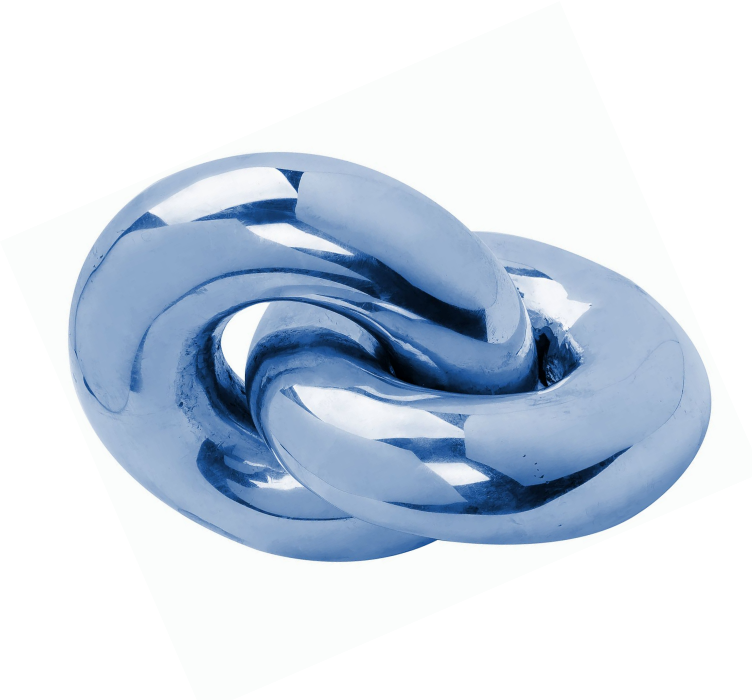
(Finite) modular symmetries and the strong CP problem

in collaboration with S.T. Petcov [2404.08032]

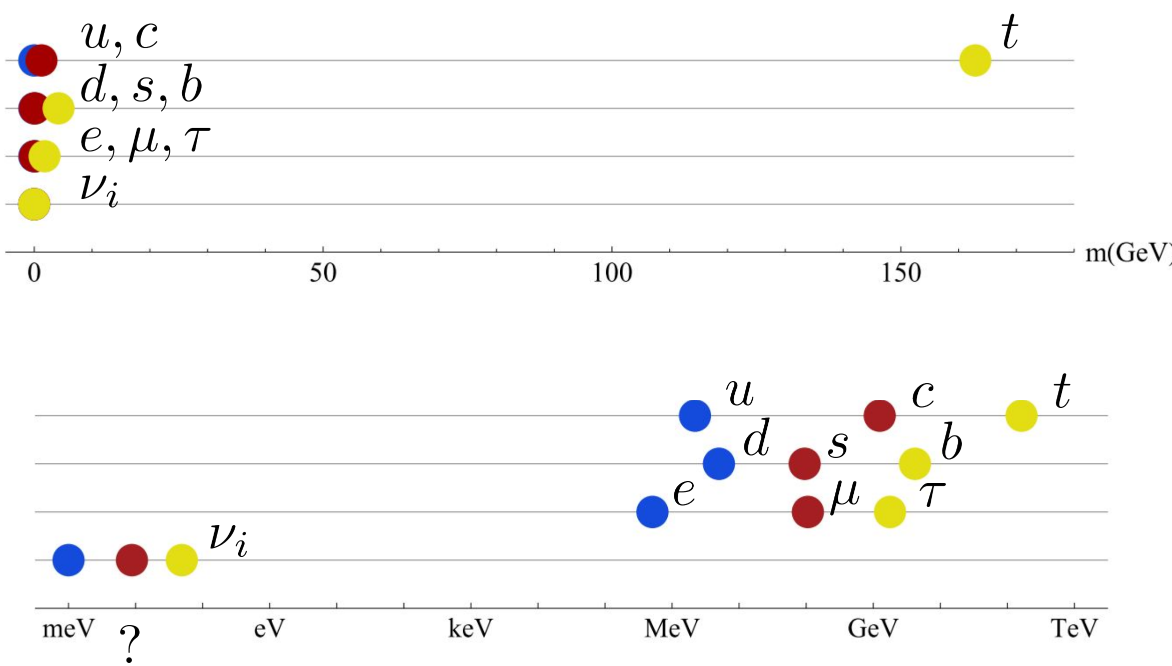


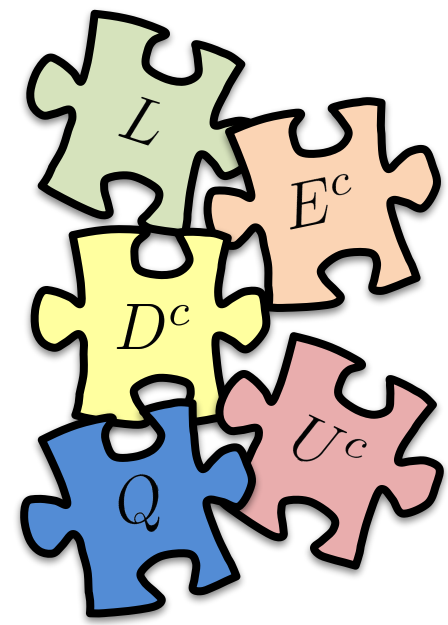
João Penedo (INFN, Roma Tre)

7 June 2024

PLANCK @ Lisbon

The flavour puzzle

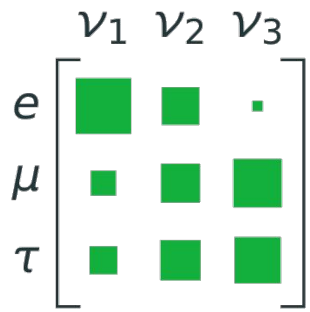


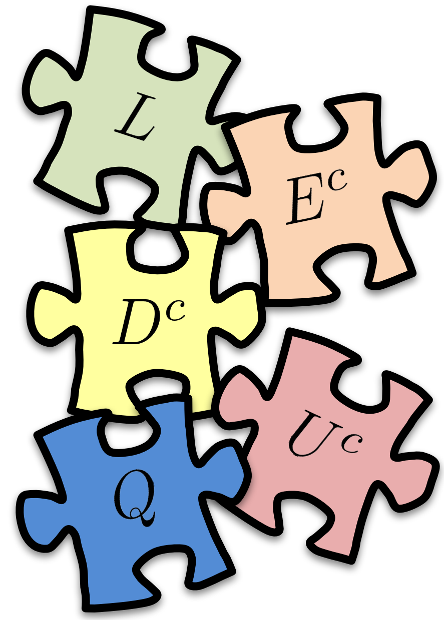


adapted from R. Toorop's PhD thesis

The flavour puzzle

$U_{\mathrm{PMNS}} \sim$





adapted from P. Novichkov's slides at PASCOS 2021

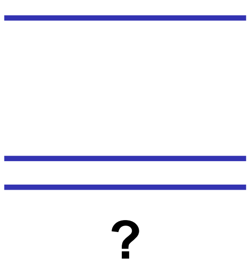
3ν flavour paradigm

Masses: ordering

$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$

Normal ordering (NO)

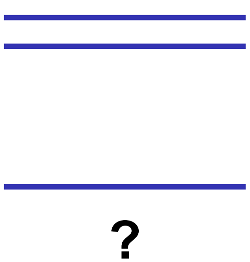
$m_1 < m_2 < m_3$



vs.

Inverted ordering (IO)

$m_3 < m_1 < m_2$



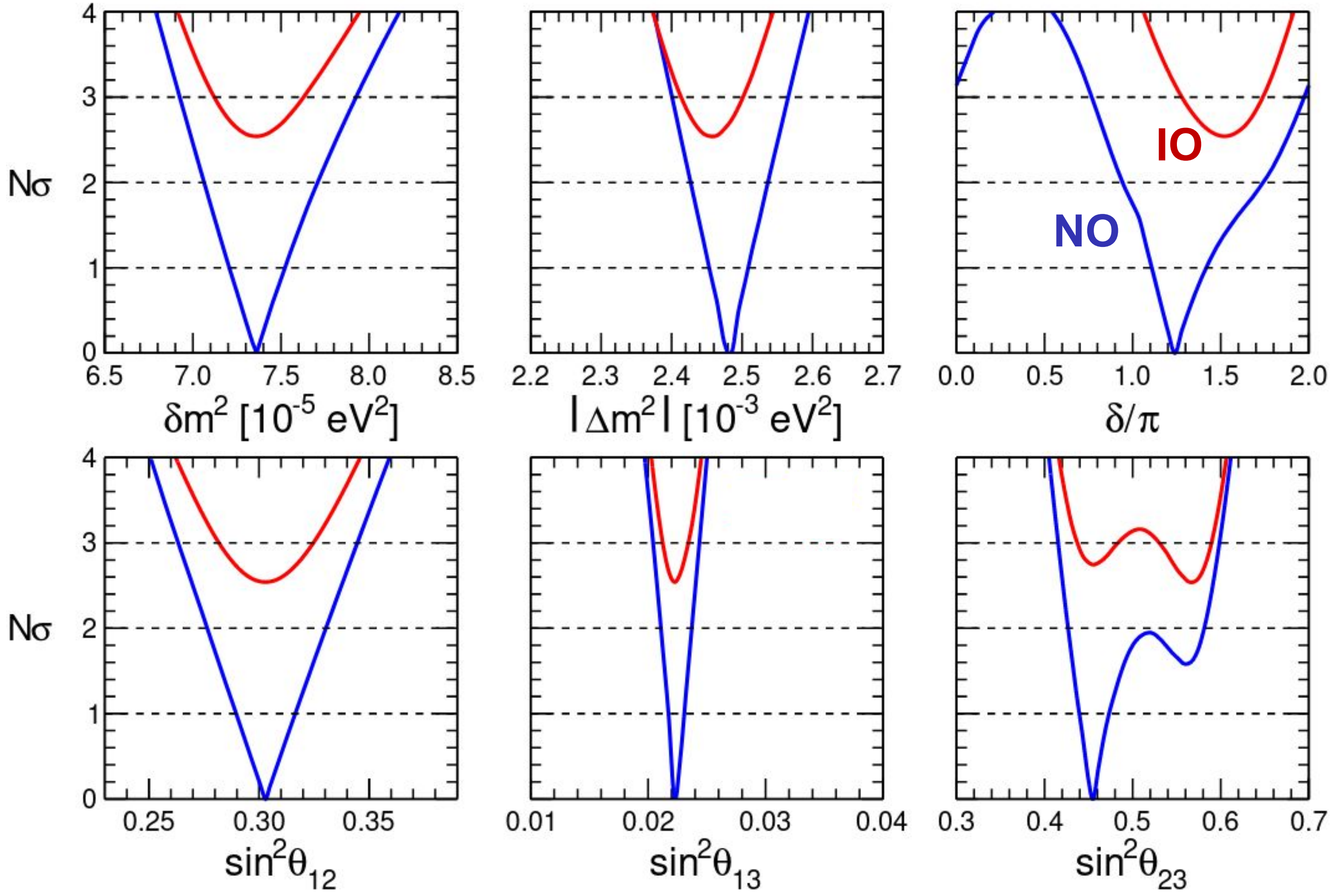
Mixing matrix parameterisation

$c_{ij} \equiv \cos\theta_{ij},\ s_{ij} \equiv \sin\theta_{ij}$

$$U_{\text{PMNS}} = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i\alpha_{21}/2} & \\ & & e^{i\alpha_{31}/2} \end{pmatrix}$$

3ν flavour paradigm

from Capozzi et al. [2107.00532],
see also València [2006.11237], NuFIT [2007.14792]

