Fermion mass hierarchy and CP violation in modular symmetry

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1 Modular Symmetry

We can discuss the flavor problem based on
" modular symmetry "

Mass hierarchy
Flavor mixing
CP violation

of quarks/leptons

Are Yukawa couplings (Mass matrix) modular forms ?

F. Feruglio, arXiv:1706.08749

Modular forms meet flavor problem !

What is Modular form ?

$$f(x) = \sin 2\pi x\,, \qquad \mathrm{T}: x \to x+1 \Rightarrow f(x+1) = f(x)$$

shift-symmetry

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (a,b,c,d) are integer and $ad - bc = 1$

$$\gamma : z \to \frac{az+b}{cz+d}$$

z is complex

Modular transformation

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad \mathrm{T}: z \to z+1$ shift-symmetry

Modular form f(z) is defined by imposing three conditions

① $f(z)$ is holomorphic @ Im Z > 0

② $f(z)$ is holomorphic @ $z \to i\infty$

③ ~~$f\left(\frac{az+b}{cz+d}\right) = f(z)$~~

Modular function only constant

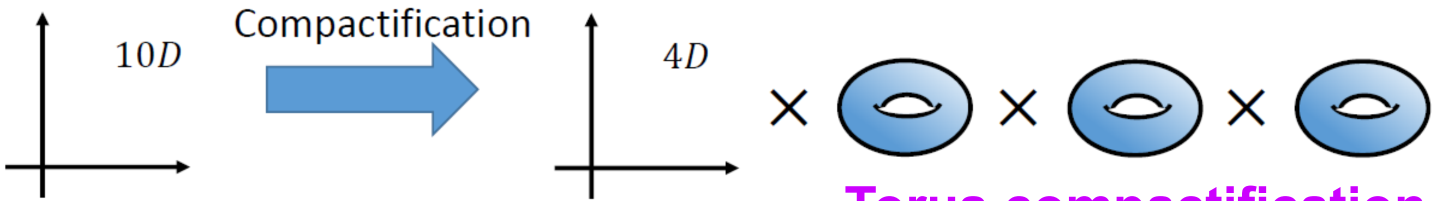
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

k: weight

保型因子

Automorphy factor

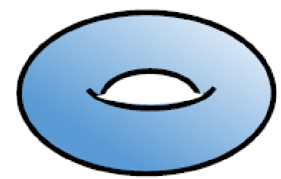
Modular forms appear naturally in top-down scenarios based on a class of string compactifications



Torus compactification

We get 4D effective Lagrangian by integrating out over 6D.

$$S = \int d^4x d^6y \, \mathcal{L}_{10D} \rightarrow \int d^4x \, \mathcal{L}_{\rm eff}$$

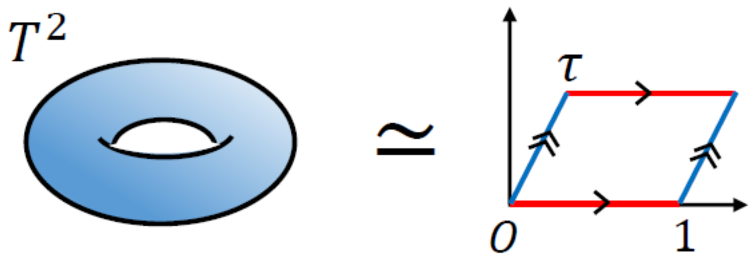
➡ $\mathcal{L}_{\rm eff}$ depends on the structure of 

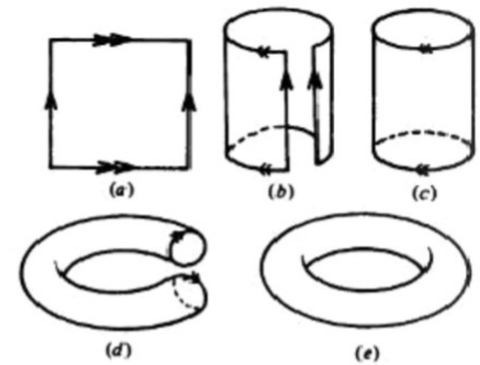
- $4D$ effective theory depends on **internal space**

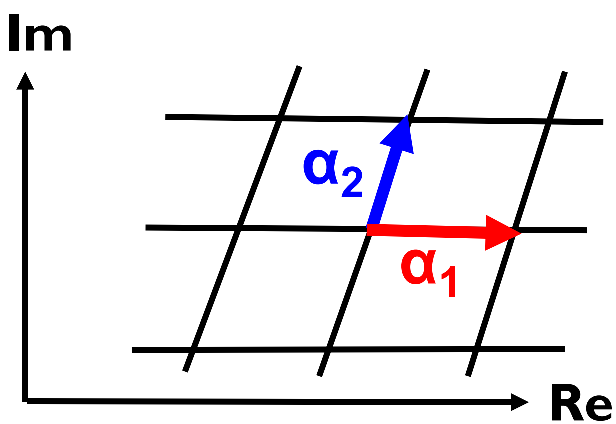
2D torus has Modular symmetry

$2D$ torus (T^2) is equivalent to
parallelogram with identification of confronted sides.

by Feruglio







$(x,y) \sim (x,y) + n_1\alpha_1 + n_2\alpha_2$

Two-dimensional torus T^2 is obtained as

$$T^2 = \mathbb{R}^2 / \Lambda$$

Λ is two-dimensional lattice,
which is spanned by two lattice vectors

$\alpha_1 = 2\pi R$ and $\alpha_2 = 2\pi R\tau$

$\tau = \alpha_2 / \alpha_1$ is a modulus parameter (complex).

