Stability of multibrane models

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Outline

- 5D models with branes
  - multibrane models
- Background configurations with Goldberger-Wise scalar
- Radion(s)
  - equations of motion and boundary conditions
  - spectrum of radions
- Conditions for stability of multibrane models
- Relation between radions and distances (between branes)
- Number of light radions
- Summary

Introduction

- **5D models with branes are very popular**
  - **Randall Sundrum 1999**
    25 years with more than 1 citation per day
- **Approach to hierarchy problem**
- **Interesting phenomenology**
  - **field profiles in 5th dimension crucial for interactions**
  - **localization of different fields on different branes**
- **AdS/CFT interpretation**
- **Approach to phase transitions**
- **Dark matter**
- **...**

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- Approach to phase transitions
- Dark matter
- ...
- More branes $\rightarrow$ more possibilities

5D models with branes

Space-time is a warped product $\mathcal{M}^5 = \mathcal{M}^4 \times S^1/\mathbb{Z}_2$

with two orbifold-branes (fixed hyper-planes of $\mathbb{Z}_2$)
one (UV) located at $y = y_1$ and another (IR) located at $y = y_2$

and additional brane(s) in between, located at y_{I_i}, $i = 1, \ldots$

All fields have well defined parities under $\mathbb{Z}_2$

$$S = \int_{\mathcal{M}^5} \mathrm{d}^5 x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} (\nabla \Phi)^2 - V(\Phi) - \sum_j \delta(y - y_j) U_j(\Phi) \right]$$

Φ is the Goldberger-Wise field $\qquad$ $j = 1, 2, I_1, \ldots$ counts all branes

We are interested in background solutions of the form
(conformally flat coordinates)

$$\mathrm{d}s^2 = e^{-2A(y)} \left(\eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2 \right) \qquad \Phi = \phi(y)$$

All results may be easily translated to often used coordinates
$\mathrm{d}s^2 = e^{-2\widetilde{A}(z)} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2$

5D models with branes

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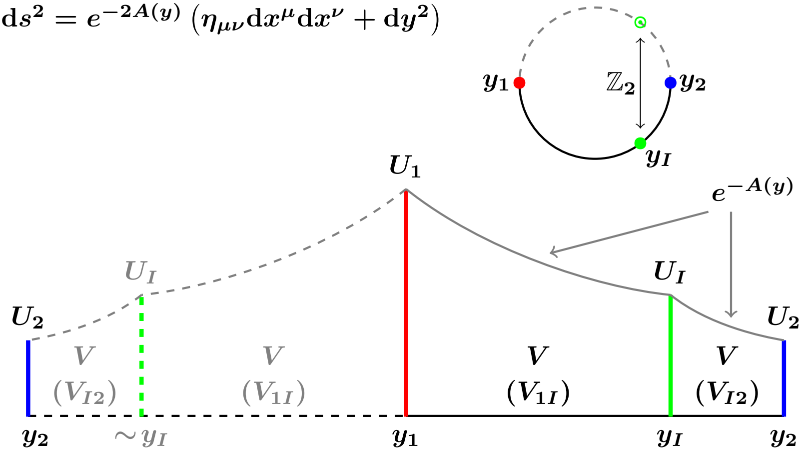

$$S = \int_{\mathcal{M}^5} \mathrm{d}^5x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}\left(\nabla\Phi\right)^2 - V(\Phi) - \sum_j \delta(y-y_j)U_j(\Phi)\right]$$

Bulk equations of motion (for one interbrane section):

$$\begin{aligned} A'' + (A')^2 - \frac{\kappa^2}{3}(\phi')^2 &= 0 \\ \phi'' - 3A'\phi' - e^{-2A}\,V' &= 0 \\ (A')^2 - \frac{\kappa^2}{12}(\phi')^2 + \frac{\kappa^2}{6}e^{-2A}\,V &= 0 \end{aligned}$$