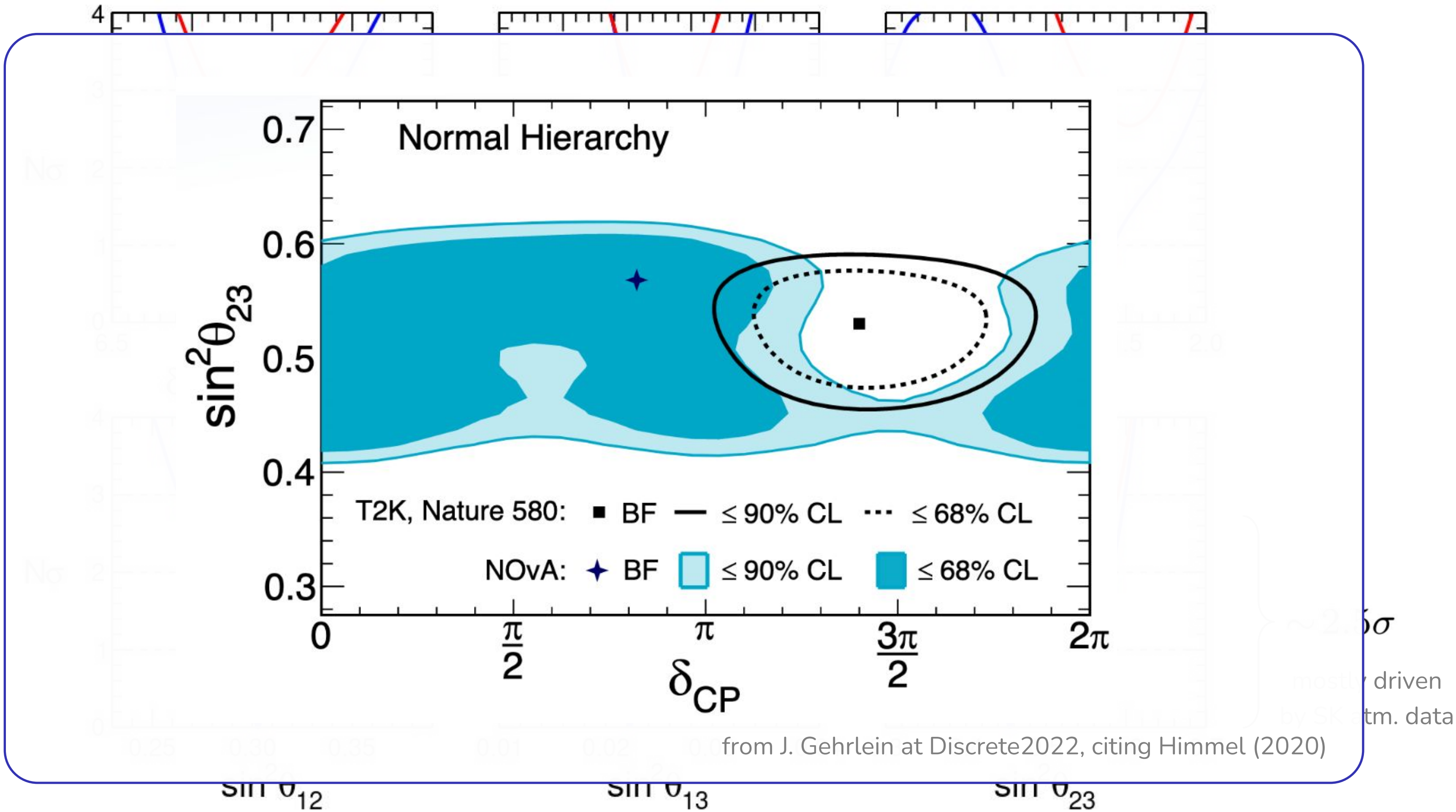


$\sim 2.5\sigma$

mostly driven by SK atm. data

3ν flavour paradigm

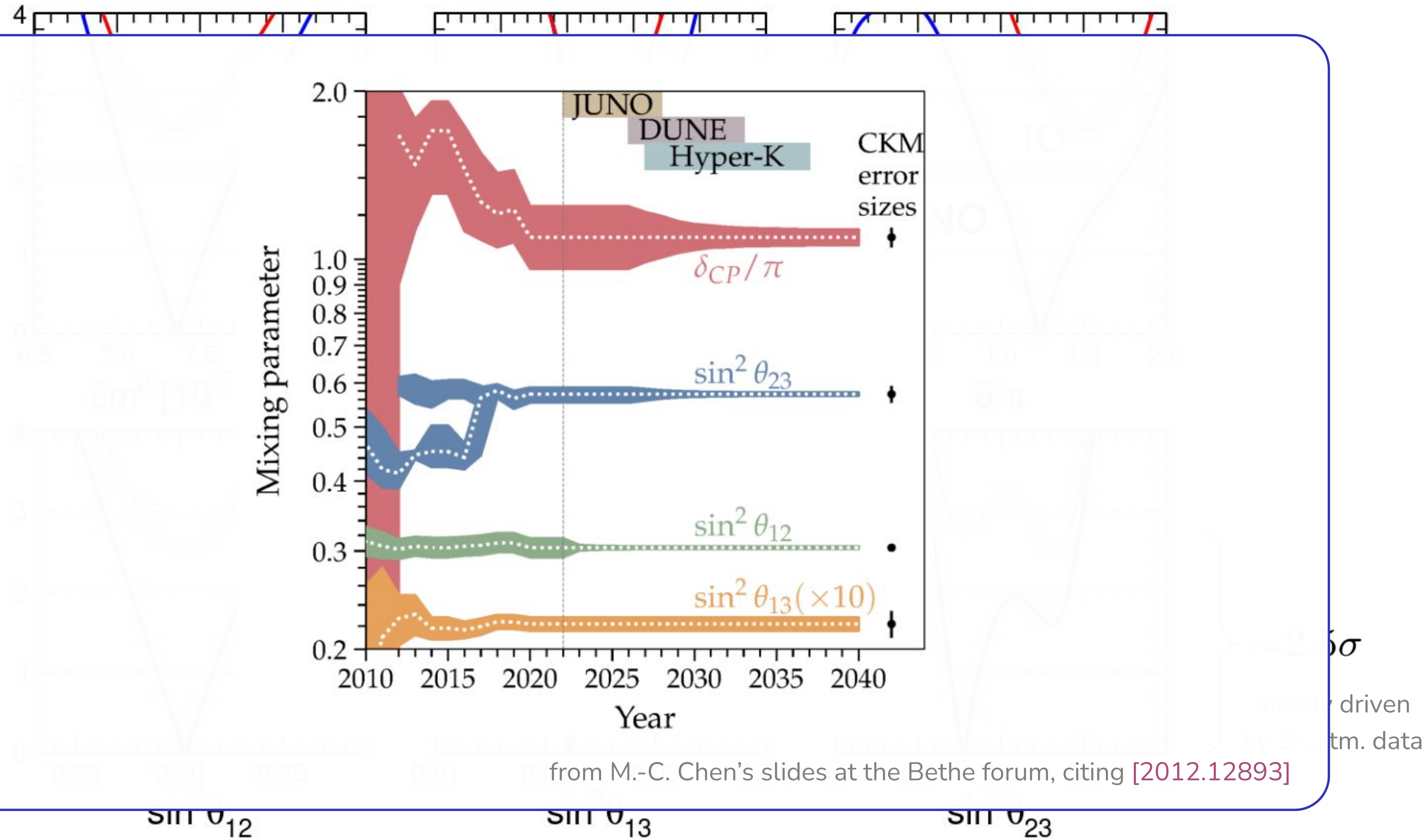
from Capozzi et al. [2107.00532],
see also València [2006.11237], NuFIT [2007.14792]



from J. Gehrlein at Discrete2022, citing Himmel (2020)

3ν flavour paradigm

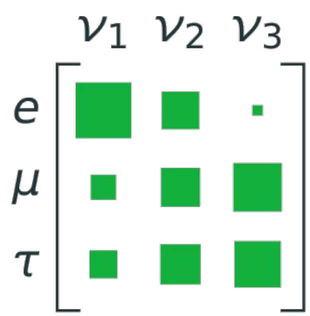
from Capozzi et al. [2107.00532],
see also València [2006.11237], NuFIT [2007.14792]



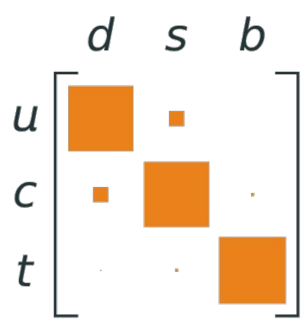
from M.-C. Chen's slides at the Bethe forum, citing [2012.12893]

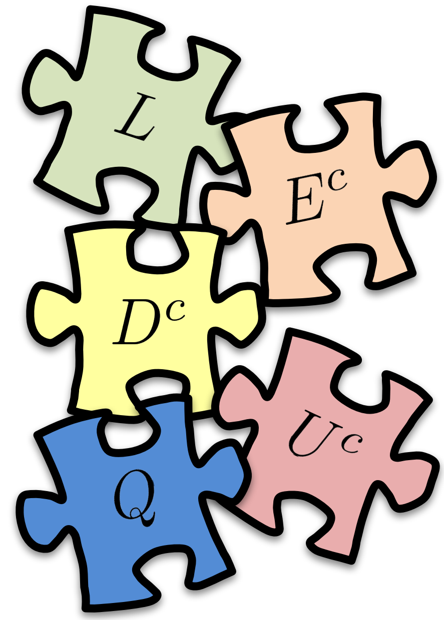
The flavour puzzle

$U_{\mathrm{PMNS}} \sim$



$V_{\mathrm{CKM}} \sim$





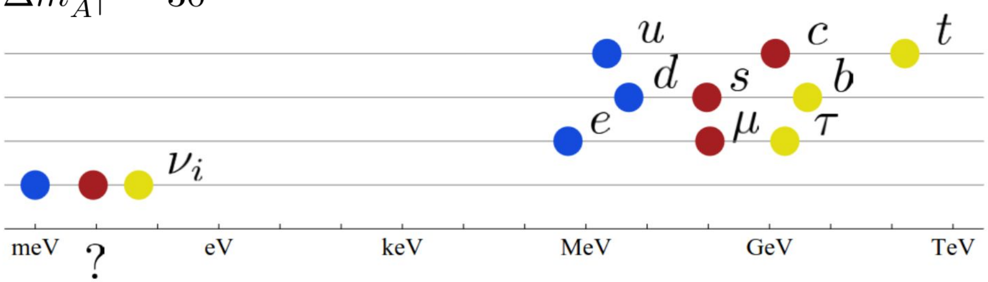
Recall talk by G. Martinelli

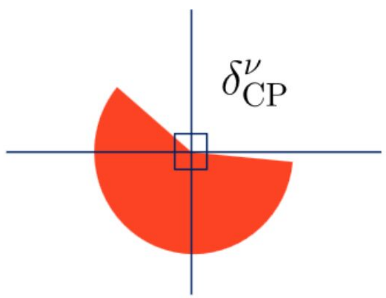
adapted from P. Novichkov's slides at PASCOS 2021

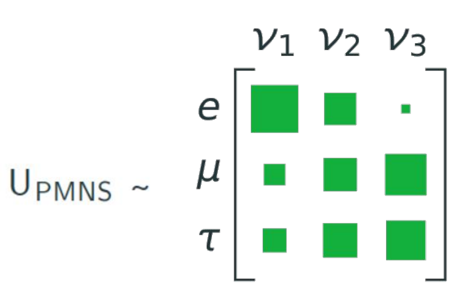
Motivation

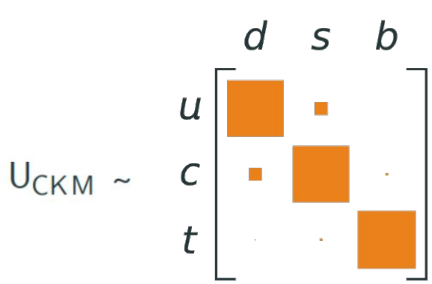
In search of an organising principle...

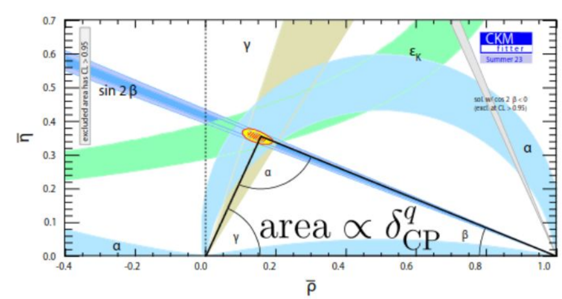
$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$





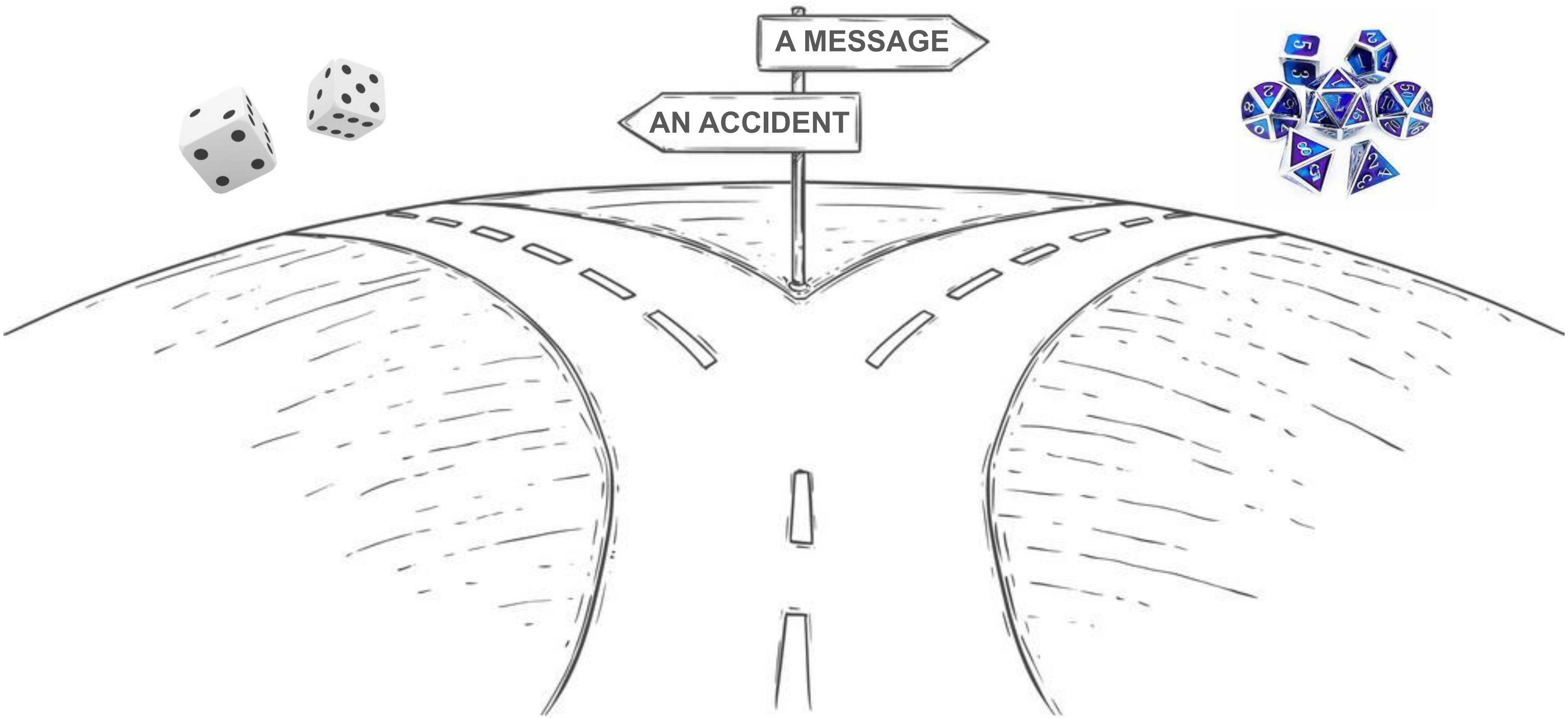




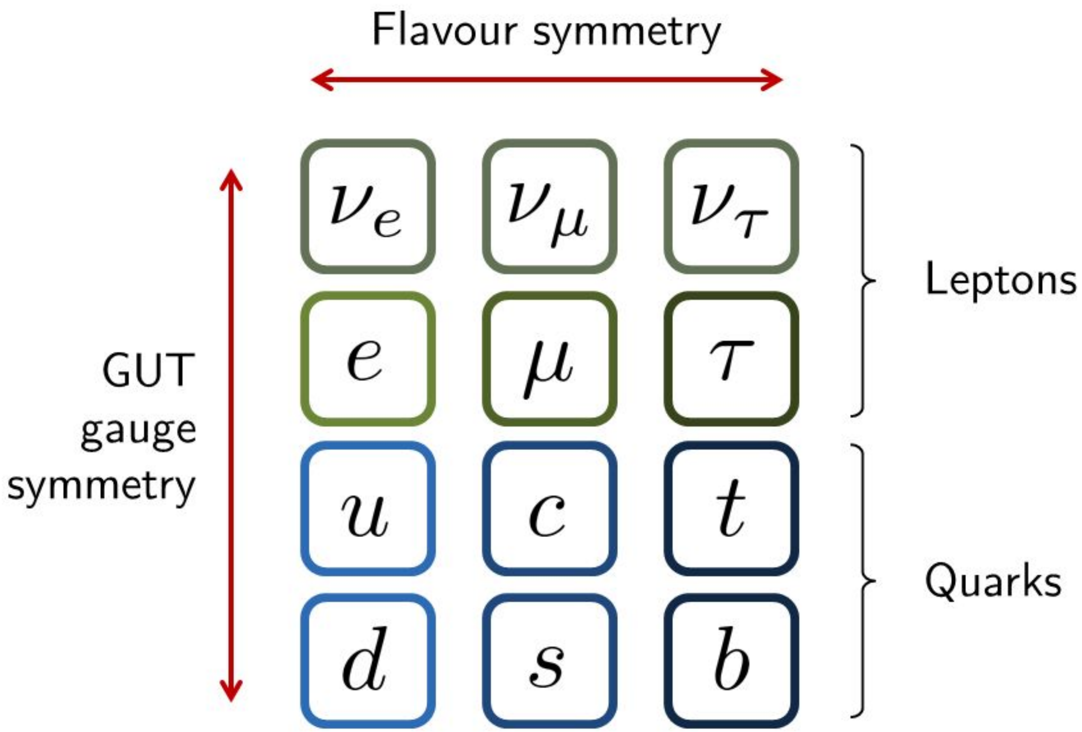


Motivation

Is there an organising principle?