The same lattice is spanned by other bases under the transformation

$$\begin{pmatrix} \alpha_2' \\ \alpha_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

ad-bc=1
a,b,c,d are integer SL(2,Z)

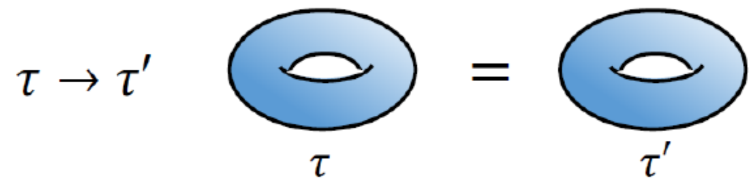
$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

ad-bc=1
a,b,c,d are integer

$\tau = \alpha_2 / \alpha_1$

$$\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}$$

Modular transformation



Modular transf. does not change the lattice (torus)

$4D$ effective theory (depends on τ)
must be invariant under modular transf.

e.g.) $\mathcal{L}_{\rm eff} \supset Y(\tau)_{ij} \phi \overline{\psi_i} \psi_j$

The modular transformation is generated by S and T .

$$\tau \xrightarrow{\gamma} \tau' = \frac{a\tau + b}{c\tau + d}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}$$

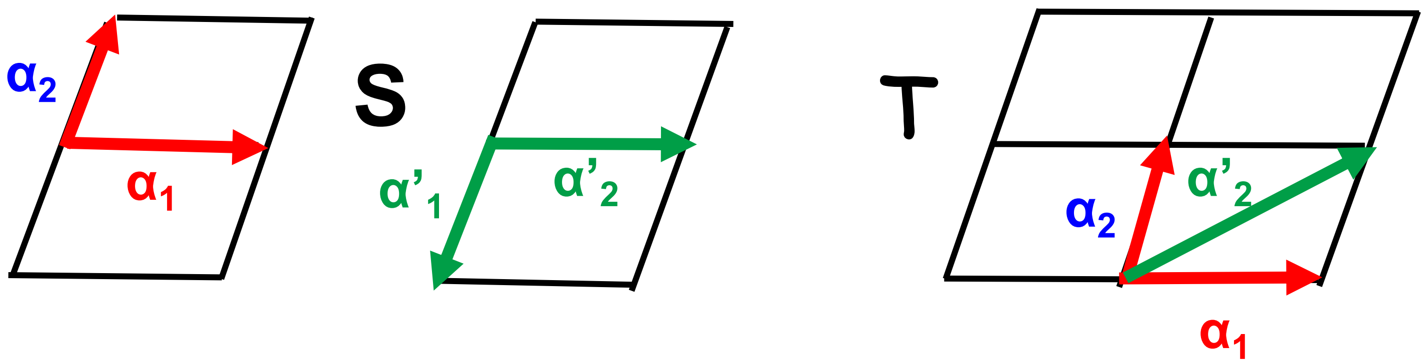
duality

$$T : \tau \longrightarrow \tau + 1$$

Dicrete shift symmetry

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} \alpha'_2 \\ \alpha'_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_1 \end{pmatrix}$$