Boundary conditions at j-th brane $\qquad [f]_j := f(y_j^+) - f(y_j^-)$

$$[A']_j = \frac{\kappa^2}{3}\,e^{-A}\,U_j \qquad\qquad [\phi']_j = e^{-A}\,U_j'$$

which at orbifold branes (because of $\mathbb{Z}_2$ symmetry) reduce to:

$$\lim_{y\to y_i^\pm} A' = \pm\frac{\kappa^2}{6}\,e^{-A}\,U_i \qquad\qquad \lim_{y\to y_i^\pm} \phi' = \pm\frac{1}{2}\,e^{-A}\,U_i'$$

5D models with branes – background solutions

Procedure do obtain background solution satisfying all EoM all BC

- Two BC and 3rd bulk EoM at the "first" brane
  $\Rightarrow A'(y_1^+)$, $\phi(y_1^+)$ and $\phi'(y_1^+)$ ($A(y_1^+)$ overall normalization)
- integration of 2 bulk EoM in the first interbrane section
  $\Rightarrow A(y)$ and $\phi(y)$ in that section
- Position of the first intermediate brane, y_{I_1}, is not a free parameter. $A(y_{I_1})$ and $\phi(y_{I_1})$ must satisfy the condition:
  $$e^A\left(4A'U_{I_1}(\phi)-\phi' U'_{I_1}(\phi)\right)+4\kappa^2 U^2_{I_1}(\phi)-\frac{1}{2}(U'_{I_1}(\phi))^2+[V(\phi)]_{I_1}=0$$
  $\Rightarrow A(y_{I_1})$, $A'(y_{I_1}^-)$, $\phi(y_{I_1})$, $\phi'(y_{I_1}^-)$
- 2 BC at this brane $\Rightarrow A'(y_{I_1}^+)$, $\phi'(y_{I_1}^+)$
- ... same procedure at each intermediate brane ...
- two BC to be fulfilled at the "last" brane but only one parameter to adjust – the position of the "last" brane y_2
- One tuning of parameters necessary to have any solution

5D models with branes – background solutions

Background solutions:

- $A(y)$ and $\phi(y)$ which satisfy bulk equations of motion and all boundary conditions
- One tuning of parameters necessary to have any solution
  - the effective 4D cosmological constant must vanish
- For a given background solution positions of all branes are fixed
- $\Rightarrow$ all distances between branes are fixed

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- $\Rightarrow$ all distances between branes are fixed

It is not guaranteed that a given solution is stable

- It is necessary to consider perturbations of the solution
- All modes of such perturbations must have positive (at least non-negative) masses squared

Radions

Scalar perturbations of a given back-ground solution

$$\begin{aligned} \mathrm{d}s^2 &= e^{-2A(y)}\{[(1+2f_1(x,y))\,\eta_{\mu\nu} + \partial_\mu\partial_\nu e(x,y)]\,\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &\qquad + (1+2f_2(x,y))\,\mathrm{d}y^2\} \\ \Phi &= \phi(y) + f_3(x,y) \end{aligned}$$

Most general scalar perturbations (in longitudinal gauge)

$e(x,y)$ must be included in models with intermediate brane(s)

[Pilo, Rattazzi, Zaffaroni 2000; Lee, Nakai, Suzuki 2022]

We expand all 5D perturbations in 4D modes

$$\begin{aligned} f_k(x,y) &= \sum_n f_n(x) F_k^n(y), \quad k = 1,2,3 \\ e(x,y) &= \sum_n f_n(x) E^n(y) \end{aligned}$$

We consider EoM for 4D mode(s) with mass squared m^2 and profile functions $F_1(y)$, $F_2(y)$, $F_3(y)$, $E(y)$

Radions

Off-diagonal Einstein equations relate the profiles F_1, F_2, F_3, E

$$\begin{aligned} 2F_2 + 4F_1 - E'' + 3A'E' &= 0 \\ 3F_1' + 3A'F_2 + \kappa^2\phi' F_3 &= 0 \end{aligned}$$

Bulk EoM for $E(y)$ is automatically fulfilled (due to other EoM)