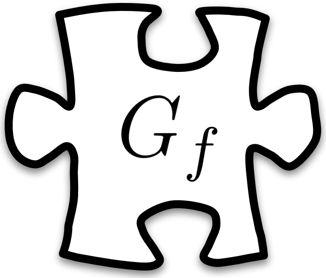
Flavour symmetries



For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010),
King and Luhn (2013), Petcov (2017), Feruglio and Romanino (2019), Ding and Valle (2024)

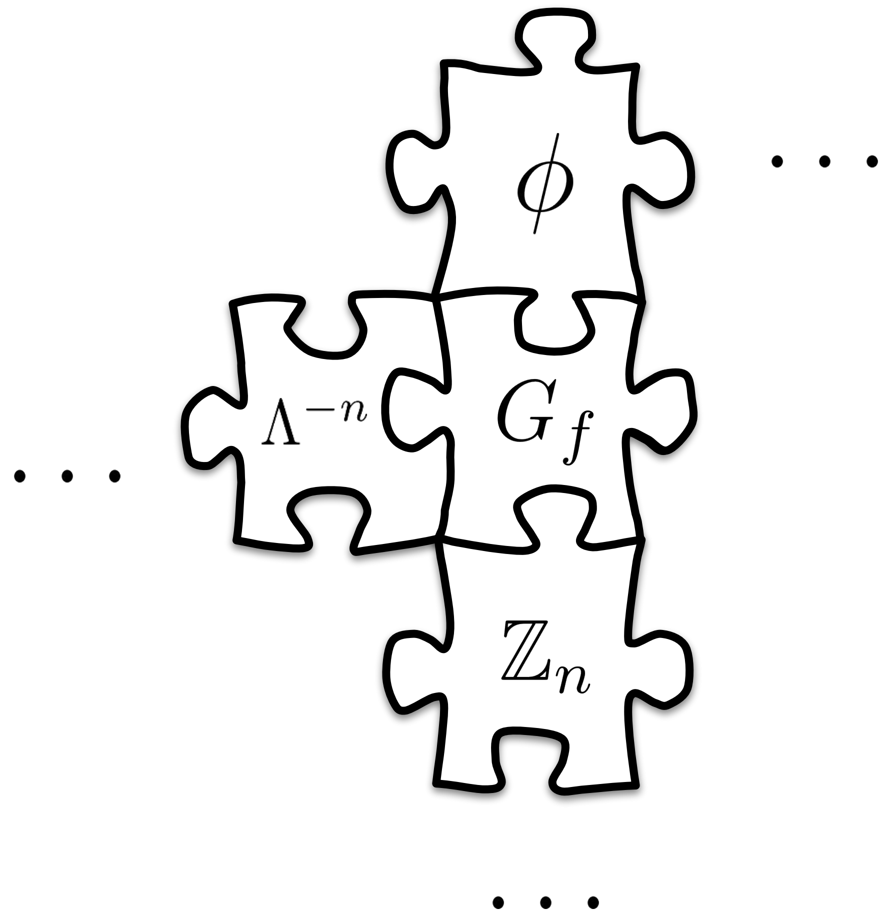
Flavour symmetries

Non-Abelian discrete flavour symmetries

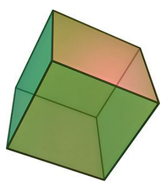


model-independent approaches relying on residual symmetries constrain mixing and the Dirac phase

Problems with the usual approach

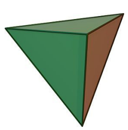


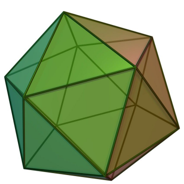
A reversal of the usual logic

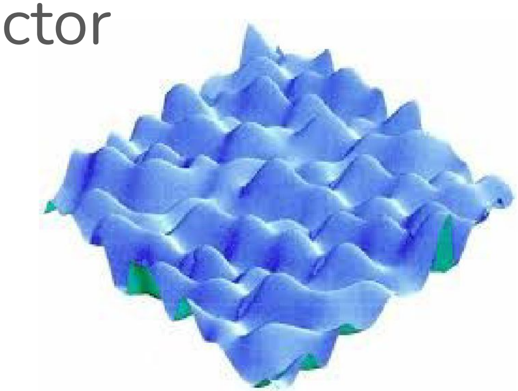


Symmetry group
and representations

Symmetry-breaking
sector

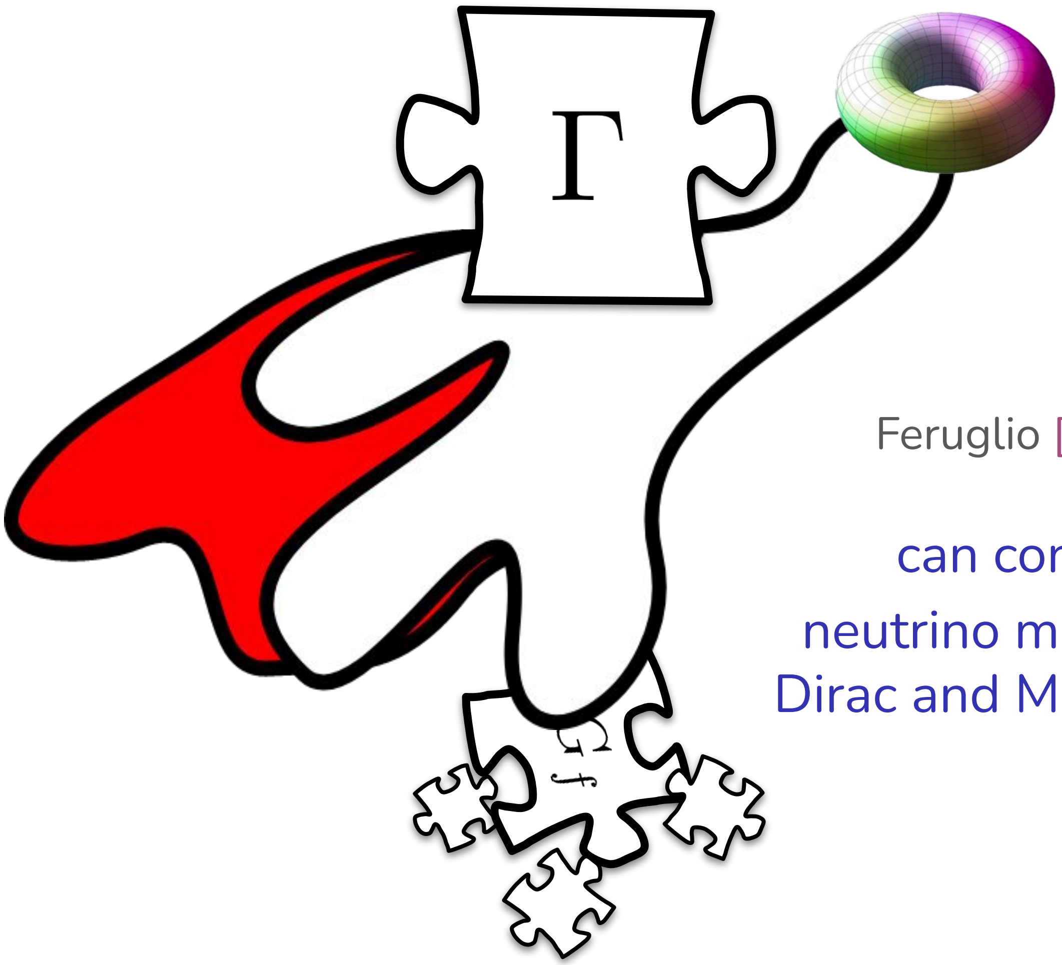






see Feruglio, PoS DISCRETE 2020-2021, 007

Modular symmetry to the rescue!



Feruglio [1706.08749]

can constrain all:

neutrino masses, mixing,
Dirac and Majorana phases

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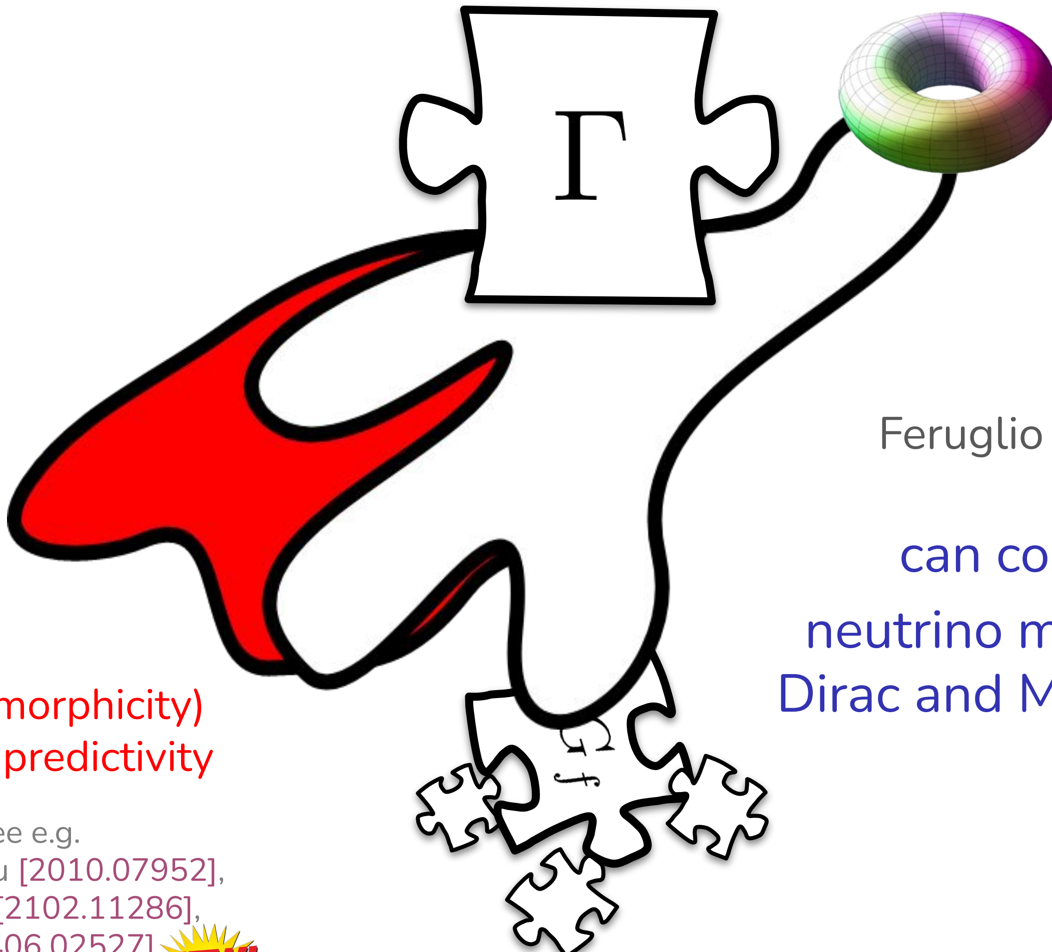


Feruglio [1706.08749]

can constrain all:

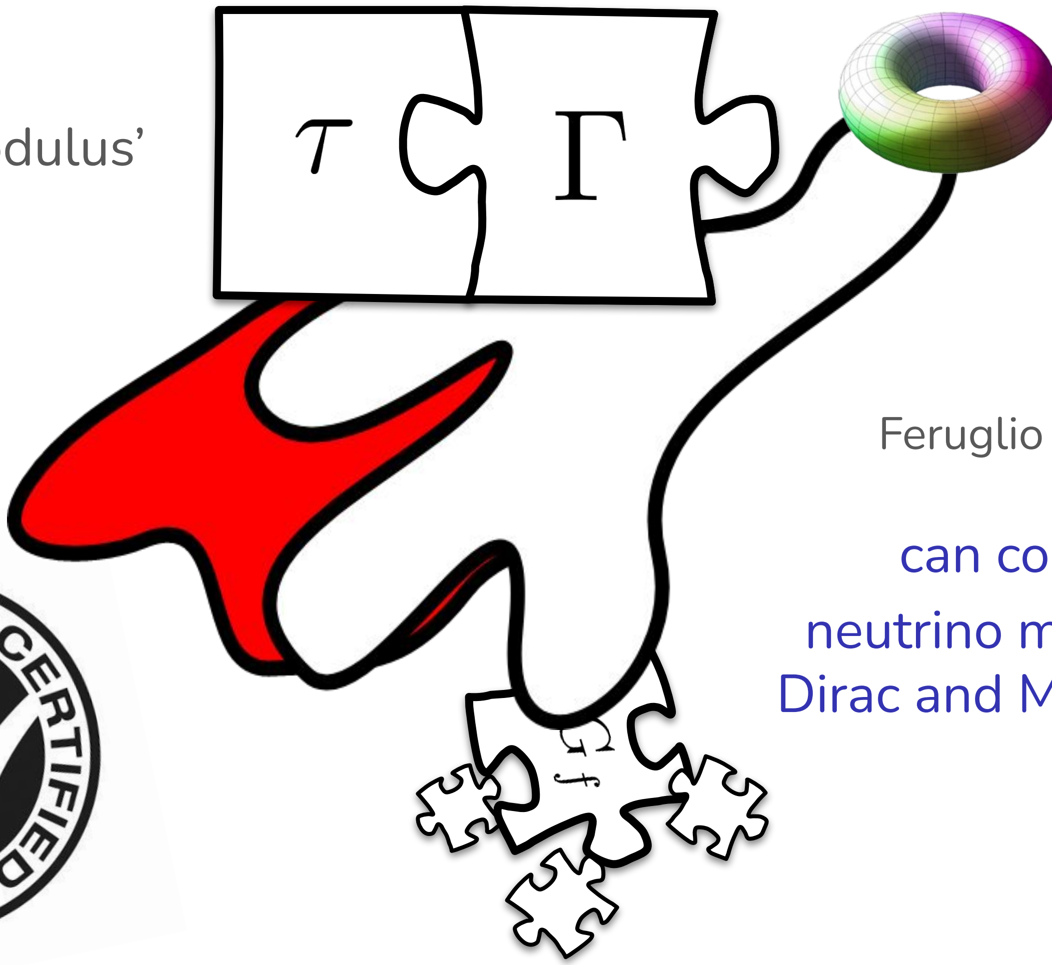
neutrino masses, mixing,
Dirac and Majorana phases

SUSY (holomorphicity)
required for predictivity

...but see e.g.
Ding, Feruglio, Liu [2010.07952],
Almumin et al. [2102.11286],
Qu, Ding [2406.02527] NEW!

Modular symmetry to the rescue!

‘modulus’



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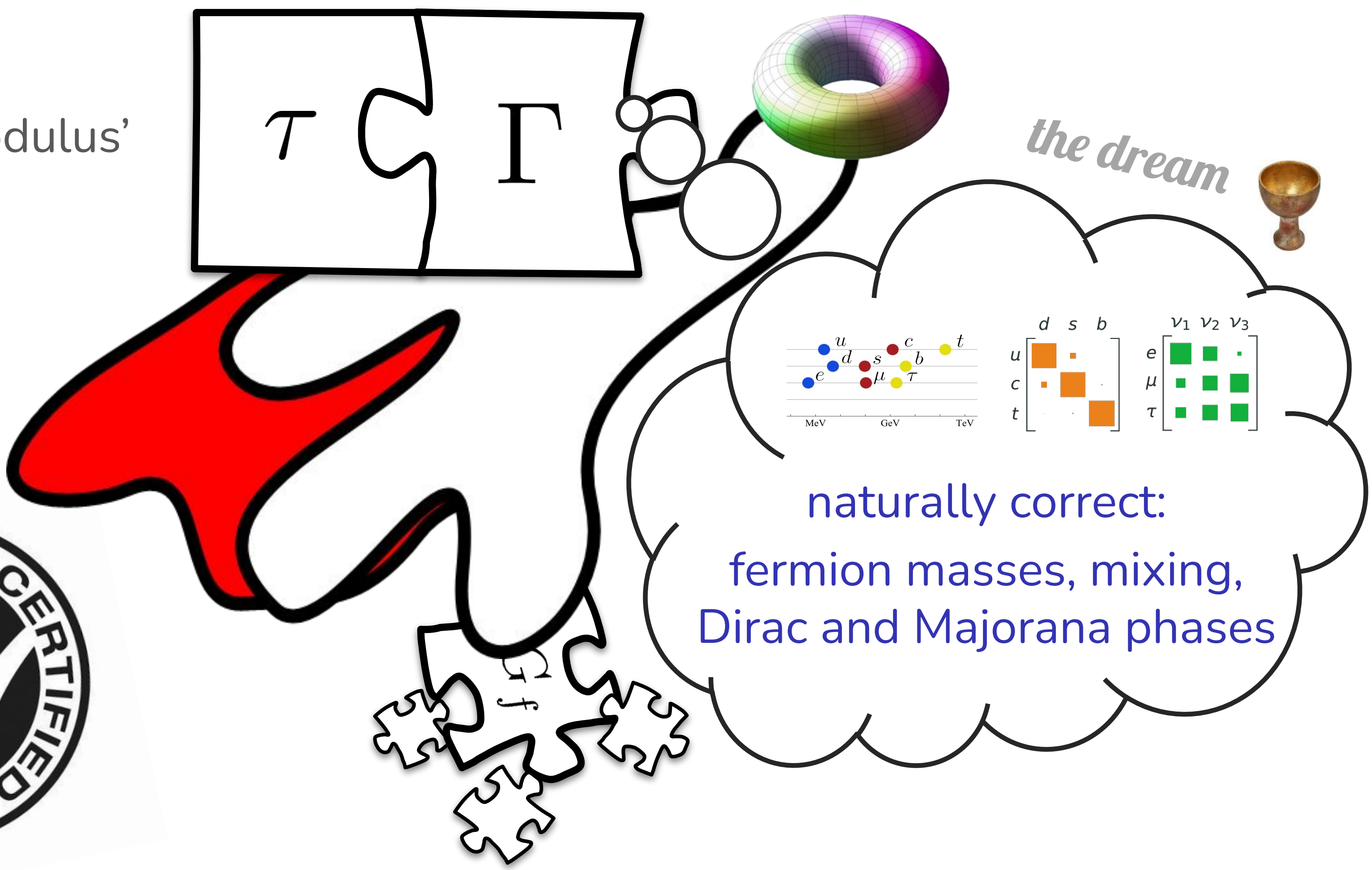


The (bottom-up) framework

Image credit: sallysbakingaddiction.com

The modulus

τ

$\neq$

τ'