$\tau = \alpha_2 / \alpha_1$

$$S : \tau \longrightarrow -\frac{1}{\tau},$$

Duality

$$T : \tau \longrightarrow \tau + 1.$$

Dicrete shift symmetry

$$S^2 = 1, \qquad (ST)^3 = 1.$$

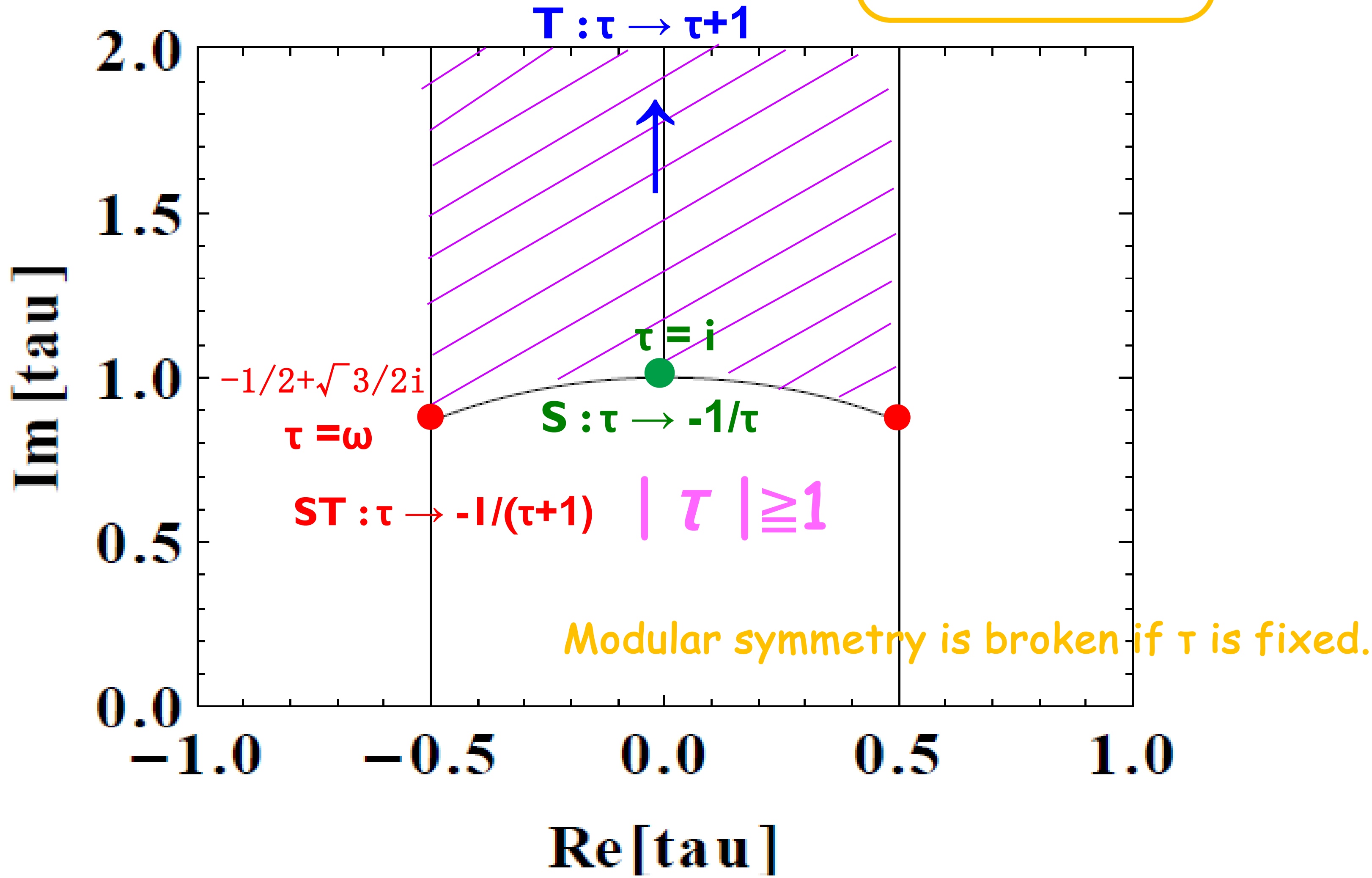
±1 is identified

generate infinite discrete group

Modular group

Fundamental Domain of τ

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$



● ● Symmetric point of τ (Residual symmetry)

Generate finite modular group

Modular group
$\Gamma \simeq \{S,T|S^2=\mathbb{I},(ST)^3=\mathbb{I}\}$ infinite discrete group

Modular group has subgroups

Impose
congruence conditio $\Gamma(N)=\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N}\}$

called principal congruence subgroups (normal subgroup)

$\Gamma_N \equiv \Gamma / \Gamma(N)$ quotient group finite group of level N

$\Gamma_N \simeq \{S,T|S^2=\mathbb{I},(ST)^3=\mathbb{I},T^N=\mathbb{I}\}$

$\Gamma_2 \simeq S_3$ $\Gamma_3 \simeq A_4$ $\Gamma_4 \simeq S_4$ $\Gamma_5 \simeq A_5$

isomorphic

Consider Yukawa couplings with Γ_N symmetry

Yukawas are given in terms of modular forms with weight k

$$Y^{(k)}(\tau)\, q^c\, Q_L\, H_q$$

modular form | modulus | k: weight

Modular transformation $Y^{(k)} \to (c\tau+d)^k Y^{(k)}$

Automorphy factor

$$Q_L \to (c\tau+d)^{-k_{Q_L}} Q_L, \quad q^c \to (c\tau+d)^{-k_{q^c}} q^c, \quad H_q \to (c\tau+d)^{-k_{H_q}} H_q$$

Modular invariance gives

$$k = k_Q + k_{q^c} + k_{H_q}$$

Weights satisfy this strictly.

Automorphy factor vanishes !

2 Modular forms with weigh k

Let us consider Level 3 (N=3)

$$\Gamma_N \simeq \{S, T | S^2 = \mathbb{I}, (ST)^3 = \mathbb{I}, T^N = \mathbb{I}\}$$

$\Gamma_3 \simeq A_4$ group 1, 1', 1'', 3

Number of modular forms depend on weight k (even)

k+1 for A_4 (2k+1 for S_4)

For k=0, the modular form is constant (modular function)

For k=2, there are 3 linealy independent modular forms,

which form a A_4 triplet.

F. Feruglio, arXiv:1706.08749

A_4 triplet of modular forms with weight 2

$$Y_1(\tau) = \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}-\frac{27\eta'(3\tau)}{\eta(3\tau)}\right),$$

$$Y_2(\tau) = \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right),$$

$$Y_3(\tau) = \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)}+\omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)}+\omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right)$$

$$Y_2^2+2Y_1Y_3=0$$

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

Dedekind eta-function

$$Y=\begin{pmatrix}Y_1(\tau)\\Y_2(\tau)\\Y_3(\tau)\end{pmatrix}=\begin{pmatrix}1+12q+36q^2+12q^3+\dots\\-6q^{1/3}(1+7q+8q^2+\dots)\\-18q^{2/3}(1+2q+5q^2+\dots)\end{pmatrix}$$

$$q=e^{2\pi i\tau}$$

$$\rho(\mathrm{S})=\frac{1}{3}\begin{pmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{pmatrix},\quad \rho(\mathrm{T})=\begin{pmatrix}1&0&0\\0&\omega&0\\0&0&\omega^2\end{pmatrix},\quad \omega=\exp(i\frac{2}{3}\pi)$$

We find easily modular forms with higher weights k=4, 6 …

# of modular forms is k+1

Weight 2
3 Modular forms

$$\mathbf{Y_3}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}_{\mathbf{3}} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}_{\mathbf{3}} = (a_1b_1 + a_2b_3 + a_3b_2)_{\mathbf{1}} \oplus (a_3b_3 + a_1b_2 + a_2b_1)_{\mathbf{1'}} \oplus (a_2b_2 + a_1b_3 + a_3b_1)_{\mathbf{1''}} \oplus \frac{1}{3}\begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix}_{\mathbf{3}} \oplus \frac{1}{2}\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix}_{\mathbf{3}}$$

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{1}\,, \quad \mathbf{1'} \otimes \mathbf{1'} = \mathbf{1''}\,, \quad \mathbf{1''} \otimes \mathbf{1''} = \mathbf{1'}\,, \quad \mathbf{1'} \otimes \mathbf{1''} = \mathbf{1}\,.$$

Modular forms with higher weights are constructed by the tensor product of modular forms of weight 2

