G_{\mu\nu} + (g_{\mu\nu}\nabla_\alpha\nabla_\beta - g_{\alpha\mu}\nabla_\beta\nabla_\nu)\,\gamma\,(\Box)\,G^{\alpha\beta} + Q_{2\mu\nu}\,(\mathrm{Ric}) = 8\pi G_{\rm N}T_{\mu\nu}$$

where $Q_{2\mu\nu}\,(\mathrm{Ric})$ is at least quadratic in the Ricci tensor and scalar, e.g.

$$\sigma\left((\sigma\Box)^n R_{\mu\alpha}\right)\left((\sigma\Box)^m R^\alpha{}_\nu\right) \quad \text{or} \quad \sigma^2\left((\sigma\Box)^n R_{\mu\alpha}\right)\left((\sigma\Box)^m R^\alpha{}_\nu\right)\left((\sigma\Box)^l R\right),$$

At classical level, **all the solutions of Einstein's gravity in vacuum are vacuum solutions of NLG**

$$G_{\mu\alpha} = 0 \Rightarrow E_{\mu\alpha} = 0$$

However, NLG has more solutions, e. g., $G_{\mu\alpha} = -\frac{1}{c}\,g_{\mu\alpha}$ for $Q_{2\mu\nu}(\mathrm{Ric}) = c\,R_{\mu\alpha}R^\alpha{}_\nu$

Dynamics of small perturbations in NLG

Let us consider small perturbations of the Minkowski metric, i.e.,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\, h_{\mu\nu}, \qquad \text{with} \qquad |\epsilon\, h_{\mu\nu}| \ll 1$$

and expand all the terms in the EoM in powers of ϵ

$$h_{\mu\nu} = \sum_{n=0}^{\infty} \epsilon^n h^{(n)}_{\mu\nu} \quad \text{and} \quad G_{\mu\nu}(g_{\mu\nu}) = \sum_{n=0}^{\infty} \epsilon^n G^{(n)}_{\mu\nu}\,, \text{with } G^{(0)}_{\mu\nu} \equiv G_{\mu\nu}(\eta) = 0\,.$$

We find the recursive equation

$$G^{(k)}_{\mu\nu} = 0 \quad \text{for} \quad k \leq n \quad \Rightarrow \mathbf{e}^{\mathbf{H}(-\sigma\Box)} G^{(n+1)}_{\mu\nu} = 0 \quad \Rightarrow G^{(n+1)}_{\mu\nu} = 0$$

The dynamics of h is the same as in GR

$$E_{\mu\nu} = 0 \quad \Rightarrow \quad G^{(n)}_{\mu\nu} = 0 \quad \forall n \quad \Rightarrow \quad G_{\mu\nu}(g_{\mu\nu}) = G_{\mu\nu}(\eta_{\mu\nu} + \epsilon h_{\mu\nu}) = \sum_{n=1}^{\infty} \epsilon^n G^{(n)}_{\mu\nu} = 0\,.$$

F. Briscese, L. Modesto, JCAP 07 (2019) **009**.

Stability of Minkowski spacetime and GW

In NLG the dynamics of small perturbations of Minkowski spacetime is the same as in Einstein's gravity

$$E_{\mu\nu} = 0 \quad \Rightarrow \quad G_{\mu\nu}(\eta_{\mu\nu} + \epsilon h_{\mu\nu}) = 0$$

Therefore

- any Strongly Asymptotically Flat initial data set satisfying a Global Smallness Assumption entails a smooth, geodesically complete, and asymptotically at solution of NLG EoM in the vacuum, which is in facts a solution of the Einstein's equations in the vacuum.
- the dynamics of GW is the same as in Einstein's theory
- This result can be generalized to Ricci-flat and maximally symmetric metrics F. Briscese, G. Calcagni, L. Modesto, Phys. Rev. D **99** (2019) 084041.