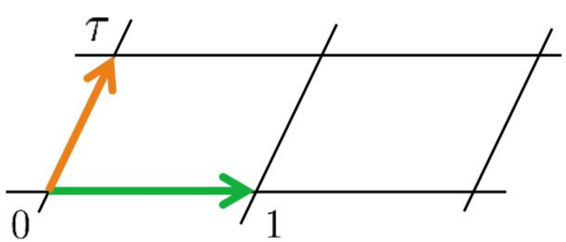
τ may describe a torus compactification

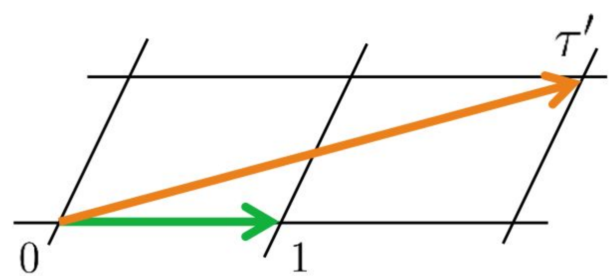
(we assume only 1 unfrozen modulus)

In the **bottom-up** modular approach τ is a dimensionless **spurion**

The modulus

$$\tau \quad = \quad \tau' = \frac{a\tau+b}{c\tau+d}$$





$$\mathrm{Im}\,\tau > 0 \qquad ad - bc = 1 \qquad a, b, c, d \in \mathbb{Z}$$

The modulus

$=$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \tau' = \frac{a\tau + b}{c\tau + d}$$

*the **modular group***

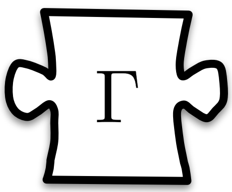
$$\det \gamma = 1$$

$$a, b, c, d \in \mathbb{Z}$$

$$\Gamma \equiv SL(2, \mathbb{Z}) = \{\gamma\}$$

The modular group

$$\tau \;\to\; \frac{a\tau + b}{c\tau + d}$$



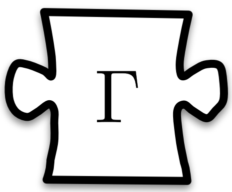
$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\, a, b, c, d \in \mathbb{Z},\ \det\gamma = 1 \right\}$$

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR$$

The modular group

$$\tau \to \frac{a\tau + b}{c\tau + d}$$



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$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} :$$

$$\tau \to -1/\tau$$

inver**S**ion

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} :$$

$$\tau \to \tau + 1$$

Translation

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} :$$

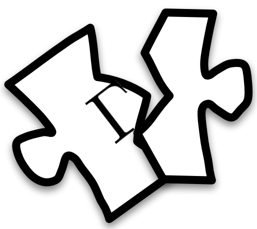
$$\tau \to \tau$$

Redundant

but can affect fields...

The modular group

$$\langle \tau \rangle \nrightarrow \frac{a\tau + b}{c\tau + d}$$



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Redundant

but can affect fields...

The field transformations

$$\psi \;\to\; (c\tau + d)^{-k}\,\rho(\gamma)\,\psi$$

The field transformations

automorphy factor

$$\psi \to \boxed{(c\tau + d)^{-k}}\, \rho(\gamma)\, \psi$$

Weight $k \in \mathbb{Z}$

The field transformations

automorphy factor

$$\psi \;\to\; \boxed{(c\tau + d)^{-k}}\;\rho(\gamma)\,\psi$$

Weight $k \in \mathbb{Z}$*

* not necessarily: rare from top-down!

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The field transformations

automorphy factor

$$\psi \to (c\tau + d)^{-k} \rho(\gamma) \psi$$

Weight $k \in \mathbb{Z}^*$

"Almost trivial" representation of the modular group

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$
$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$
$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$
$$\cdots$$

Feruglio [1706.08749]

* not necessarily: rare from top-down!

The field transformations

NEW!

automorphy factor

$$\psi \to \boxed{(c\tau + d)^{-k}}\,\boxed{\rho(\gamma)}\,\psi$$

Weight $k \in \mathbb{Z}^*$

"Almost trivial" representation of the modular group

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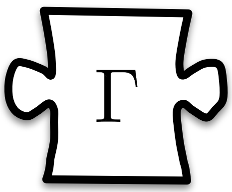
$$\cdots$$

Feruglio [1706.08749]

$\Gamma(N) \subset$



Principal congruence subgroup of level *N*

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$\rho(\gamma)$ **is effectively a representation of** $\Gamma'_N \equiv \Gamma / \Gamma(N)$

other choices are possible: in general, ***vector-valued modular forms***, see e.g. [2112.14761, 2311.10136]

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$

← drop the **R** generator

(in general there in TD)

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR, \quad T^N = \mathbb{1}$$

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2, \mathbb{Z}_4)$	$A'_5 \equiv SL(2, \mathbb{Z}_5)$

← drop the **R** generator

(in general there in TD)

$\Gamma_2 \simeq S_3$
Kobayashi et al. [1803.10391]
Meloni, Parriciatu [2306.09028]

$\Gamma_3 \simeq A_4$
Feruglio [1706.08749]

$\Gamma_4 \simeq S_4$
JP, Petcov [1806.11040]

$\Gamma_5 \simeq A_5$
Novichkov et al. [1812.02158]

summary in Appendices of Novichkov, JP, Petcov, Titov [1905.11970]

$\Gamma'_3 \simeq A'_4$
Liu, Ding [1907.01488]

$\Gamma'_4 \simeq S'_4$
Novichkov, JP, Petcov [2006.03058]

$\Gamma'_5 \simeq A'_5$
Wang, Yu, Zhou [2010.10159]

For early top-down, see e.g.:

Kobayashi et al. [1804.06644];
Kobayashi, Tamba [1811.11384];
de Anda et al. [1812.05620];
Baur et al. [1901.03251, 1908.00805];
Kariyazono et al. [1904.07546];
Nilles et al. [2001.01736, 2004.05200, 2006.03059]; Kobayashi, Otsuka [2001.07972, 2004.04518];
Abe et al. [2003.03512];
Ohki et al. [2003.04174];
Kikuchi et al. [2005.12642]

by now, O(200) papers...

A lot of model building...

- models based on finite modular groups of higher *N*
- modular models of unification (also without GUTs)
- modular models of leptogenesis
- models with multiple moduli

$$\tau \to \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$$

based on symplectic modular invariance (**Siegel modular group**) and automorphic forms

- models relating modular flavour symmetries and inflation
- models exploring the interplay of modular and gCP symmetries

...and a vast literature...

- models based on finite modular groups of higher *N*
  [2004.12662, 2108.02181, 2307.01419]
- modular models of unification (also without GUTs)
  [1906.10341, 2012.01397, 2101.02266, 2101.12724, 2103.02633, 2103.16311, 2108.09655, 2206.14675, 2312.09255]
- modular models of leptogenesis
  [1909.06520, 2007.00545, 2103.07207, 2201.10429, 2204.08338, 2205.08269, 2206.14675, 2402.18547, 2405.09363]
- models with multiple moduli
  [1811.04933, 1812.11289, 1906.02208, 1908.02770, 2304.05958]

$$\tau \to \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$$

  based on symplectic modular invariance (**Siegel modular group**)
  and automorphic forms: Ding, Feruglio, Liu [2010.07952, 2402.14915]
  From TD, see e.g.: Nilles et al. [2105.08078], Baur et al. [2012.09586], Kikuchi et al. [2305.16709]

- models relating modular flavour symmetries and inflation   Recall talk by X. Wang
  [2208.10086, 2303.02947, 2405.06497, 2405.08924]
- models exploring the interplay of modular and gCP symmetries
  [1901.03251, 1905.11970, 1910.11553, 2006.03058, 2012.01688, 2012.13390, 2102.06716, 2106.11659]

But how does it work?

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(\psi_1 \dots \psi_n)_{\mathbf{1}}$$

$$\psi \to (c\tau + d)^{-k} \rho_{\mathbf{r}}(\gamma)\, \psi$$

Assuming rigid SUSY (*W* not invariant in SUGRA)

Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(Y(\tau)\,\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \;\to (c\tau + d)^{-k}\,\rho_{\mathbf{r}}(\gamma)\,\psi$$

Assuming rigid SUSY (*W* not invariant in SUGRA)

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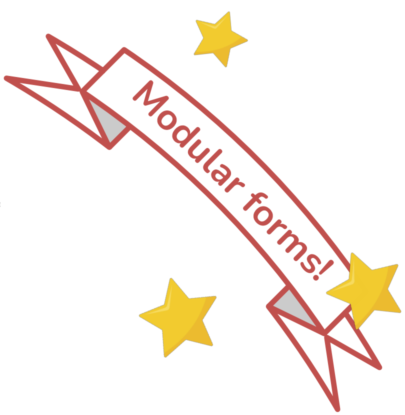
$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(Y(\tau)\, \psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \;\rightarrow (c\tau + d)^{-k}\, \rho_{\mathbf{r}}(\gamma)\, \psi$$

$$Y(\tau) \;\rightarrow (c\tau + d)^{k_Y}\, \rho_Y(\gamma)\, Y(\tau)$$



Assuming rigid SUSY (*W* not invariant in SUGRA)

Need modular forms

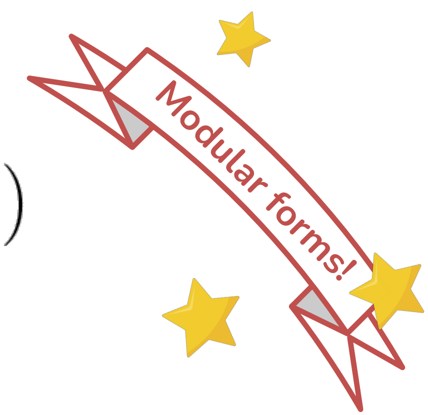
$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g\left(Y(\tau)\, \psi_1 \ldots \psi_n\right)_{\mathbf{1}}$$

$$\psi \to (c\tau + d)^{-k}\, \rho_{\mathbf{r}}(\gamma)\, \psi$$

$$Y(\tau) \to (c\tau + d)^{k_Y}\, \rho_Y(\gamma)\, Y(\tau)$$

$$\begin{cases} k_Y = k_1 + \ldots + k_n \\ \rho_Y \otimes \rho_1 \otimes \ldots \otimes \rho_n \supset \mathbf{1} \end{cases}$$



Assuming rigid SUSY (*W* not invariant in SUGRA)

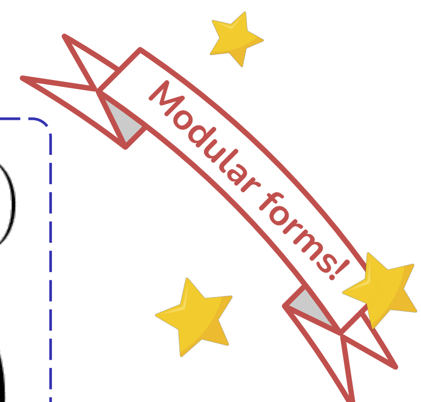
Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(Y(\tau)\,\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \;\to (c\tau + d)^{-k}\,\rho_{\mathbf{r}}(\gamma)\,\psi$$

$$Y(\tau) \;\to (c\tau + d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau) = Y\left(\frac{a\tau + b}{c\tau + d}\right)$$



Assuming rigid SUSY (W not invariant in SUGRA)

The modular forms

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

Not so many available!

*A **finite set** of functions for each k_Y*

The modular forms

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Γ_N Γ'_N	S_3 S_3	A_4 $A'_4 \equiv T'$	S_4 $S'_4 \equiv SL(2,\mathbb{Z}_4)$	A_5 $A'_5 \equiv SL(2,\mathbb{Z}_5)$
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Not so many available!

*A **finite set** of functions for each k_Y*

Lowest-weight *k* modular forms for each group:

$\Gamma_N^{(\prime)}$	$Y_{\mathbf{r}}^{(k)}$
$\Gamma_2 \simeq S_3$	$Y_{\mathbf{2}}^{(2)}$
$\Gamma'_3 \simeq A'_4$	$Y_{\hat{\mathbf{2}}}^{(1)}$
$\Gamma_3 \simeq A_4$	$Y_{\mathbf{3}}^{(2)}$
$\Gamma'_4 \simeq S'_4$	$Y_{\hat{\mathbf{3}}}^{(1)}$
$\Gamma_4 \simeq S_4$	$Y_{\mathbf{2}}^{(2)}, Y_{\mathbf{3'}}^{(2)}$
$\Gamma'_5 \simeq A'_5$	$Y_{\hat{\mathbf{6}}}^{(1)}$
$\Gamma_5 \simeq A_5$	$Y_{\mathbf{3}}^{(2)}, Y_{\mathbf{3'}}^{(2)}, Y_{\mathbf{5}}^{(2)}$

non-singular, unlike modular *functions*. Can still have an interpretation, see Feruglio, Strumia, Titov [2305.08908]

can generalize modular group to e.g. the larger metaplectic group and get half-integer weight forms, see Liu et al. [2007.13706]

Example

Let's build a modular-invariant term!

$$W \supset N N$$

Example

$$W \supset NN$$

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$$\Gamma_3 \simeq A_4$$

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$$Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

Example

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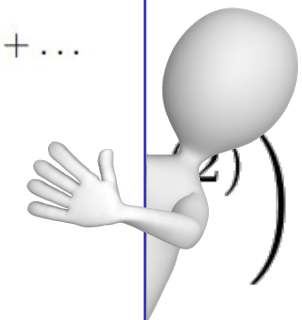

$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right] = 1 + 12q + 36q^2 + 12q^3 + \dots$$

$$Y_2(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -6q^{1/3}(1 + 7q + 8q^2 + \dots)$$

$$Y_3(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -18q^{2/3}(1 + 2q + 5q^2 + \dots)$$

with

$$\omega = e^{2\pi i/3} \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

obey $Y_2^2 + 2Y_1Y_3 = 0$

Feruglio [1706.08749]

$$= \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & \dots \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

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so now we can build models...

Example: an S_4 lepton model

Novichkov, JP, Petcov, Titov [1811.04933]

Ingredients: **Choose group, field content** $\psi \sim (\mathbf{r}, k)$

Example: an S4 lepton model

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$$N^c \sim (\mathbf{3}', 0)\,, \quad L \sim (\mathbf{3}, 2)$$
$$E^c \sim (\mathbf{1}', 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1}', 2)$$

Ingredients: **Choose group, field content** $\psi \sim (\mathbf{r}, k)$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$
$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}\,,$$

$g' \in \mathbb{C}$ only physical phase

Procedure: **Fit couplings and τ** $\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$

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$$\text{gCP} \Rightarrow g' \in \mathbb{R}$$

Novichkov, JP, Petcov, Titov [1905.11970]

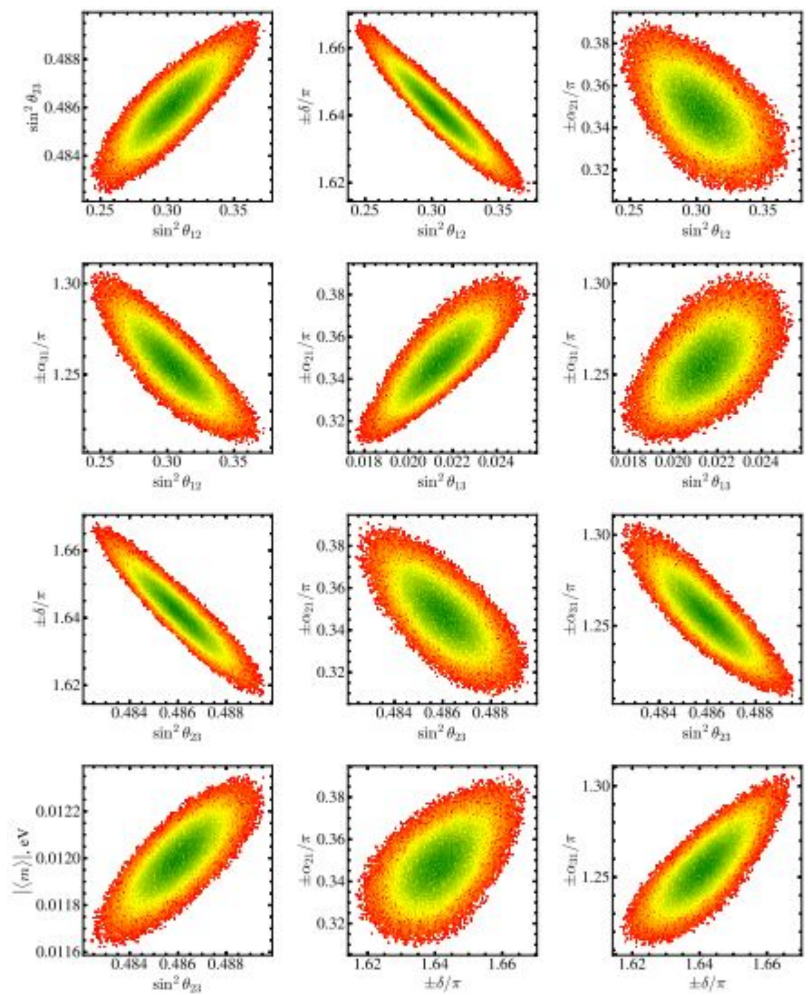
τ can be the only source of CPV

Example: an S4 lepton model (results)

Novichkov, JP, Petcov, Titov
[1811.04933, 1905.11970]

$$\sin^2\theta_{23} \sim 0.49$$
$$\delta \sim 1.6\pi$$
$$\alpha_{21} \sim 0.3\pi$$
$$\alpha_{31} \sim 1.3\pi$$

$$|\langle m \rangle|_{\beta\beta} \sim 12 \text{ meV}$$
$$\sum_i m_i \sim 0.08 \text{ eV}$$



7 (4) parameters

vs.