J.T.Penedo, S.T.Petcov, Nucl.Phys.B939(2019)292

$$\mathbf{Y_3^{(2)}} \times \mathbf{Y_3^{(2)}} \quad \Rightarrow \quad \mathbf{Y_1^{(4)}} = Y_1^2 + 2Y_2Y_3\,, \quad \mathbf{Y_{1'}^{(4)}} = Y_3^2 + 2Y_1Y_2\,, \quad \mathbf{Y_{1''}^{(4)}} = Y_2^2 + 2Y_1Y_3 = 0$$

Weight 4
5 Modular forms

$$\mathbf{Y_3^{(4)}} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}\,,$$

Modular forms at nearby symmetric points

Consider A_4 triplet modular forms with weigh k=2. (N=3)

$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

$\varepsilon = 6\,|q|^{1/3}$

$\tau \to \infty i$ $\quad (Y_1, Y_2, Y_3)^T \to (1, -\varepsilon, -1/2\,\varepsilon^2)^T \to (1, 0, 0)^T$

A_4 triplet $\quad |\varepsilon| << 1$

k=4 $\quad \mathbf{Y}_{\mathbf{3}}^{(4)} = Y_0^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{1}}^{(4)} = Y_0^2, \quad \mathbf{Y}_{\mathbf{1'}}^{(4)} = 0,$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

k=6 $\quad \mathbf{Y}_{\mathbf{3}}^{(6)} = Y_0^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(6)} = 0, \quad \mathbf{Y}_{\mathbf{1}}^{(6)} = Y_0^3,$

Z_3 symmetry

k=8 $\quad \mathbf{Y}_{\mathbf{3}}^{(8)} = Y_0^4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} = 0, \quad \mathbf{Y}_{\mathbf{1}}^{(8)} = Y_0^4, \quad \mathbf{Y}_{\mathbf{1'}}^{(8)} = 0, \quad \mathbf{Y}_{\mathbf{1''}}^{(8)} = 0$

$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots , \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

$$\varepsilon = 6\,|q|^{1/3}$$

Modular forms are also hierarchical at $\tau=\omega$

$$\rho(ST) = \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$\tau = \omega$ **k=2** $\mathbf{Y}_{\mathbf{3}}^{(2)} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ Z_3 symmetry

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \frac{9}{4}Y_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \qquad \mathbf{Y}_{\mathbf{3}}^{(6)} = 0 , \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} = \frac{27}{8}\omega^2 Y_0^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

k=4 **k=6**

3 Mass hierarchy in modular invariance

P.P.Novichkov, J.T.Penedo, S.T.Petcov, JHEP 04(2021)206, arXiv:2102.07488

We can construct the mass matrix with hierarchical masses by using the hierarchical modular forms at nearby $\tau=\infty i$ and ω

$$\mathcal{M}_q \sim v_q \begin{pmatrix} \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}_{RL}$$

This hierarchical structure is not accidental.
Thanks to Residual symmetry Z_3 (N=3)

F. Feruglio, V. Gherardi, A. Romanino,A. Titov,
S.T.Petcov, M.Tanimoto
S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, H. Uchida
Y. Abe, T. Higaki, J. Kawamurab,T. Kobayashi,
S. Kikuchi, T. Kobayashi, K. Nasu, S. Takada, H. Uchida
Y. Abe, T. Higaki, J. Kawamura ,T. Kobayashi

Modular invariant mass matrix

$$M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau)\rho(\gamma)^\dagger \qquad K = k^c + k$$

τ=i∞ **γ=T : τ → τ+1** cτ+d = 1 $\quad M_{ij}(T\tau) = (\rho_i^c\,\rho_j)^*\, M_{ij}(\tau)$

$$q \xrightarrow{T} q\,\xi \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$q \equiv exp(i2\pi\tau/N) \qquad \xi = \exp(i\,2\pi/N)$

n-th derivative

$$M_{ij}(\xi\ddot{q}) = (\rho_i^c\,\rho_j)^* M_{ij}(\ddot{q}) \rightarrow \xi^n M_{ij}^{(n)}(0) = (\rho_i^c\,\rho_j)^* M_{ij}^{(n)}(0)$$

$$M_{ij}(q) = a_0\, q^\ell + a_1\, q^{\ell+N} + a_2\, q^{\ell+2N} + \dots, \qquad \ell = 0, 1, 2, \dots, N-1,$$

For N=3 $\quad M(\tau) \sim \mathcal{O}(\epsilon^\ell) \qquad \ell = 0, 1, 2 \qquad |q| = \varepsilon$ **Z_3 symmetry**

Mass hierarchy is also realized close to τ=ω

$$M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau)\rho(\gamma)^\dagger \qquad K = k^c + k$$

mass matrix is invariant under ST transformation (Z_3 symmetry)

Near τ=ω $$u = \frac{\tau-\omega}{\tau-\omega^2} \quad (u = 0 \; @ \; \tau = \omega) \qquad |u| = \epsilon$$

ST transformation : $u \to \omega^2 u$

$$M(ST\tau)_{ij} = M(\omega^2 u)_{ij} = (-(\tau+1))^K [\rho^c(\gamma)_i \rho(\gamma)_j]^* M(u)_{ij}$$

$$M(\tau) \sim \mathcal{O}(\epsilon^\ell) \quad \ell = 0, 1, 2$$

due to residual symmetry Z_3

Observed Yukawa ratios at GUT scale with tanβ=10

S. Antusch, V. Maurer, JHEP 1311 (2013) 115 [arXiv:1306.6879].

$$\frac{y_d}{y_b} = 9.21 \times 10^{-4}\ (1 \pm 0.111)\,, \qquad \frac{y_s}{y_b} = 1.82 \times 10^{-2}\ (1 \pm 0.055)$$

$$\frac{y_u}{y_t} = 5.39 \times 10^{-6}\ (1 \pm 0.311)\,, \qquad \frac{y_c}{y_t} = 2.80 \times 10^{-3}\ (1 \pm 0.043)$$

$$m_{b\,(t)} : m_{s\,(c)} : m_{d\,(u)} \sim 1 : |\epsilon| : |\epsilon|^2$$

For down quark sector $\varepsilon_d = 0.02 \sim 0.03$

For up quark sector $\varepsilon_u = 0.002 \sim 0.003$

We have only one ε because of one modulus T $|q| = \varepsilon$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