$\Rightarrow$ only one independent scalar bulk perturbation

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One tower of scalar 4D KK modes

- infinitely many radions (scalar KK modes)
  - similarly to infinitely many gravitons (tensor KK modes)
- the number of radions does not depend on the number of branes

Radions

It is convenient to define $Q := e^{-2A}\left(F_1 + \frac{1}{2}A'E'\right)$

$$p := \frac{3}{2\kappa^2}\frac{e^A}{(\phi')^2}$$
$$q := e^A$$

Bulk EoM $\boxed{-(p\,Q')' + q\,Q = m^2 p\,Q}$

Radions

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Bulk EoM $$-(p\,Q')' + q\,Q = m^2 p\,Q$$

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Boundary conditions at j-th brane (non-trivial E' plays role)

$$\left[\frac{Q'}{\phi'}\right]_j = \frac{U_j'}{U_j}\,[Q]_j$$

$$B_j\left(\left\langle\frac{Q'}{\phi'}\right\rangle_j - \left[\frac{Q'}{\phi'}\right]_j\frac{\langle\phi'\rangle_j}{[\phi']_j}\right) + m^2\left[\frac{Q}{\phi'}\right]_j = 0$$

$$B_j := e^{-A}U_j'' - [A']_j - \left[\frac{\phi''}{\phi'}\right]_j \qquad \langle f\rangle_j = \frac{1}{2}\left(f(y_j^-) + f(y_j^+)\right)$$

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For orbifold branes boundary conditions simplify to

$$0 = 0 \qquad \frac{1}{2}B_i Q'(y_i^\pm) \pm m^2 Q(y_i^\pm) = 0 \qquad +(-) \text{ for } i = 1(2)$$

Conditions for stability of multibrane models

Background configuration is stable (unstable)
$\Leftrightarrow$
the smallest radion mass squared is positive (negative)

Using solution of bulk EoM with $m^2 = 0$ satisfying BC at the orbifold brane at y_i:

$$\begin{aligned} Q_0(y) &= e^{-2A(y)} + 2A'(y)e^{A(y)} \int_{y_i}^{y} e^{-3A(y')} \mathrm{d}y' \\ Q_0'(y) &= \frac{2}{3}\kappa^2 (\phi'(y))^2 e^{A(y)} \int_{y_i}^{y} e^{-3A(y')} \mathrm{d}y' \end{aligned}$$

one can prove that the necessary and sufficient condition for the existence of at least one massless radion is

$$\prod_j B_j \phi'(y_j^-)\phi'(y_j^+) = 0$$

Analogous result for 2-brane models e.g. [Lesgourgues, Sorbo 2004]

Conditions for stability of multibrane models

It is also possible to show that the smallest radion m^2 is given by

$$m^2 = \min_Q \frac{\int p(Q')^2 + \int qQ^2 + \sum_I q\frac{[Q]_I^2}{[A']_I}}{\int pQ^2 + \frac{2}{B_1}pQ^2\Big|_{y_1^+} + \frac{2}{B_2}pQ^2\Big|_{y_2^-} + \sum_I \frac{3e^A}{2\kappa^2}\frac{1}{B_I}\left[\frac{Q}{\phi'}\right]_I^2}$$

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At least one radion is tachyonic if any of the following conditions is satisfied

- $B_j < 0$ at any brane
- $[A']_I \propto U_I < 0$ negative tension of any intermediate brane
- $\phi'(y) = 0$ at any bulk point

The last point is subtle because $p = \frac{3e^A}{2\kappa^2(\phi')^2}$ diverges for $\phi' = 0$.