12 (9) observables

Minimal model found with one less parameter:
Ding, Liu, Yao [2211.04546]

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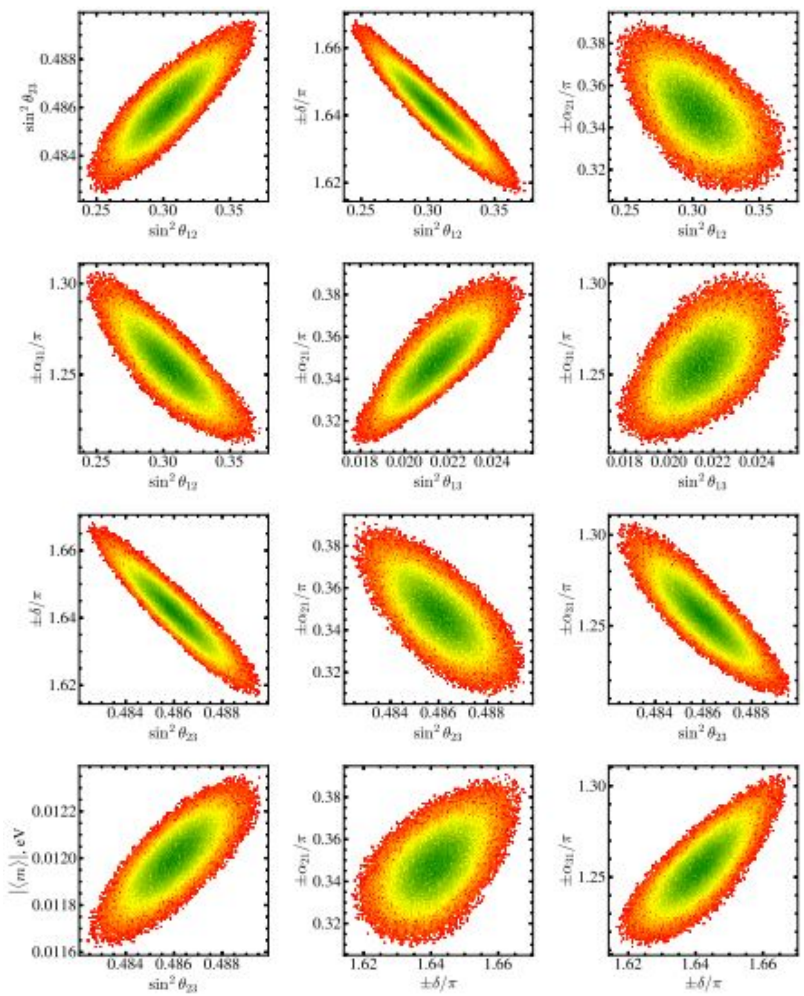
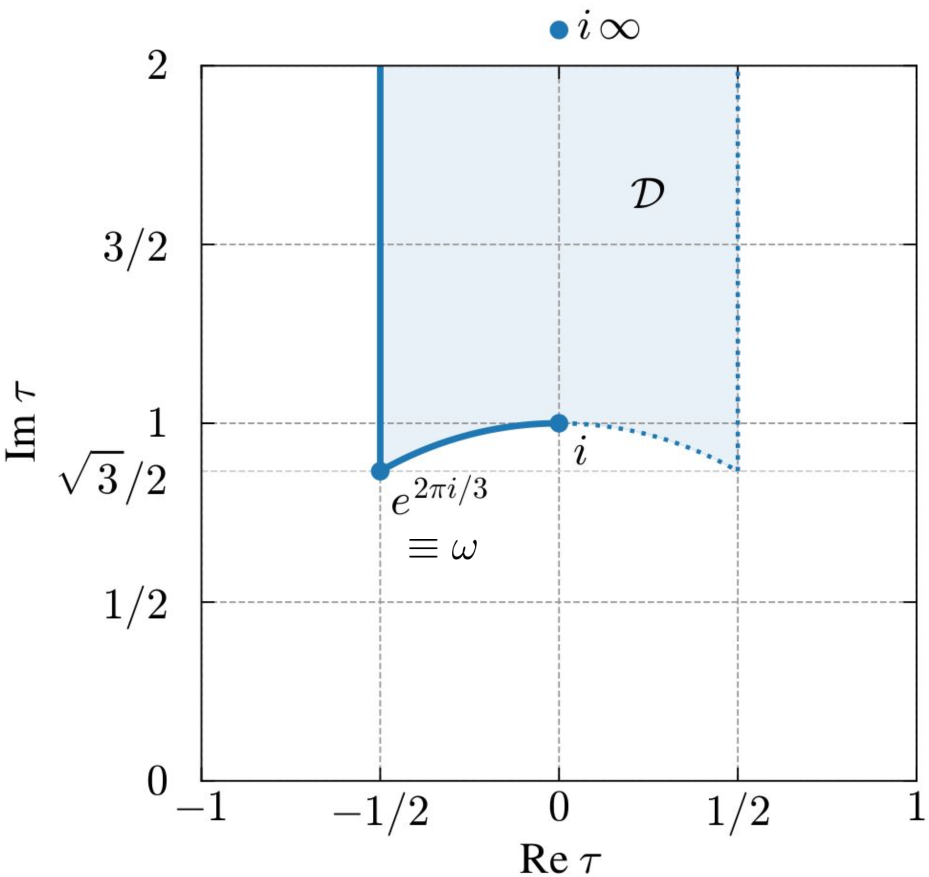


...and one does not need to consider the whole ½ plane

The fundamental domain



- **Any** $\boldsymbol{\tau}$ breaks the full modular symmetry
- To **fit** a model which is invariant under the full modular group, it is enough to scan τ in the **fundamental domain**

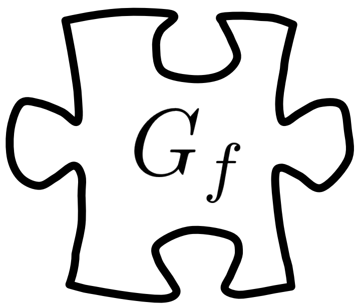
In some cases, can avoid fit by looking at invariants see Chen et al. [2211.04546]

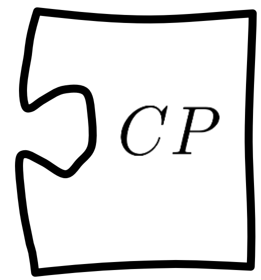
...and one does not need to consider the whole ½ plane

Image credit: www.davidharber.co.uk

CP violation

Flavour symmetries + gCP (general~~ized~~ CP)





$$\psi(x) \to \rho_{\mathbf{r}}(g)\, \psi(x)$$

$$\psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\, \overline{\psi}(x_{\mathrm{P}})$$

Branco, Lavoura, Rebelo (1986)
Harrison, Scott (2002)
Grimus, Lavoura (2003)
Farzan, Smirnov (2006)
Ferreira et al. (2012)
...

Flavour symmetries + gCP (general~~ized~~ CP)

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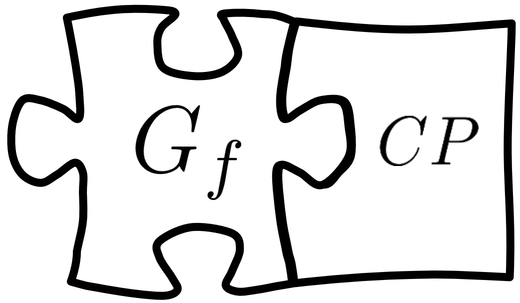


$\psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}})$

can constrain mixing, the Dirac **and the Majorana phases**

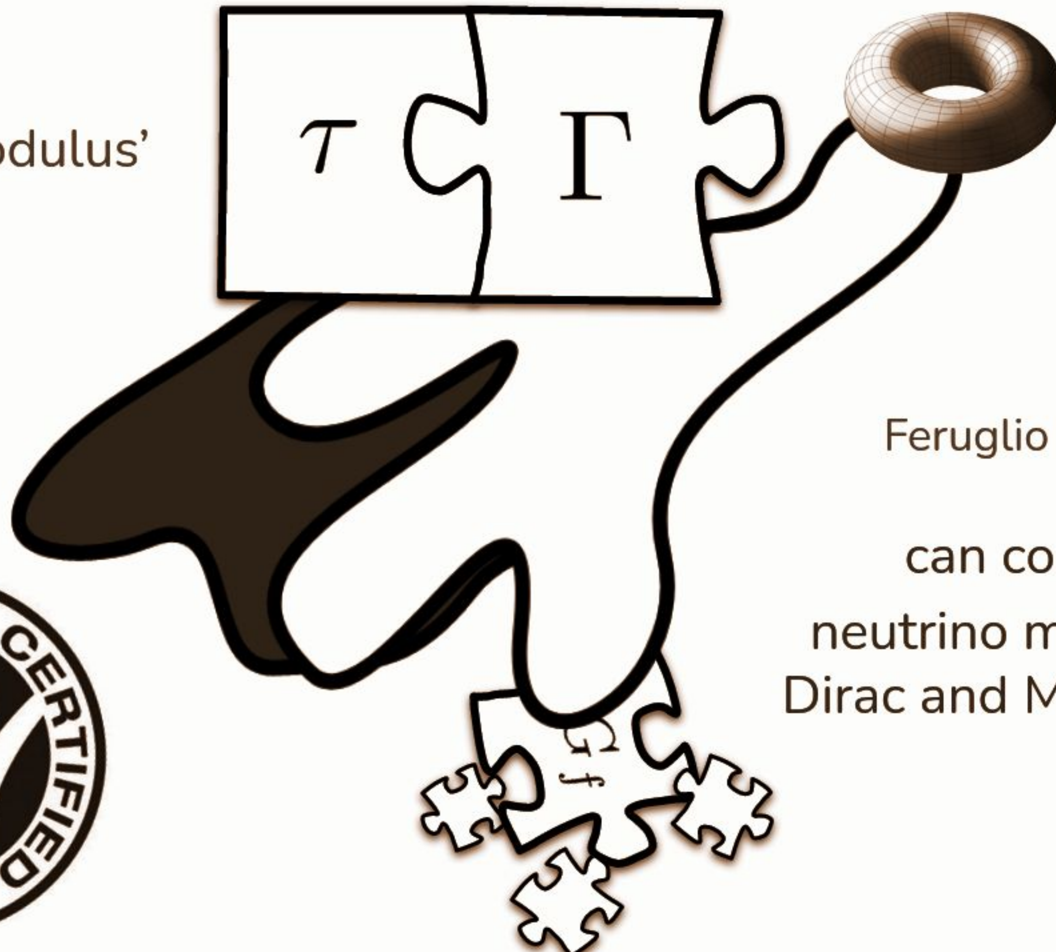
Consistency condition [Feruglio et al. (2012), Holthausen et al. (2012)]

$$X_{\mathbf{r}}^{\mathrm{CP}}\,\rho_{\mathbf{r}}^{*}(g)\,\left(X_{\mathbf{r}}^{\mathrm{CP}}\right)^{-1} = \rho_{\mathbf{r}}(u(g))$$

u is a class-inverting outer automorphism [Chen et al. (2014)]

Modular symmetry to the rescue!

'modulus'



Feruglio [1706.08749]

can constrain all:

neutrino masses, mixing,
Dirac and Majorana phases

Modular symmetry + gCP

'modulus'

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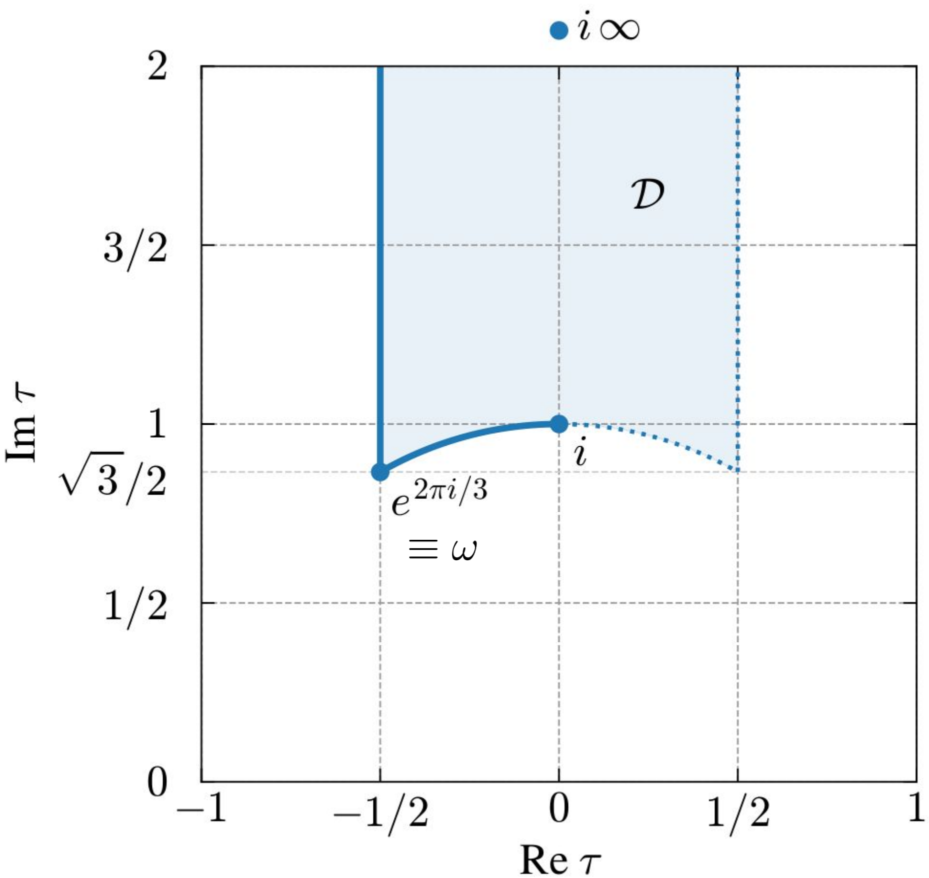
Modular symmetry + gCP (back to the fundamental domain)



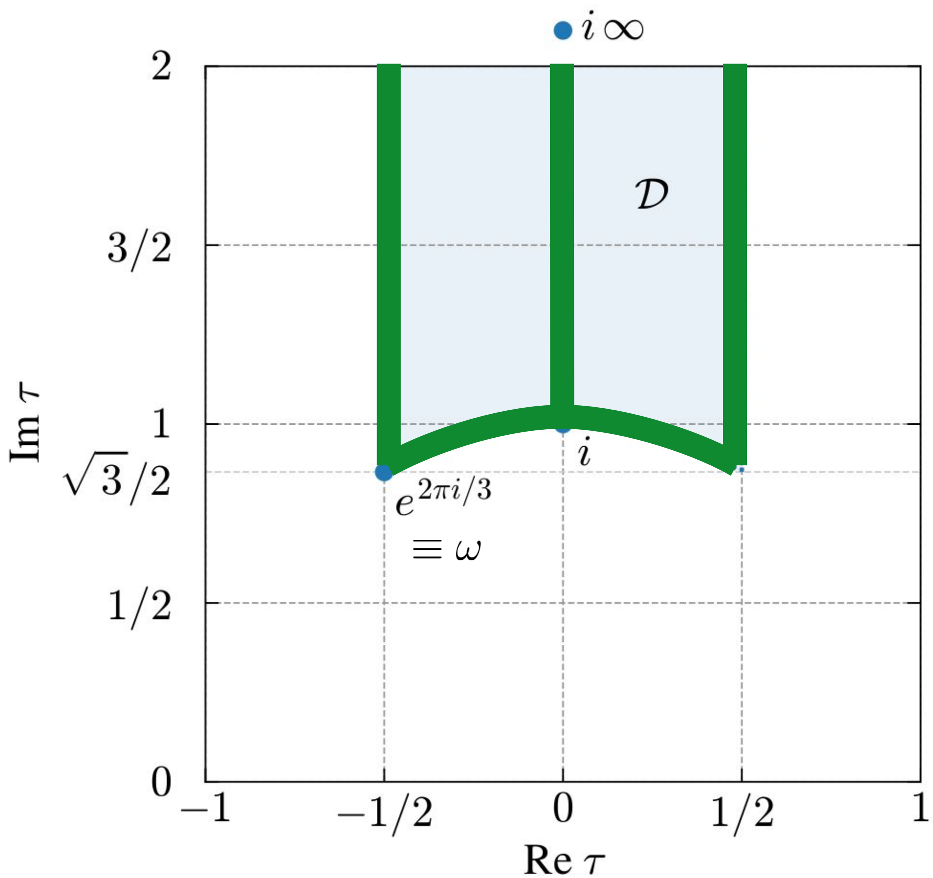
- **Any τ** breaks the full modular symmetry
- Special values of τ preserve the CP symmetry
- The modulus can be the **only source of CP violation**! (recall the S4 model of slide 50...)