4 Examples in A_4 modular symmetry

@ $\tau=\omega$

	Q	(u^c, c^c, t^c), (d^c, s^c, b^c)	H_q	$\mathbf{Y}_{\mathbf{3}}^{(6)}$, $\mathbf{Y}_{\mathbf{3'}}^{(6)}$	$\mathbf{Y}_{\mathbf{3}}^{(4)}$	$\mathbf{Y}_{\mathbf{3}}^{(2)}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1}, \mathbf{1''}, \mathbf{1'})$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
k_I	2	(4, 2, 0)	0	$k=6$	$k=4$	$k=2$

$$W_d = \left[\alpha_d(\mathbf{Y}_{\mathbf{3}}^{(6)}Q)_{\mathbf{1}}d_1^c + \alpha'_d(\mathbf{Y}_{\mathbf{3'}}^{(6)}Q)_{\mathbf{1}}d_1^c + \beta_d(\mathbf{Y}_{\mathbf{3}}^{(4)}Q)_{\mathbf{1'}}s_{1'}^c + \gamma_d(\mathbf{Y}_{\mathbf{3}}^{(2)}Q)_{\mathbf{1''}}b_{1'}^c\right] H_d$$

Suppose all coefficients are same order.

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1^{\prime(6)} & Y_3^{(6)} + g_q Y_3^{\prime(6)} & Y_2^{(6)} + g_q Y_2^{\prime(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}_{RL}$$

$g_q = \alpha'_q / \alpha_q$

S.T.Petcov, M.Tanimoto, Eur. Phys. J. C 83(2023)579 [arXiv:2212.13336]

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1'^{(6)} & Y_3^{(6)} + g_q Y_3'^{(6)} & Y_2^{(6)} + g_q Y_2'^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}_{RL}$$

At τ = ω **in the diagonal base of ST**

$$\mathbf{Y}_3^{(2)} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Y}_3^{(4)} = \frac{9}{4} Y_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{Y}_3^{(6)} = 0\,, \qquad \mathbf{Y}_{3'}^{(6)} = \frac{27}{8}\omega^2 Y_0^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathcal{M}_q^{(0)} = M_q V_{\rm ST}^\dagger = v_q \begin{pmatrix} 0 & 0 & \frac{27}{8}\,\hat{\alpha}_q\, g_q\, \omega \\ 0 & 0 & \frac{9}{4}\,\hat{\beta}_q\, \omega^2 \\ 0 & 0 & \frac{3}{2}\,\hat{\gamma}_q \end{pmatrix}$$

rank one matrix

very small

$$\tau = \omega + \epsilon$$

$$\frac{Y_2(\tau)}{Y_1(\tau)} \simeq -\frac{2}{3}\epsilon_1\,, \qquad \frac{Y_3(\tau)}{Y_1(\tau)} \simeq \frac{2}{9}\epsilon_1^2$$

$$\epsilon_1 \simeq 2.1\, i\, \epsilon$$

In the diagonal base of ST

$$\mathcal{M}_q \sim v_q \begin{pmatrix} \hat{\alpha}_q \omega Y_1^3 & 0 & 0 \\ 0 & \hat{\beta}_q \omega^2 Y_1^2 & 0 \\ 0 & 0 & \hat{\gamma}_q Y_1 \end{pmatrix} \begin{pmatrix} (-3 + \frac{3}{4} g_q)\epsilon_1^2 & -\frac{9}{2}\epsilon_1 (1 + \frac{g_q}{2}) & g_q \frac{27}{8} \\ -\frac{3}{2}\epsilon_1^2 & \frac{3}{2}\epsilon_1 & \frac{9}{4} \\ \frac{1}{3}\epsilon_1^2 & -\epsilon_1 & \frac{3}{2} \end{pmatrix}$$

$g_q \sim 1$

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : |\epsilon_1| : |\epsilon_1|^2 \simeq 1 : |\epsilon| : |\epsilon|^2$$

$g_q >> 1$

$$m_{q3} : m_{q2} : m_{q1} \simeq \quad 1 : \frac{|\epsilon_1|}{|g_q|} : \left(\frac{|\epsilon_1|}{|g_q|}\right)^2$$

$$\tau = \omega + \epsilon$$

Real parameters except for τ

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	g_d	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	g_u
$0.01779 + i\,0.02926$	3.26	0.43	-1.40	1.05	0.80	-16.1

$|g_d| \sim 1$ $|g_u| \sim 10$

	$\frac{m_s}{m_b}\times10^2$	$\frac{m_d}{m_b}\times10^4$	$\frac{m_c}{m_t}\times10^3$	$\frac{m_u}{m_t}\times10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	J_{CP}
Fit	1.52	8.62	2.50	5.43	0.2230	0.0786	0.00368	-2.9×10^{-8}
Exp 1 σ	1.82 ±0.10	9.21 ±1.02	2.80 ±0.12	5.39 ±1.68	0.2250 ±0.0007	0.0400 ±0.0008	0.00353 ±0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times10^{-5}$

CPV is very small !

Why CPV is so small ?