It is necessary to use the Mukhanov-Sasaki variable $v(y)$ satisfying in this case: $Q = \lambda^{-1}e^{-2A}(\phi')^2(A')^{-1}(e^{3A/2}A'(\phi')^{-1}v)'$ for which bulk EoM is regular also at points at which $\phi' = 0$

Conditions for stability of multibrane models

Necessary and sufficient conditions for stability of (multibrane) configuration

- each intermediate brane must have positive tension

$$U_{I_i}(\phi(y_{I_i})) > 0$$

- at each brane

$$e^{-A(y_j)}U_j''(\phi(y_j)) - [A']_j - \left[\frac{\phi''}{\phi'}\right]_j > 0$$

- everywhere

$$\phi'(y) \neq 0$$

Background configurations DO NOT depend on quantities U_j''

For any given background solution decreasing of any U_j'' may result in arbitrarily light radion and finally destabilize the system

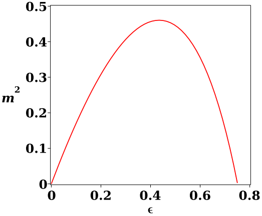
Upper bounds on radion(s) masses are obtained in the stiff potential approximation i.e. $U_j'' \to \infty$

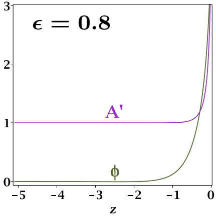
However, even $U_j'' \to \infty$ DOES NOT guarantee stability

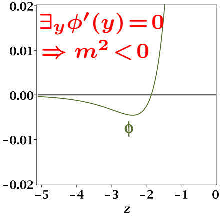
Conditions for stability of multibrane models

Example: simple 2-brane model with quadratic potentials

$$V(\Phi) = -\frac{6k^2}{\kappa^2} - \frac{1}{2}\epsilon k^2 \Phi^2 \qquad U_i(\Phi) = \lambda_i + \mu_i(\Phi - v_i)^2$$







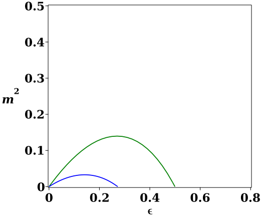
$$\frac{e^{-A}U_2''}{|[A']_2|} \approx 10^5 \gg 1$$

$$B_j = e^{-A}U_j'' - [A']_j - \left[\frac{\phi''}{\phi'}\right]_j$$

$\sim$ stiff potential approximation

$$\frac{e^{-A}U_2''}{|[A']_2|} \approx 20$$

$$\frac{e^{-A}U_2''}{|[A']_2|} \approx 10$$



Relation between radions and distances

- **Distance between points A and B with coordinates (x, y_A) and (x, y_B) is given by:** $d_{AB} = \int_{y_A}^{y_B} \sqrt{g_{55}}\,\mathrm{d}y$
- **In our metric**

$$\sqrt{g_{55}} = \sqrt{e^{-2A(y)}\left(1 + 2f_2(x,y)\right)} \approx e^{-A(y)}\left(1 + f_2(x,y)\right)$$

$$\boxed{d_{AB} \approx \int_{y_A}^{y_B} e^{-A(y)} + \sum_k f_k(x) \int_{y_A}^{y_B} e^{-A(y)} F_2^k(y)}$$

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Excitations of each radion field change distances between fixed points on two different branes.

Changes depend on t and $\vec{x}$

Unless for given k, y_A and y_B $\quad \int_{y_A}^{y_B} e^{-A(y)} F_2^k(y) = 0$

Typically lighter radions (less zeros of $F_2(y)$) have stronger impact on distances than heavier radions (more zeros of $F_2(y)$)

We should express $F_2(y)$ in terms of the radion field $Q(y)$

Relation between radions and distances

Excitations of one radion with the profile $Q_{m^2}(y)$ change distances as follows:

$$\Delta d_{12} = -2 f_{m^2}(t,\vec{x}) \int_{y_1}^{y_2} e^{A}\, Q_{m^2}\, \mathrm{d}y$$

$$\Delta d_{1I(I2)} = f_{m^2}(t,\vec{x}) \left(-2 \int_{y_1(y_I)}^{y_I(y_2)} e^{A}\, Q_{m^2}\, \mathrm{d}y + (-)\, e^{A(y_I)} \frac{[Q_{m^2}]_I}{[A']_I} \right)$$