- CP is violated by the modulus unless

$$-\tau^* = \gamma\tau$$

special regions of the fundamental domain

Modular symmetry + gCP (back to the fundamental domain)



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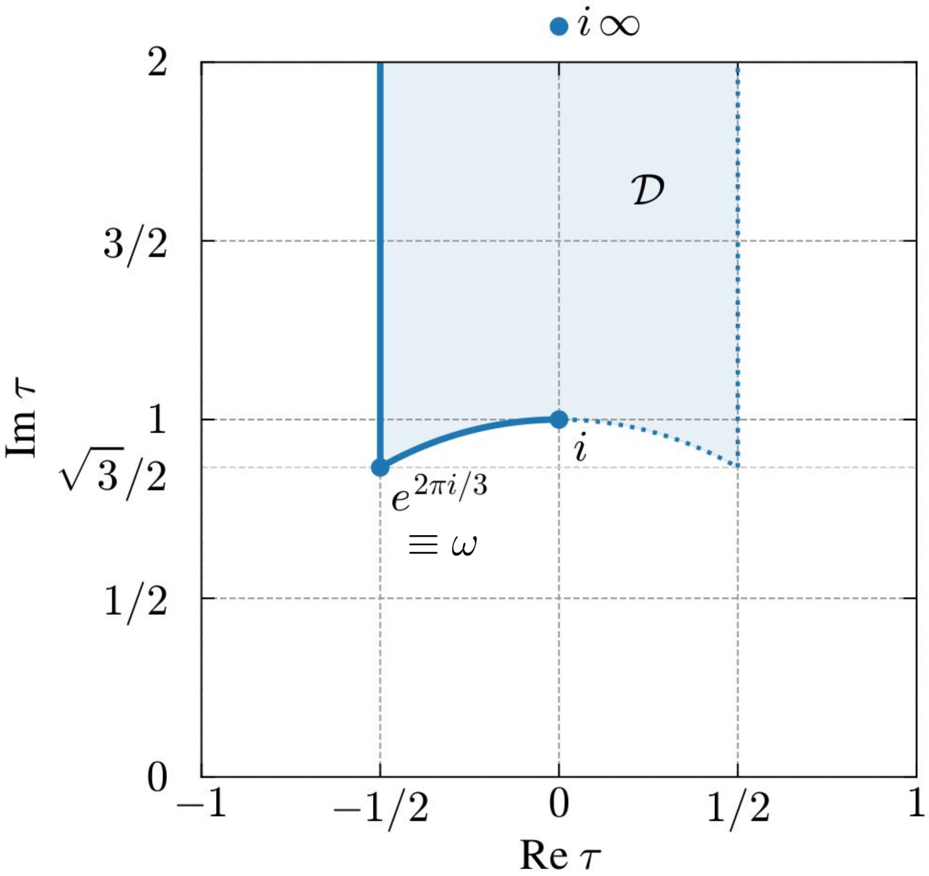
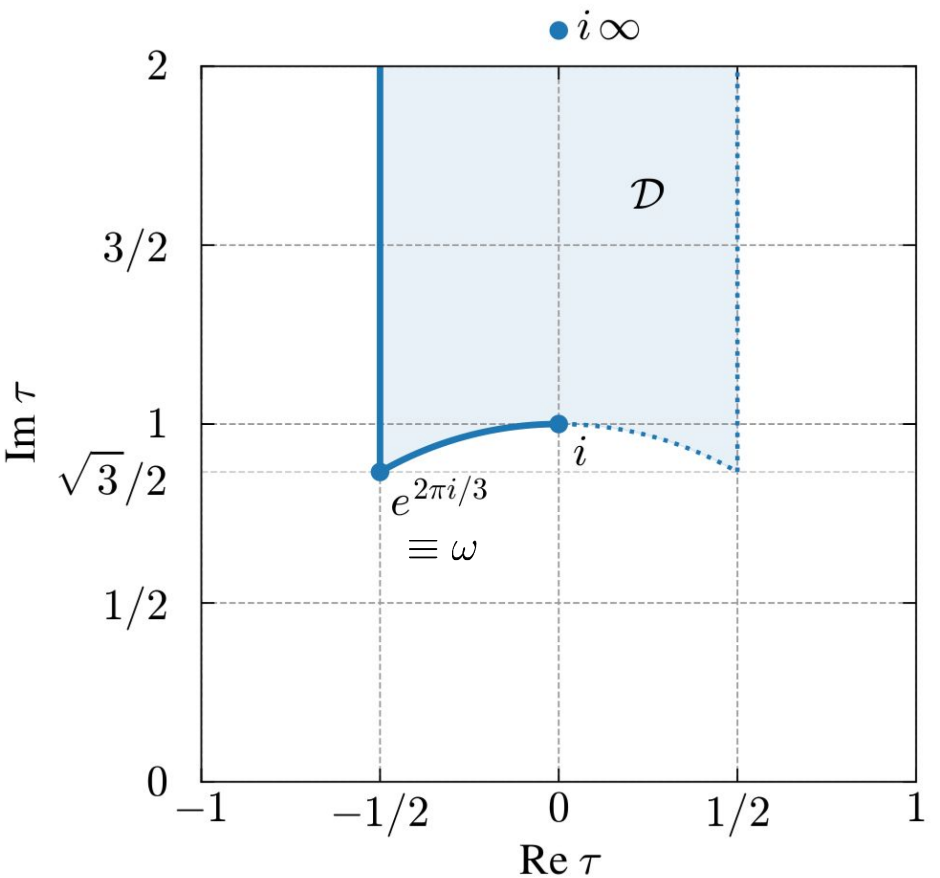
Intermezzo: Residual modular symmetries



Recall talks by M. Tanimoto, M. Levy

Intermezzo: Residual modular symmetries



- At special values of τ, some residual symmetry remains

  see e.g. Novichkov et al. [1811.04933]; Novichkov et al.' [1812.11289]

- Near them, these symmetries are linearly realized

  see e.g. Feruglio [2302.11580]

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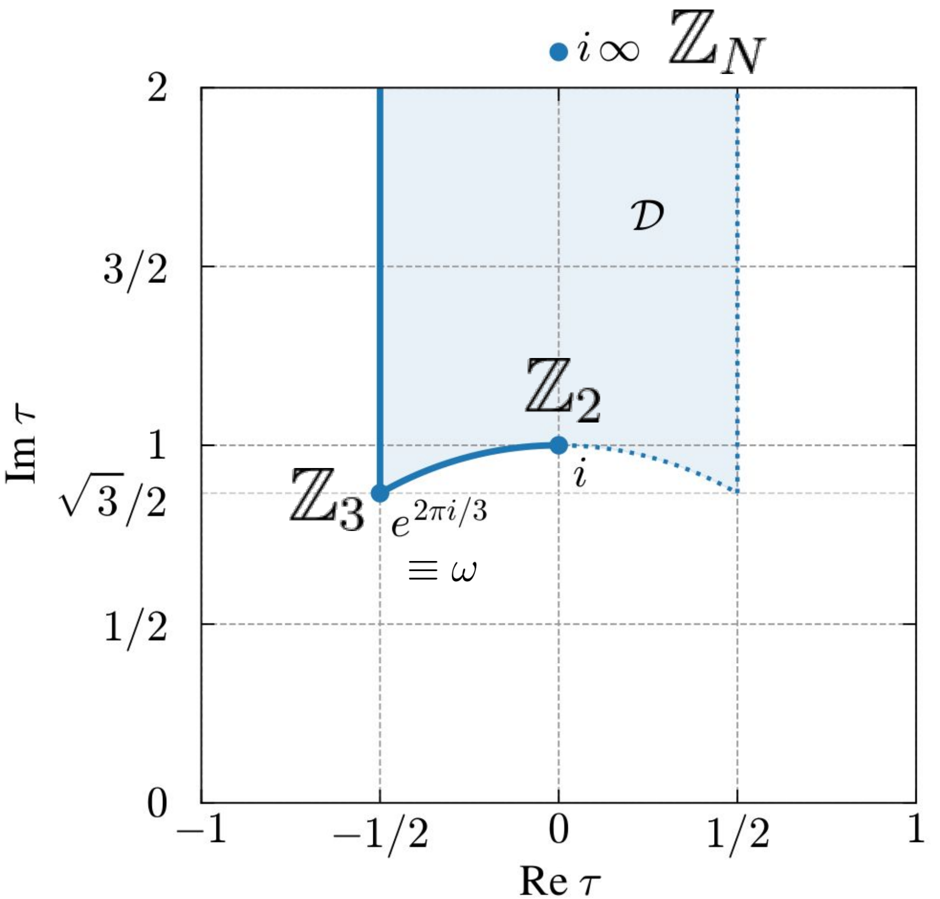
Intermezzo: Residual modular symmetries



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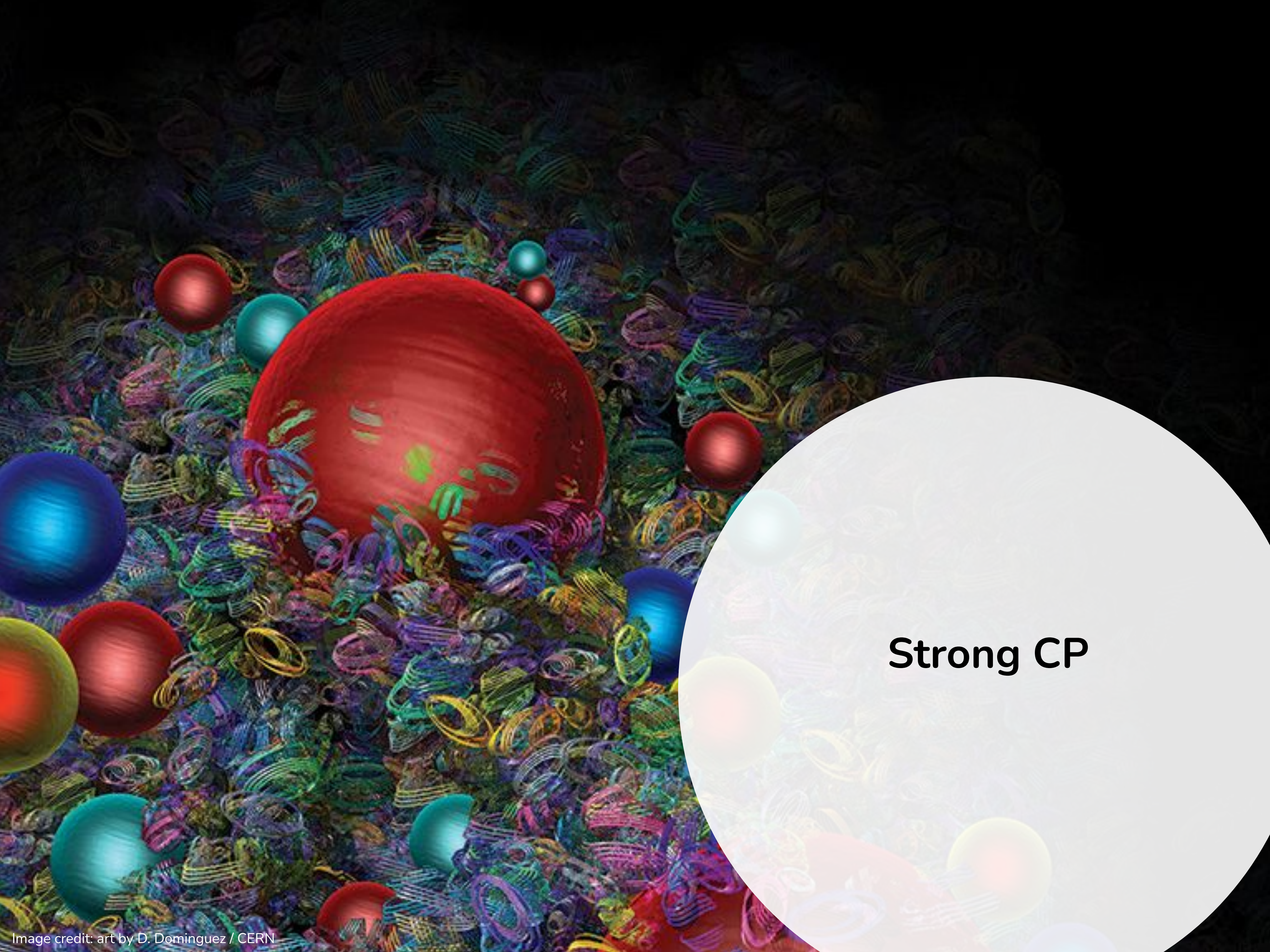
Recall talks by M. Tanimoto, M. Levy

if the base symmetry is smaller, more stabilizers arise, see e.g. Varzielas, Levy, Zhou [2008.05329]

Key idea for hierarchies:

some couplings vanish as we approach a symmetric point

Novichkov, JP, Petcov [2102.07488]



Strong CP

Image credit: art by D. Dominguez / CERN

The problem

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\not{D} - M_q)q - \frac{1}{4}G^{a\,\mu\nu}G^{a}_{\mu\nu} + \theta\frac{g_s^2}{32\pi^2}G^{a\,\mu\nu}\tilde{G}^{a}_{\mu\nu}$$

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The physical quantity:

$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_u M_d$$

O(1) a priori

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The physical quantity:

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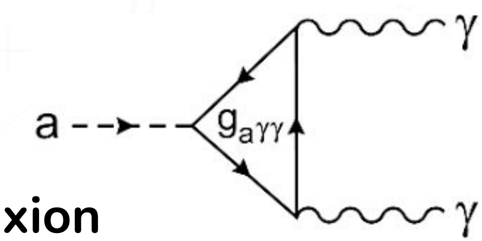
O(1) a priori

$$|d_{\text{n}}| < 1.8 \times 10^{-26}\, e\cdot\text{cm} \;\; (90\% \;\; \text{C.L.}) \quad \Rightarrow \quad |\bar{\theta}| \lesssim 10^{-10}$$

[2001.11966] [hep-ph/9908508]

The problem...