CP phase structure of mass matrix

$$\tau = \omega + \epsilon \qquad \frac{Y_2(\tau)}{Y_1(\tau)} \simeq -\frac{2}{3}\epsilon_1\,, \qquad \frac{Y_3(\tau)}{Y_1(\tau)} \simeq \frac{2}{9}\epsilon_1^2 \qquad \epsilon_1 \simeq 2.1\, i\, \epsilon$$

$$\mathcal{M}_q^{gen} = v_q \begin{pmatrix} i^2\,\epsilon^2 & i\,\epsilon & 1 \\ i^2\,\epsilon^2 & i\,\epsilon & 1 \\ i^2\,\epsilon^2 & i\,\epsilon & 1 \end{pmatrix}, \quad q = d, u$$

$$(\mathcal{M}_q^{gen})^\dagger \mathcal{M}_q^{gen} = v_q^2 \begin{pmatrix} -i\,e^{-i\,\kappa_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\,e^{i\,\kappa_q} \end{pmatrix} \begin{pmatrix} |\epsilon_q|^4 & |\epsilon_q|^3 & |\epsilon_q|^2 \\ |\epsilon_q|^3 & |\epsilon_q|^2 & |\epsilon_q| \\ |\epsilon_q|^2 & |\epsilon_q| & 1 \end{pmatrix} \begin{pmatrix} i\,e^{i\,\kappa_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i\,e^{-i\,\kappa_q} \end{pmatrix}$$

$P(\kappa_q)$ $\qquad$ $\epsilon_q = |\epsilon_q| e^{i\,\kappa_q}$ $\qquad$ $P(\kappa_q)^*$

$$U^{gen}_{CKM} = O^T_u P^*(\kappa_u) P(\kappa_d) O_d$$

$$P(\kappa_q) = \mathrm{diag}(e^{-i(\kappa_q+\pi/2)}, 1, e^{i(\kappa_q+\pi/2)})$$

Common τ $\epsilon_{1d} = \epsilon_{1u}$ $\kappa_d = \kappa_u$ $P^*(\kappa_u)P(\kappa_d) = 1$

CP conserving if other parameters are real

Two different τ $\epsilon_d \neq \epsilon_u$ $P^*(\kappa_u)P(\kappa_d) \neq 1$

CP violation even if other parameters are real

Spontaneous CP violation

S.T.Petcov, M.Tanimoto, JHEP 08 (2023)086 [arXiv:2306.05730]

@ τ= i∞

putting $|g_d| \sim |g_u| \sim 1$

	Q	(d^c, s^c, b^c) , (u^c, c^c, t^c)		H_u	H_d
$SU(2)$	2	1		2	2
A_4	3	$(1', 1', 1')$	$(1', 1', 1')$	1	1
k	2	$(4, 2, 0)$	$(6, 2, 0)$	0	0

Irreducible representations

A_4 : 1, 1', 1'', 3

Weight k is set to vanish automorphy factor $(c\tau + d)^k$

$$W_d = \left[\alpha_d (\mathbf{Y}_3^{(6)} Q)_1 d_1^c + \alpha'_d (\mathbf{Y}_{3'}^{(6)} Q)_1 d_1^c + \beta_d (\mathbf{Y}_3^{(4)} Q)_{1'} s_{1'}^c + \gamma_d (\mathbf{Y}_3^{(2)} Q)_{1''} b_{1'}^c \right] H_d$$

A_4: 3 x 3 x 1'   3 x 3 x 1'   3 x 3 x 1'

Weight: 6 -2 -4   4 -2 -2   2 -2 0

$$M_d = v_d \begin{pmatrix} \hat{\alpha}'_d & 0 & 0 \\ 0 & \hat{\beta}_d & 0 \\ 0 & 0 & \hat{\gamma}_d \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(6)} & \tilde{Y}_2^{(6)} & \tilde{Y}_1^{(6)} \\ \tilde{Y}_3^{(4)} & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix} , \quad M_u = v_u \begin{pmatrix} \hat{\alpha}'_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)} & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ \tilde{Y}_3^{(4)} & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$$\tilde{Y}_i^{(6)} = g_d Y_i^{(6)} + Y_i'^{(6)} , \qquad \tilde{Y}_i^{(8)} = f_u Y_i^{(8)} + Y_i'^{(8)} , \qquad g_d \equiv \alpha_d / \alpha'_d \qquad f_u \equiv \alpha_u / \alpha'_u$$

$$\mathrm{Det}\,[\mathcal{M}_u^2] = 0$$

due to

$$Y_2^2 + 2Y_1 Y_3 = 0$$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$q \equiv \exp\left(2i\pi\tau\right) = \left(p\,\epsilon\right)^3$$

$$\epsilon = \exp\left(-\frac{2}{3}\pi\,\mathrm{Im}\left[\tau\right]\right)\,, \qquad p = \exp\left(\frac{2}{3}\pi\,i\,\mathrm{Re}\left[\tau\right]\right)$$

$\tau = i\infty$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{Y}_{\mathbf{3}}^{(4)} = Y_0^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} = Y_0^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} = 0 \qquad \mathbf{Y}_{\mathbf{3}}^{(8)} = Y_0^4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\,, \qquad \mathbf{Y}_{\mathbf{3'}}^{(8)} = 0$$

kinetic terms

Simplest Modular invariant kinetic terms of matters

$$\sum_I \frac{|\partial_\mu \psi^{(I)}|^2}{\langle -i\tau + i\bar{\tau} \rangle^{k_I}}$$

This is not canonical form.

We need overall renomarization $\psi^{(I)} \to \sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\, \psi^{(I)}$

comment

Possible non-minimal additions to Kaehler potential, compatible with the modular symmetry including modular forms Y and $\bar{Y}$ reduces the predictive power of flavor models, and often assumed to be negligible.