Emergence of R^2 gravity in SRQG

Consider the SRQG Lagrangian in 4 dimensionsfor $\gamma_4 = 0$

$$\mathcal{L} = R + R\,\gamma_0(\Box)R - R_{\mu\nu}\gamma_2(\Box)R^{\mu\nu}$$

At lowest order SRQG reduces to Starobinsky theory

$$\gamma(\Box) = \frac{e^{H(\sigma\Box)} - 1}{\Box} \simeq \sigma \implies \mathcal{L} = R + \frac{\sigma}{6}R^2 + O\left(\sigma^2 R\Box R\right)$$

the model reduces to Starobinsky $f(R)$ gravity $1/\sqrt{\sigma}\,M_{\rm pl} = 1.3 \times 10^{-5}\left(\frac{55}{N}\right)$

FB, A. Marciano, L. Modesto, E. N. Saridakis, PRD 87 (2013) no.8, 083507

Maximal content on Minkowski spacetime

To reproduce Starobinsky inflation, the nonlocal model must contain (and in facts it contains at most) one extra scalar degree of freedom without breaking unitarity. This is achieved introducing a form factor with one zero by the replacement

$$e^{H_0(-\sigma\Box)} \to e^{H_0(-\sigma\Box)} \frac{-\Box - m^2}{m^2}$$

$$\mathcal{O}^{-1} = \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2\, w(\sigma k^2)} + \frac{P^{(2)}}{k^2 e^{H_2}} - \frac{m^2 P^{(0)}}{2k^2 e^{H_0}(k^2 - m^2)} =$$

$$= \frac{\xi(2P^{(1)} + \bar{P}^{(0)})}{2k^2\, w(\sigma k^2)} + \frac{P^{(2)}}{k^2 e^{H_2}} - \frac{P^{(0)}}{2k^2 e^{H_0}} + \frac{P^{(0)}}{2e^{H_0}(k^2 - m^2)}$$

FB, L. Mosesto, S. Tsujikawa, PRD 89 (2014) no.2, 024029

Exact Starobinsky inflation

Action in Weyl basis

$$\mathcal{L} = M_P^2 R + \lambda \left[R\,\mathcal{F}_R(\sigma\Box)R + W_{\alpha\beta\mu\nu}\mathcal{F}_W(\sigma\Box)W^{\alpha\beta\mu\nu} \right]$$

- Scalar power spectra and spectral index are the same as in local R^2 theory
- $\mathcal{P}_t$ and n_t are modified $\sim e^{-H(R/\Lambda_N^2)}$
- The single field inflation constraint $r = -8n_t$ is violated
- n_t can be both positive or negative
- non-gaussianities have been calculated

S. Koshelev, L. Modesto, L. Rachwal, A. A. Starobinsky, JHEP **11** (2016) 067
S. Koshelev, S. Kumar, A. Mazumdar, A. A. Starobinsky, JHEP **06**(2020) 152
S. Koshelev, K. S. Kumar, A. A. Starobinsky, I. Jour. of Mod. Phys. D. **29** (2020) 2043018.

Asymptotic freedom

- Quadratic gravity
  E. S. Fradkin and A. A. Tseytlin, PLB, 1981, vol. **104**, 5; Nucl. Phys. B,1982, **201**, 3;PLB, 1982, **110**,2. I. G. Avramidi, A. O. Barvinsky, Phys. Lett. B **159**, 269 (1985). I. G. Avramidi, (In Russian) Sov. J. Nucl. Phys. **44**, 255 (1986).
- Higher-derivatives and nonlocal gravity
  M. Asorey, J.L. Lopez, I.L. Shapiro, Int. J. Mod. Phys. A **12** (1997) 5711.
  L. Modesto, L. Rachwaland, I.L. Shapiro, Eur. Phys. J. C **78** (2018) 555.
  M.B. Einhorn, D.R.T. Jones, Phys. Rev. D **96** (2017) 124025.

Implications

- asymptotic freedom could alleviate the ultraviolet trans-Planckian problem: FB, L. Modesto, JHEP **09**(2020)056.
- asymptotic freedom in the primordial universe, arXiv:2206.06384.

Conclusions

- The theory is quantum renormalizable on flat space-time
- Nonlocality can be introduced preserving the basic requirements for a meaningful theory (unitarity, Lorentz invariance etc.)
- Classical nonlocal gravity has nice properties (BH solutions, GW, stability of classical solutions, Starobinsly inflation etc.)
- Asymptotically freedom could alleviate the cosmological trans-Planckian problem
- Observable effects in $\mathcal{P}_t$ and n_t
- As in $f(R)$ gravity, the specific form of the form factor cannot be derived from first principles
- All these properties depend in a crucial way on the specific form factor
- Open problems: nonlocality in the SM sector, signatures of nonlocality, removal of singularities of Einstein's theory, conformal invariance, bouncing cosmologies etc.

Thank you!