...and some solutions



Promotion to a dynamical field: the (invisible) **axion**

Peccei and Quinn [1977], Wilczek, Weinberg [1978]

Solutions with **spontaneously broken CP symmetry**

- Nelson-Barr [1984] (minimal e.g.: Bento-Branco-Parada [1991])
- NHDM with Abelian symmetries — see talk by C. Manzari
- the Feruglio-Strumia-Titov modular solution [2305.08908]

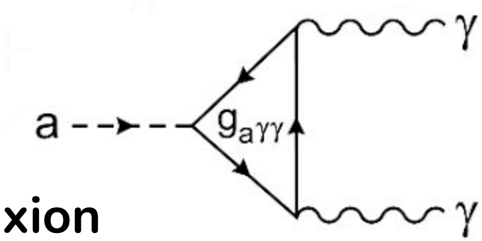
The problem

- We want to impose CP in the UV
- We need to break it to get the CKM phase
- We need to ensure that after breaking $\arg\det(Y_u Y_d) = 0$

from C. Manzari slides, on Tuesday

The problem

...and some solutions



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[2001.11966] [hep-ph/9908508]

The modular idea by Feruglio, Strumia, Titov [2305.08908]

- No axions! see instead [2002.06931, 2402.02071] for a modular origin of the axion
- Need to **produce quark CPV phase** in the CKM mixing matrix
- Need to **suppress**:

$$\bar{\theta} = \theta_{\mathrm{QCD}} + \arg\det M_u M_d$$

vanishes due to **imposed gCP** (θ_{QCD})

vanishes due to **special structure** ($\arg\det M_u M_d$)

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$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_u M_d$$

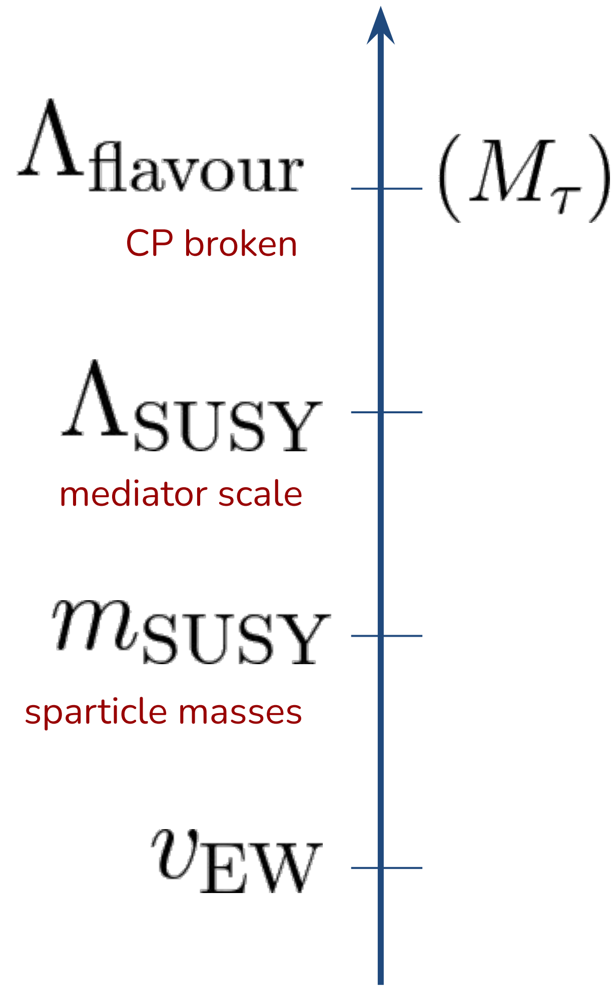
θ_{QCD}: vanishes due to **imposed gCP**

$\arg\det M_u M_d$: vanishes due to **special structure**

- It turns out to be holomorphic → insensitive to Kähler!
- Relies on the fact that mass matrix **determinants are modular forms**

already noted e.g. in Ding, Liu, Yao [2211.04546]
derivation in app. A of JP, Petcov [2404.08032]

...its quality...

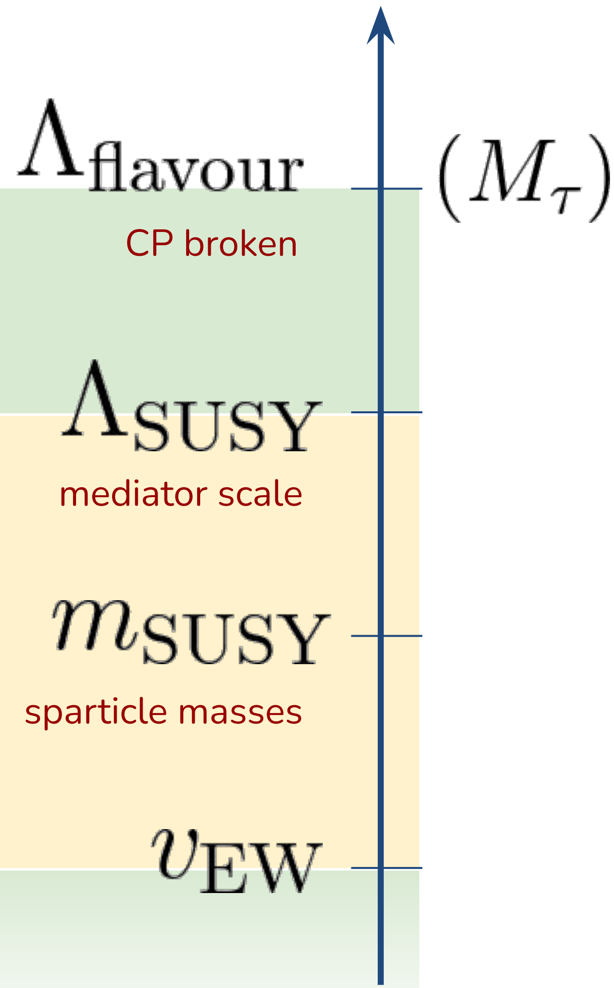


Separation of scales also needed to trust model predictions

see Feruglio and Criado [1807.01125]

see also A. Titov slides @ Cosmic WISPers 2024

…its quality…



- **No more operators** in the limit of unbroken SUSY
- SUSY **non-renormalization** theorems

- **gluino mass real** if SUSY-breaking sector is CP conserving w/ zero modular charges
- assume gauge or anomaly mediation: controls dangerous **threshold corrections**
- consistency will automatically give **real μ term**

- SM contribution is **negligible** (4 loops)

Separation of scales also needed to trust model predictions

see Feruglio and Criado [1807.01125]

see also A. Titov slides @ Cosmic WISPers 2024

...& the consequences

$$Q_i \xrightarrow{\gamma} (c\tau + d)^{-k_i} \rho_{ij}(\gamma) \, Q_j$$

$$q_i^c \xrightarrow{\gamma} (c\tau + d)^{-k_i^c} \rho_{ij}^c(\gamma) \, q_j^c$$

- The determinants have weights

$$k^q_{\text{det}} = 3k_{H_q} + \sum_i k_i + k_i^c$$

...& the consequences

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...& the consequences

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- The determinants have weights

$$k^q_{\det} = 3k_{H_q} + \sum_i k_i + k_i^c$$

- To avoid massless quarks, must have $k^q_{\det} \geq 0$

- Then, to make $\arg\det M_u M_d$ vanish we require

$$k^u_{\det} = k^d_{\det} = 0$$

so that both determinants are τ-independent **constants** (weight 0) and **real**, due to the imposed gCP!

...& the consequences

$$Q_i \xrightarrow{\gamma} (c\tau + d)^{-k_i}\, \rho_{ij}(\gamma)\, Q_j$$
$$q_i^c \xrightarrow{\gamma} (c\tau + d)^{-k_i^c}\, \rho_{ij}^c(\gamma)\, q_j^c$$

- The determinants have weights

$$k^q \;=\; 3k \;+\; \sum k \;+\; k^c$$

Consistent w/ cancellation of the QCD **modular anomaly**, which further implies

$$k_{H_u} + k_{H_d} = 0$$

$$k^u_{\text{det}} = k^d_{\text{det}} = 0$$

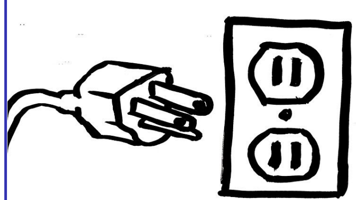
so that both determinants are τ-independent **constants** (weight 0) and **real**, due to the imposed gCP!

...& the consequences

$$Q_i \xrightarrow{\gamma} (c\tau + d)^{-k_i} \rho_{ij}(\gamma)\, Q_j$$
$$q_i^c \xrightarrow{\gamma} (c\tau + d)^{-k_i^c} \rho_{ij}^c(\gamma)\, q_j^c$$

- The determinants have weights

$$k_{\text{det}}^q = 3k_{H_q} + \sum_i k_i + k_i^c$$



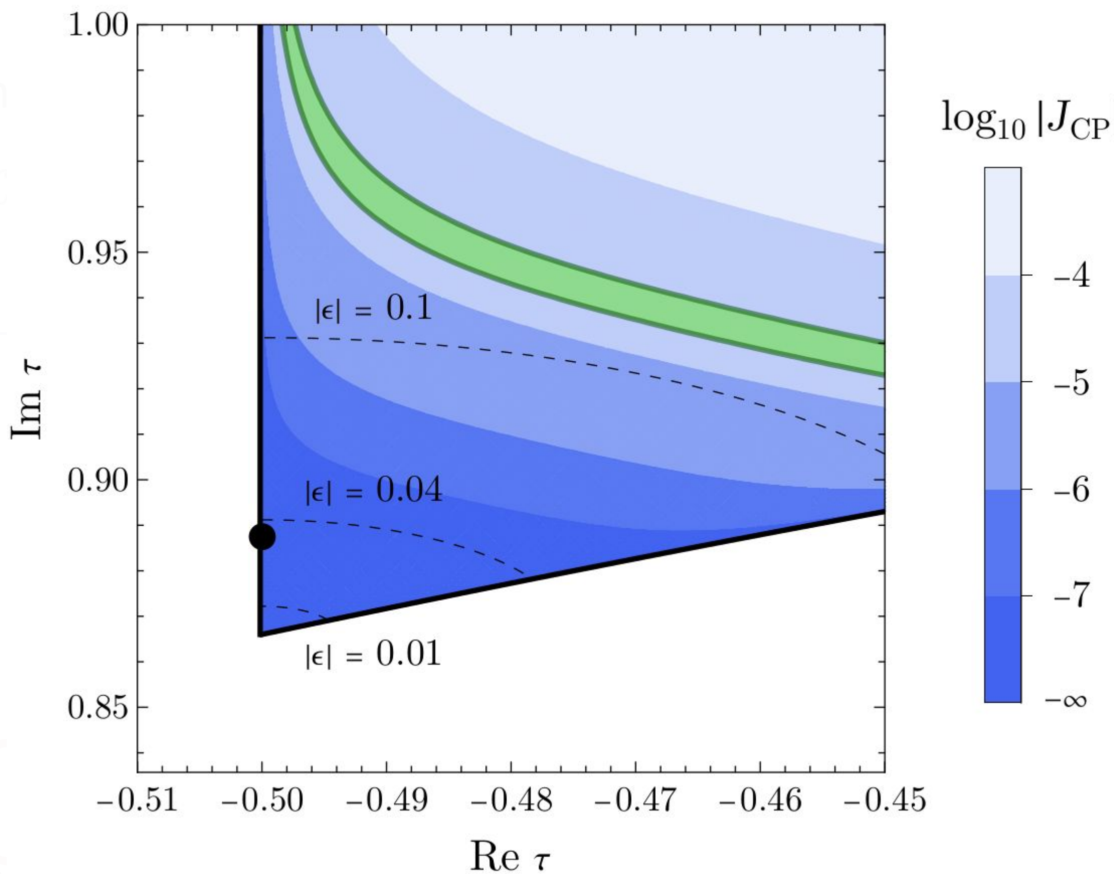
incompatible w/ modular hierarchy mechanism of Novichkov, JP, Petcov [2102.07488] :(

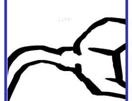
Recall talk by M. Tanimoto

so that both determinants are τ-independent **constants** (weight 0) and **real**, due to the imposed gCP!

…& the consequences

NB.: it was **already difficult** to marry CKM CPV with hierarchies…





Recall talk by M. Levy

Varzielas, Levy, JP, Petcov [2307.14410]

JP, Petcov [2404.08032]

Which matrices work?

(weight structures)

$$M \;:\; \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix} \qquad k_{ij} = k_i + k_j^c$$

$$k_{\text{det}} = k_{11}+k_{22}+k_{33} = k_{12}+k_{23}+k_{31} = \ldots \overset{!}{=} 0$$

JP, Petcov [2404.08032]

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(weight structures)

$$M \ : \ \begin{pmatrix} k+k' & k' & 0 \\ k & 0 & -k' \\ 0 & -k & -k-k' \end{pmatrix}$$

at least one of *k*, *k*’ non-zero

JP, Petcov [2404.08032]

Which matrices work?

(weight structures)

$$M \; : \; \begin{pmatrix} k+k' & k' & 0 \\ k & 0 & -k' \\ 0 & -k & -k-k' \end{pmatrix}$$

e.g. $k.\, k' < 0$

$$M = v_q \begin{pmatrix} 0 & 0 & \alpha_{13} \\ 0 & \alpha_{22} & \alpha_{23}\, \mathcal{Y}_1^{(|k'|)} \\ \alpha_{31} & \alpha_{32}\, \mathcal{Y}_2^{(|k|)} & \alpha_{33}\, \mathcal{Y}_3^{(|k+k'|)} \end{pmatrix}$$

at least one of *k*, *k*’ non-zero

$$\det M = -v_q^3\, \alpha_{13}\alpha_{22}\alpha_{31}$$

JP, Petcov [2404.08032]

Which matrices work?

(weight structures in both sectors)

$$M_u : \begin{pmatrix} k+k' & k' & 0 \\ k & 0 & -k' \\ 0 & -k & -k-k' \end{pmatrix}$$

$$M_d : \begin{pmatrix} k+k' & k' & 0 \\ k & 0 & -k' \\ 0 & -k & -k-k' \end{pmatrix}$$

at least one of *k*, *k*’ non-zero

up to **simultaneous** permutations from the left
and independent permutations from the right (weak basis transformations)

Which irreps work?

- Irreps beyond 1D imply **extra relations** between weights
- **Cannot have triplet** irreps
- Potentially viable non-singlet case: **2+1**

$$Q \sim (\mathbf{2}_Q, k_2) \oplus (\mathbf{1}_Q, k_1)\,,$$
$$u^c \sim (\overline{\mathbf{2}_Q}, -k_2) \oplus (\overline{\mathbf{1}_Q}, -k_1)\,,$$
$$d^c \sim (\overline{\mathbf{2}_Q}, -k_2) \oplus (\overline{\mathbf{1}_Q}, -k_1)\,,$$

$$M_q \propto \begin{pmatrix} |\alpha_1^q/\beta_q| & 0 & \cos\theta_q\, e^{i\phi_1^q} \\ 0 & |\alpha_1^q/\beta_q| & \sin\theta_q\, e^{i\phi_2^q} \\ 0 & 0 & |\alpha_2^q/\beta_q| \end{pmatrix}^{(T)}$$

JP, Petcov [2404.08032]

Which matrices work?

(weight structures in both sectors)

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at least one of k, k' non-zero

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$$\begin{aligned} Q &\sim (\mathbf{2}_Q, k_2) \oplus (\mathbf{1}_Q, k_1)\,, \\ u^c &\sim (\overline{\mathbf{2}_Q}, -k_2) \oplus (\overline{\mathbf{1}_Q}, -k_1)\,, \\ d^c &\sim (\overline{\mathbf{2}_Q}, -k_2) \oplus (\overline{\mathbf{1}_Q}, -k_1)\,, \end{aligned}$$

$$M_q \propto \begin{pmatrix} |\alpha_1^q/\beta_q| & 0 & \cos\theta_q\, e^{i\phi_1^q} \\ 0 & |\alpha_1^q/\beta_q| & \sin\theta_q\, e^{i\phi_2^q} \\ 0 & 0 & |\alpha_2^q/\beta_q| \end{pmatrix}^{(T)}$$

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too much strain on the model, does not work → **quarks must furnish 1D irreps**

$$k_2 > k_1$$

in the interval [0, 1], implying the upper bound $|V_{us}| \lesssim m_u/m_c \simeq 0.002$. This is two orders of magnitude smaller than the value $|V_{us}| \simeq 0.225$ [45] required by quark data. By a similar procedure, one can also show that

$$\left| |V_{ub}| - \frac{m_d}{m_s} \right| \lesssim \frac{m_u}{m_c}\,, \qquad (41)$$

i.e. that $|V_{ub}| \in [0.048, 0.053]$, again in contradiction with the data, which requires the much smaller $|V_{ub}| \simeq 0.003$.

$$k_2 < k_1$$

once again leading to an upper bound on the magnitude of a mixing matrix element. In this case one has $|V_{cb}| \lesssim m_s/m_b \simeq 0.014$, while data requires $|V_{cb}| \simeq 0.036$, a value more than twice as large.