In general, the distance between fixed points on two different branes is changed by a radion even if (the integral of) its profile between those branes vanishes

Exception: the total distance between the "end-of-the-world" orbifold branes

$$e^{A(y_I)} \frac{[Q_{m^2}]_I}{[A']_I} = e^{2A(y_I)} \frac{3[Q_{m^2}]_I}{\kappa^2 U_I}$$

Brane with smaller tension U_I is more easy to bend (distance becomes x-dependent)

Non-zero radion field Q_{m^2}

- $F_2 \neq 0$
  g_{55} is changed $\Rightarrow$ distances between branes are affected
- $F_1 \neq 0$ $E \neq 0$
  $g_{\mu\nu}$ is changed $\Rightarrow$ distances along branes are also affected

Distance between points C and D with coordinates (t, x^1_C, x^2, x^3, y) and (t, x^1_D, x^2, x^3, y)

$$
\begin{aligned}
d_{CD} \approx \; & e^{-A(y)}(x^1_D - x^1_C) \\
& + e^{+A(y)} \left[\frac{Q_{m^2}}{A'}\right]_j \left[\frac{1}{A'}\right]_j^{-1} \int_{x^1_C}^{x^1_D} f_{m^2}(t, \vec{x})\, \mathrm{d}x^1 \\
& + \frac{1}{2} e^{-A(y)} E_{m^2}(y) \int_{x^1_C}^{x^1_D} \partial_{xx} f_{m^2}(t, \vec{x})\, \mathrm{d}x^1
\end{aligned}
$$

Excitations of (each) radion field modify distances along branes

Number of light radions

How many radions may be light?

- **number of massless radions in 2-brane models:**
  - **0**
    if $B_1 \cdot B_2 \cdot \phi'(y_1^+) \cdot \phi'(y_2^-) \neq 0$
  - **1**
    if $B_1 \cdot B_2 \cdot \phi'(y_1^+) \cdot \phi'(y_2^-) = 0$ but at least one of $B_i \neq 0$
  - **2**
    if $B_1 = B_2 = 0$

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  - **2**
    if $B_1 = B_2 = 0$
- **number of massless radions in 3-brane models**
  - **0**
    if $B_1 \cdot B_I \cdot B_2 \cdot \phi'(y_1^+) \cdot \phi'(y_I^-) \cdot \phi'(y_I^+) \cdot \phi'(y_2^-) \neq 0$
  - **1**
    if $B_1 \cdot B_I \cdot B_2 \cdot \phi'(y_1^+) \cdot \phi'(y_I^-) \cdot \phi'(y_I^+) \cdot \phi'(y_2^-) = 0$
    but at least two of $B_j \neq 0$
  - **2**
    if exactly two $B_j = 0$
    one localized in one interbrane space if $B_I \cdot \phi'(y_I^\pm) = 0$
  - **4**
    if $B_1 = B_I = B_2 = 0$
    two localized in interbrane spaces

Summary

- 5D multibrane models
  - one fine tuning necessary for solutions with flat 4D sections
  - all distances between branes are fixed by a given background solution – warp factor $e^{-2A(y)}$ and GW scalar $\phi(y)$
- Radions: 4D modes of scalar perturbations
  - one more 5D scalar perturbation must be taken into account (as compared to 2-brane models)
  - still one (infinite) tower of KK modes – 4D radions
- Bulk EoM and especially BC for radions
  are generalized in the presence of intermediate branes
- Conditions for stability of a given background configuration

  $$U_{I_i} > 0 \qquad e^{-A}U_j'' - [A']_j - \left[\frac{\phi''}{\phi'}\right]_j > 0 \qquad \phi' \neq 0$$

  - background configurations do not depend on U_j''
  - radion(s) may be light or even tachyonic also for $U_j'' \to \infty$
- Each radion field excitation deforms branes
  changing distances in all directions: between and along branes
  - in most cases in a non-trivial way
- There may be up to 2^{N-1} massless (light) radions in a model with N branes