Superpotential

$$W_d = \left[\alpha_d(\mathbf{Y}_3^{(6)}Q)_1 d_1^c + \alpha'_d(\mathbf{Y}_{3'}^{(6)}Q)_1 d_1^c + \beta_d(\mathbf{Y}_3^{(4)}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y}_3^{(2)}Q)_{1''} b_{1'}^c\right] H_d$$

Kinetic terms

$$\sum_I \frac{|\partial_\mu \psi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

We renormalize superfields to get canonical kinetic terms

$$\psi^{(I)} \to \sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\, \psi^{(I)}$$

$$\alpha_u \to \hat{\alpha}_u = \alpha_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha_u\,(2\mathrm{Im}\tau)^4, \qquad \alpha'_u \to \hat{\alpha}'_u = \alpha'_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha'_u\,(2\mathrm{Im}\tau)^4\,,$$

$$\beta_u \to \hat{\beta}_u = \beta_u \sqrt{(2\mathrm{Im}\tau)^4} = \beta_u\,(2\mathrm{Im}\tau)^2\,, \qquad \gamma_u \to \hat{\gamma}_u = \gamma_u \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_u\,(2\mathrm{Im}\tau)\,,$$

$$\alpha_d \to \hat{\alpha}_d = \alpha_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha_d\,(2\mathrm{Im}\tau)^3, \qquad \alpha'_d \to \hat{\alpha}'_d = \alpha'_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha'_d\,(2\mathrm{Im}\tau)^3\,,$$

$$\beta_d \to \hat{\beta}_d = \beta_d \sqrt{(2\mathrm{Im}\tau)^4} = \beta_d\,(2\mathrm{Im}\tau)^2\,, \qquad \gamma_d \to \hat{\gamma}_d = \gamma_d \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_d\,(2\mathrm{Im}\tau)\,.$$

$2\mathrm{Im}\,\tau$ is large $\tau \longrightarrow i\infty$ compared with the case of at $\tau = \omega$

Down type quark mass matrix

At τ=i∞

$$M_q = v_q \begin{pmatrix} g_q\,\hat\alpha'_q & 0 & 0 \\ 0 & \hat\beta_q & 0 \\ 0 & 0 & \hat\gamma_q \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{RL}$$

rank one

$$\mathcal{M}_q^{2(0)} \equiv M_q^\dagger M_q = v_q^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |g_q|^2\,\hat\alpha'^2_q + \hat\beta_q^2 + \hat\gamma_q^2 \end{pmatrix}$$

In the vicinity of τ=i∞ $|\alpha'_q| \sim |\beta_q| \sim |\gamma_q|$

$$\mathcal{M}_q = v_q \begin{pmatrix} \hat\alpha'_q & 0 & 0 \\ 0 & \hat\beta_q & 0 \\ 0 & 0 & \hat\gamma_q \end{pmatrix} \begin{pmatrix} 18\,(\epsilon\,p)^2(4-g_q) & -6\,(\epsilon\,p)(2+g_q) & g_q \\ 54\,(\epsilon\,p)^2 & 6\,(\epsilon\,p) & 1 \\ -18\,(\epsilon\,p)^2 & -6\,(\epsilon\,p) & 1 \end{pmatrix}$$

$$\mathcal{M}_q^2 \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 p^* & \epsilon^2 p^{*2} \\ \epsilon^3 p & \epsilon^2 & \epsilon p^* \\ \epsilon^2 p^2 & \epsilon p & 1 \end{pmatrix}$$

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : \left|\frac{12\epsilon}{I_\tau g_q}\right| : \left|\frac{12\epsilon}{I_\tau g_q}\right|^2$$

$I_\tau = 2\mathrm{Im}\,\tau$

$g_q > \mathcal{O}(1)$

Up type quark mass matrix

In order to protect a massless quark, we can consider dimesuion 6 mass operator

$(u^c Q H_u)(H_u H_d)/\Lambda^2$ with $k_Q = 2 - k_{Hd}\,,\quad k_{u^c} = 6 + k_{Hd} - k_{Hu}$

or SUSY breaking by F term F/Λ^2

F. Feruglio, V. Gherardi, A. Romanino and A. Titov, JHEP 05 (2021), 242; arXiv:2101.08718

$$M_u = v_u \begin{pmatrix} \hat{\alpha}'_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)}(1+C_{u1}) & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ \tilde{Y}_3^{(4)}(1+C_{u2}) & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)}(1+C_{u3}) & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$$m_t : m_c : m_u \simeq \left[1 : \left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right) : \frac{3}{2}\left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right)^2 f_u^3 I_\tau |C_u|\right] I_\tau^4 f_u$$

$$C_u = 3 f_u (C_{u1} - C_{u2}) + (-4C_{u1} + 3C_{u2} + C_{u3}) \qquad I_\tau = 2\,\mathrm{Im}\,\tau$$

I_τ is a overall normalization factor for canonical kinetic terms

Down type quark masses   k=2, 4, 6 modular forms

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : \left|\frac{12\epsilon}{I_\tau g_q}\right| : \left|\frac{12\epsilon}{I_\tau g_q}\right|^2$$

Up type quark masses   k=2, 4, 8 modular forms

$$m_t : m_c : m_u \simeq \left[1 : \left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right) : \frac{3}{2}\left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right)^2 f_u^3 I_\tau |C_u|\right] I_\tau^4 f_u$$

$$I_\tau = 2\mathrm{Im}\,\tau$$

$$\tilde{Y}_i^{(6)} = g_d Y_i^{(6)} + Y_i'^{(6)}\,, \qquad \tilde{Y}_i^{(8)} = f_u Y_i^{(8)} + Y_i'^{(8)}\,, \qquad g_d \equiv \alpha_d/\alpha'_d \qquad f_u \equiv \alpha_u/\alpha'_u$$

A successful numerical result

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\|f_u\|$	$\arg[f_u]$	C_{u1}
$-0.3952 + i\,2.4039$	3.82	1.17	-0.677	1.72	3.21	1.68	127.3°	-0.07147

$$q = e^{2\pi i \tau}$$

8 real parameters + 2 phase

enough J_{CP}

Order 1 parameters, β_q/α_q , γ_q/α_q , g_d , f_u

$C_{u1} \sim (F/\Lambda^2) / \varepsilon^2$

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\|V_{us}\|$	$\|V_{cb}\|$	$\|V_{ub}\|$	$\|J_{CP}\|$	δ_{CP}
Fit	1.89	8.78	2.81	5.52	0.2251	0.0390	0.00364	2.94×10^{-5}	70.7°
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$	66.2° $^{+3.4°}_{-3.6°}$

8 output $\quad$ Nσ=2.0

5 Summary

- Quark mass hierarchy is obtained at nearby symmetric points $\tau=i\infty$ and ω thanks to the residual symmetry.

  Im τ is important for $\tau=i\infty$.

- Spontaneous CP violation ?

  τ is origin of both CP violation and mass hierarchy ?

- One modulus or multi-modulei ?

Flavor theory with modular forms is developing !

**Talks by M. Levy, X. Wang: 6.June,
J. Penedo: 7.June**

Back-up slides

Modular forms at τ=i

Z_2 symmetry

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Unitary transformation

$\tau = i$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} 1 \\ 1-\sqrt{3} \\ -2+\sqrt{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \sqrt{6}-3/\sqrt{2} \\ \sqrt{3/2} \end{pmatrix}$$

k=4

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}$$

F. Feruglio, V. Gherardi, A. Romanino and A. Titov, JHEP 05 (2021), 242; arXiv:2101.08718

Consider effective theories with Γ_N symmetry

$$\mathcal{L}_{\rm eff} \in f(\tau)\phi^{(1)}\cdots\phi^{(n)}$$

$f(\tau), \phi^{(I)}$: non-trivial rep. of Γ_N

Modular form of Level N

$$\tau \longrightarrow \tau' = \gamma\tau = \frac{a\tau+b}{c\tau+d}$$

Modular transformation

Automorphy factor

$$f_i(\tau) \longrightarrow f_i(\gamma\tau) = (c\tau+d)^k \rho(\gamma)_{ij} f_j(\tau)$$

modular form of weight k

k is modular weight

Representation matrix
for finite groups of N
Phase for N=1 full modular group

Modular transformation of chiral superfields

$$(\phi^{(I)})_i(x) \longrightarrow (c\tau+d)^{-k_I}\rho(\gamma)_{ij}(\phi^{(I)})_j(x)$$

CP invariance and Lepton model

CP transformation in modular invariant theory

P.P.Novichkov, J.T.Penedo, S.T.Petcov, A.V.Titov, JHEP 07(2019)165 [arXiv:1905.11970].

$$\tau \xrightarrow{\mathrm{CP}} -\tau^* , \qquad \psi(x) \xrightarrow{\mathrm{CP}} \overline{\psi}(x_P) , \qquad \mathbf{Y}_{\mathrm{r}}^{(\mathrm{k})}(\tau) \xrightarrow{\mathrm{CP}} \mathbf{Y}_{\mathrm{r}}^{(\mathrm{k})}(-\tau^*) = \mathbf{Y}_{\mathrm{r}}^{(\mathrm{k})*}(\tau)$$

bar denotes hermitian conjugation

We can construct CP invariant mass matrices in modular invariant flavor theory.

example

$$M_E(-\tau^*) = M_E(\tau)^* , \qquad M_\nu(-\tau^*) = M_\nu(\tau)^*$$

CP violation could be realized by fixing τ.

Modular transformation is the transformation of modulus τ

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

$$S : \tau \longrightarrow -\frac{1}{\tau}, \quad T : \tau \longrightarrow \tau + 1.$$

weight 2; k=2

3 modular forms

S $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

T $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$f_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} f_j(\tau)$$

S transformation

$$\begin{pmatrix} Y_1(-1/\tau) \\ Y_2(-1/\tau) \\ Y_3(-1/\tau) \end{pmatrix} = \tau^2 \rho(S) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix},$$

T transformation

$$\begin{pmatrix} Y_1(\tau+1) \\ Y_2(\tau+1) \\ Y_3(\tau+1) \end{pmatrix} = \rho(T) \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}.$$

$(c\tau + d)^k$ cτ+d = -τ

$(c\tau + d)^k$ cτ+d = 1

$$\rho(\mathrm{S}) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(\mathrm{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \omega = \exp(i\frac{2}{3}\pi)$$

Flavor symmetry acts non-linealy (Modular forms).

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} , \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} \equiv \begin{pmatrix} Y_1^{(8)} \\ Y_2^{(8)} \\ Y_3^{(8)} \end{pmatrix} = (Y_1^2 + 2Y_2 Y_3) \begin{pmatrix} Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \\ Y_2^2 - Y_1 Y_3 \end{pmatrix} , \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} \equiv \begin{pmatrix} Y_1'^{(8)} \\ Y_2'^{(8)} \\ Y_3'^{(8)} \end{pmatrix} = (Y_3^2 + 2Y_1 Y_2) \begin{pmatrix} Y_2^2 - Y_1 Y_3 \\ Y_1^2 - Y_2 Y_3 \\ Y_3^2 - Y_1 Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} = (Y_1^2 + 2Y_2 Y_3)\mathbf{Y}_{\mathbf{3}}^{(4)}$$

Modular group

Three matrices construct γ (Modular transformation)

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ f(z+1) = f(z)$$ z → z+1

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \ f\left(\frac{1}{-z}\right) = (-z)^k f(z)$$ z → -1/z

$$I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} : \ f\left(\frac{-z}{-1}\right) = (-1)^k f(z)$$ ⇒ k=even

$$S : \tau \longrightarrow -\frac{1}{\tau},$$
$$T : \tau \longrightarrow \tau + 1.$$

τ : modulus

$$S^2 = 1, \qquad (ST)^3 = 1.$$

generate infinite discrete group PSL(2,Z)