JP, Petcov [2404.08032]

What can we do with 1D?

$$Q \sim (\mathbf{1}_{Q_1}, k_1) \oplus (\mathbf{1}_{Q_2}, k_2) \oplus (\mathbf{1}_{Q_3}, k_3)\,,$$
$$u^c \sim (\overline{\mathbf{1}_{Q_1}}, -k_1) \oplus (\overline{\mathbf{1}_{Q_2}}, -k_2) \oplus (\overline{\mathbf{1}_{Q_3}}, -k_3)$$
$$d^c \sim (\overline{\mathbf{1}_{Q_1}}, -k_1) \oplus (\overline{\mathbf{1}_{Q_2}}, -k_2) \oplus (\overline{\mathbf{1}_{Q_3}}, -k_3)$$

$$k \equiv k_2 - k_3, \quad k' \equiv k_1 - k_2$$

What can we do with 1D?

$$\mathrm{I}: \begin{pmatrix} \alpha_1^q & 0 & \tilde{\alpha}_{13}^q \, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & \tilde{\alpha}_{23}^q \, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \; k > 0, \, k' \geq 0 \,,$$

$$\mathrm{II}: \begin{pmatrix} \alpha_1^q & \tilde{\alpha}_{12}^q \, \mathcal{Y}_{q,12}^{(k')} & \tilde{\alpha}_{13}^q \, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & 0 \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \; k \geq 0, \, k' > 0 \,,$$

$$\mathrm{III}: \begin{pmatrix} \alpha_1^q & \tilde{\alpha}_{12}^q \, \mathcal{Y}_{q,12}^{(k')} & \tilde{\alpha}_{13}^q \, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & \tilde{\alpha}_{23}^q \, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \; k, k' > 0 \,,$$

$$\mathrm{IV}: \begin{pmatrix} \alpha_{11}^q & \alpha_{12}^q & \tilde{\alpha}_{13}^q \, \mathcal{Y}_{q,13}^{(k)} \\ \alpha_{21}^q & \alpha_{22}^q & \tilde{\alpha}_{23}^q \, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \; k > 0 \,,$$

$$\mathrm{V}: \begin{pmatrix} \alpha_1^q & \tilde{\alpha}_{12}^q \, \mathcal{Y}_{q,12}^{(k')} & \tilde{\alpha}_{13}^q \, \mathcal{Y}_{q,13}^{(k')} \\ 0 & \alpha_{22}^q & \alpha_{23}^q \\ 0 & \alpha_{32}^q & \alpha_{33}^q \end{pmatrix}, \; k' > 0 \,,$$

$$Q \sim (\mathbf{1}_{Q_1}, k_1) \oplus (\mathbf{1}_{Q_2}, k_2) \oplus (\mathbf{1}_{Q_3}, k_3) \,,$$
$$u^c \sim (\overline{\mathbf{1}_{Q_1}}, -k_1) \oplus (\overline{\mathbf{1}_{Q_2}}, -k_2) \oplus (\overline{\mathbf{1}_{Q_3}}, -k_3)$$
$$d^c \sim (\overline{\mathbf{1}_{Q_1}}, -k_1) \oplus (\overline{\mathbf{1}_{Q_2}}, -k_2) \oplus (\overline{\mathbf{1}_{Q_3}}, -k_3)$$

$$k \equiv k_2 - k_3, \quad k' \equiv k_1 - k_2$$

- **New irreps** and **forms** beyond Γ
- Zeros from **non-trivial 1D** or **missing form**
- Several forms may contribute to each entry

$$\mathcal{Y}_{q,ij}^{(w)} \equiv \sum_{n=1} g_n^q \, Y_{\mathbf{1}_{ij},n}^{(w)}$$

- Canonical rescaling may help with hierarchies

$$\tilde{\alpha}_{ij}^q \equiv \alpha_{ij}^q \, (2 \, \mathrm{Im} \, \tau)^{k_{ij}/2}$$

- Forms are related, so getting CPV is tricky!

How “minimal” can we be?

$$\mathrm{I}: \begin{pmatrix} \alpha_1^q & 0 & \tilde{\alpha}_{13}^q\, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & \tilde{\alpha}_{23}^q\, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \quad k>0,\ k'\geq 0,$$

≥6 params.

$$\mathrm{II}: \begin{pmatrix} \alpha_1^q & \tilde{\alpha}_{12}^q\, \mathcal{Y}_{q,12}^{(k')} & \tilde{\alpha}_{13}^q\, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & 0 \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \quad k\geq 0,\ k'> 0,$$

≥6 params.

$$\mathrm{III}: \begin{pmatrix} \alpha_1^q & \tilde{\alpha}_{12}^q\, \mathcal{Y}_{q,12}^{(k')} & \tilde{\alpha}_{13}^q\, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & \tilde{\alpha}_{23}^q\, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \quad k, k'> 0,$$

≥7 params.

$$\mathrm{IV}: \begin{pmatrix} \alpha_{11}^q & \alpha_{12}^q & \tilde{\alpha}_{13}^q\, \mathcal{Y}_{q,13}^{(k)} \\ \alpha_{21}^q & \alpha_{22}^q & \tilde{\alpha}_{23}^q\, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \quad k>0,$$

≥8 params.

$$\mathrm{V}: \begin{pmatrix} \alpha_1^q & \tilde{\alpha}_{12}^q\, \mathcal{Y}_{q,12}^{(k')} & \tilde{\alpha}_{13}^q\, \mathcal{Y}_{q,13}^{(k')} \\ 0 & \alpha_{22}^q & \alpha_{23}^q \\ 0 & \alpha_{32}^q & \alpha_{33}^q \end{pmatrix}, \quad k'>0,$$

≥8 params.

$$\begin{aligned} Q &\sim (\mathbf{1}_{Q_1}, k_1) \oplus (\mathbf{1}_{Q_2}, k_2) \oplus (\mathbf{1}_{Q_3}, k_3)\,, \\ u^c &\sim (\overline{\mathbf{1}_{Q_1}}, -k_1) \oplus (\overline{\mathbf{1}_{Q_2}}, -k_2) \oplus (\overline{\mathbf{1}_{Q_3}}, -k_3) \\ d^c &\sim (\overline{\mathbf{1}_{Q_1}}, -k_1) \oplus (\overline{\mathbf{1}_{Q_2}}, -k_2) \oplus (\overline{\mathbf{1}_{Q_3}}, -k_3) \end{aligned}$$

$$k \equiv k_2 - k_3, \quad k' \equiv k_1 - k_2$$

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- Canonical rescaling may help with hierarchies

$$\tilde{\alpha}_{ij}^q \equiv \alpha_{ij}^q\, (2\,\mathrm{Im}\,\tau)^{k_{ij}/2} \in \mathbb{R}$$

- Forms are related, so getting CPV is **tricky**!

need more params. then naively expected

The minimal & next-to-minimal landscapes

\# params. = 6+6+2 = 14 # params. = 7+7+2 = 16

(k,k')	Minimal models (I and II)	Next-to-minimal models (I and II)	Next-to-minimal models (III)
All Γ'_N	$(10, 12), (12, 14), (14, 16)$	$(16, 18), (18, 20), (20, 22)$	$(4, 8), (4, 14), (6, 6), (6, 10), (6, 14), (8, 8),$ $(8, 10), (8, 14), (10, 10)$
S_3 only	$(10', 12), (10, 18'), (10', 18), (12, 12'),$ $(12, 14'), (12, 16'), (12', 16), (12, 20'),$ $(12', 20), (14', 16), (14, 18'), (14', 18),$ $(14, 22'), (14', 22), (16, 16'), (16', 18),$ $(16', 18'), (16, 20'), (16', 20), (18, 20'),$ $(18', 20'), (20, 20'), (20', 22), (20', 22')$	$(16, 18'), (16, 24'), (16', 24), (18, 18'),$ $(18', 20), (18, 22'), (18', 22), (18, 26'),$ $(18', 26), (20, 22'), (20, 24'), (20', 24),$ $(20, 28'), (20', 28), (22, 22'), (22, 24'),$ $(22', 24'), (22, 26'), (22', 26), (24', 26),$ $(24', 26'), (26, 26'), (26, 28'), (26', 28')$	$(4, 14'), (4, 20'), (6', 6'), (6', 10'), (6, 12'),$ $(6', 12'), (6', 14'), (6, 16'), (6', 16'),$ $(6, 20'), (6', 20'), (8, 10'), (8, 14'), (8, 16'),$ $(8, 20'), (10', 10'), (10, 12'), (10', 12'),$ $(10, 14'), (10', 14), (10, 16'), (10', 16'),$ $(12', 14), (12', 14'), (14, 14')$
A'_4 only	$(8', 12), (8', 18), (10', 12), (10, 16'),$ $(10', 16), (10, 20''), (10', 20), (12, 12'),$ $(12, 12''), (12, 14'), (12, 14''), (12, 16''),$ $(12', 16), (12'', 16'), (12, 18'), (12, 18''),$ $(12', 18), (12'', 18), (12, 22''), (12', 22),$ $(12'', 22'), (14, 16'), (14', 16), (14', 16'),$ $(14'', 16), (14'', 16'), (14', 18), (14, 20'),$ $(14, 20''), (14', 20), (14', 20''), (14'', 20),$ $(14'', 20'), (14, 24''), (14'', 24'), (16, 16''),$ $(16', 16''), (16, 18'), (16, 18''), (16', 18'),$ $(16', 18''), (16'', 18), (16'', 20'), (16, 22''),$ $(16', 22''), (16'', 22), (16'', 22'),$ $(16'', 26'), (18, 18'), (18, 18''), (18', 20),$ $(18', 20'), (18', 20''), (18'', 20), (18'', 20'),$ $(18'', 20''), (18, 22''), (18', 22), (18'', 22'),$ $(18', 24''), (18'', 24'), (20, 22''), (20', 22''),$ $(20'', 22''), (22, 22''), (22', 22''),$ $(22'', 24'), (22'', 24''), (22'', 26')$	$(14', 24), (16, 16'), (16', 18), (16, 20''),$ $(16', 20), (16, 22'), (16', 22), (16, 26''),$ $(16', 26), (18, 20'), (18, 20''), (18, 24'),$ $(18, 24''), (18', 24), (18'', 24), (18, 28''),$ $(18', 28), (18'', 28'), (20, 20'), (20, 20''),$ $(20', 20''), (20, 22'), (20', 22), (20', 22'),$ $(20'', 22), (20'', 22'), (20, 24''), (20'', 24'),$ $(20, 26'), (20, 26''), (20', 26), (20', 26''),$ $(20'', 26), (20'', 26'), (20, 30''), (20'', 30'),$ $(22, 22'), (22, 24'), (22, 24''), (22', 24'),$ $(22', 24''), (22'', 24), (22, 26''), (22', 26),$ $(22, 28''), (22', 28''), (22'', 28), (22'', 28'),$ $(22'', 32'), (24', 24''), (24', 26), (24', 26'),$ $(24', 26''), (24'', 26), (24'', 26'),$ $(24'', 26''), (24', 30''), (24'', 30')$	$(4', 8''), (4, 12'), (4', 12''), (4', 14''),$ $(4, 16''), (4', 16''), (4, 18'), (4', 18''),$ $(4, 22''), (4', 22''), (6, 10'), (6, 14'),$ $(6, 14''), (6, 18'), (6, 18''), (6, 22''), (8, 8'),$ $(8', 8''), (8'', 8''), (8'', 10'), (8, 12'),$ $(8, 12''), (8', 12'), (8', 12''), (8'', 12'),$ $(8'', 12''), (8, 14'), (8', 14), (8', 14''),$ $(8'', 14'), (8'', 14''), (8, 16''), (8'', 16''),$ $(8, 18'), (8, 18''), (8', 18'), (8', 18''),$ $(8'', 18'), (8'', 18''), (8, 22''), (8'', 22''),$ $(10, 10'), (10', 10'), (10, 12'), (10', 12''),$ $(10, 14'), (10, 14''), (10', 14), (10', 14'),$ $(10, 16''), (10', 16''), (10, 18''), (10', 18'),$ $(12', 12'), (12'', 12''), (12', 14), (12', 14'),$ $(12', 14''), (12'', 14), (12'', 14'),$ $(12'', 14''), (12', 18'), (12'', 18''),$ $(14, 14''), (14', 14'), (14, 16''), (14'', 16'')$

• • •

listed all 462 one can get with Γ'N

The minimal & next-to-minimal landscapes

params. = 6+6+2 = 14 | # params. = 7+7+2 = 16

(k,k')	Minimal models (I and II)	Next-to-minimal models (I and II)	Next-to-minimal models (III)
All Γ'_N	$(10,12)$, $(12,14)$, $(14,16)$	$(16,18)$, $(18,20)$, $(20,22)$	$\boxed{(4,8)}$, $(4,14)$, $\boxed{(6,6),\ (6,10)}$, $(6,14)$, $\boxed{(8,8)}$,
S_3 only	$(10',12)$, $(10,18')$, $(10',18)$, $(12,12')$, $(12,14')$, $(12,16')$, $(12',16)$, $(12,20')$, $(12',20)$, $(14',16)$, $(14,18')$, $(14',18)$, $(14,22')$, $(14',22)$, $(16,16')$, $(16',18)$, $(16',18')$, $(16,20')$, $(16',20)$, $(18,20')$, $(18',20')$, $(20,20')$, $(20',22)$, $(20',22')$	$(16,18')$, $(16,24')$, $(16',24)$, $(18,18')$, $(18',20)$, $(18,22')$, $(18',22)$, $(18,26')$, $(18',26)$, $(20,22')$, $(20,24')$, $(20',24)$, $(20,28')$, $(20',28)$, $(22,22')$, $(22,24')$, $(22',24')$, $(22,26')$, $(22',26)$, $(24',26)$, $(24',26')$, $(26,26')$, $(26,28')$, $(26',28')$	$(4,$ ... $(6,$ ... $(8,20')$, $(10',10')$, $(10,12')$, $(10',12')$, $(10,14')$, $(10',14)$, $(10,16')$, $(10',16')$, $(12',14)$, $(12',14')$, $(14,14')$
A'_4 only	$(8',12)$, $(8',18)$, $(10',12)$, $(10,16')$, $(10',16)$, $(10,20'')$, $(10',20)$, $(12,12')$, $(12,12'')$, $(12,14')$, $(12,14'')$, $(12,16'')$, $(12',16)$, $(12'',16')$, $(12,18')$, $(12,18'')$, $(12',18)$, $(12'',18)$, $(12,22'')$, $(12',22)$, $(12'',22')$, $(14,16')$, $(14',16)$, $(14',16')$, $(14'',16)$, $(14'',16')$, $(14',18)$, $(14,20')$, $(14,20'')$, $(14',20)$, $(14',20'')$, $(14'',20)$, $(14'',20')$, $(14,24'')$, $(14'',24')$, $(16,16'')$, $(16',16'')$, $(16,18')$, $(16,18'')$, $(16',18')$, $(16',18'')$, $(16'',18)$, $(16'',20')$, $(16,22'')$, $(16',22'')$, $(16'',22)$, $(16'',22')$, $(16'',26')$, $(18,18')$, $(18,18'')$, $(18',20)$, $(18',20')$, $(18',20'')$, $(18'',20)$, $(18'',20')$, $(18'',20'')$, $(18,22'')$, $(18',22)$, $(18'',22')$, $(18',24'')$, $(18'',24')$, $(20,22'')$, $(20',22'')$, $(20'',22'')$, $(22,22'')$, $(22',22'')$, $(22'',24')$, $(22'',24'')$, $(22'',26')$	$(14',24)$, $(16,16')$, $(16',18)$, $(16,20'')$, $(16',20)$, $(16,22')$, $(16',22)$, $(16,26'')$, $(16',26)$, $(18,20')$, $(18,20'')$, $(18,24')$, $(18,24'')$, $(18',24)$, $(18'',24)$, $(18,28'')$, $(18',28)$, $(18'',28')$, $(20,20')$, $(20,20'')$, $(20',20'')$, $(20,22')$, $(20',22)$, $(20',22')$, $(20'',22)$, $(20'',22')$, $(20,24'')$, $(20'',24')$, $(20,26')$, $(20,26'')$, $(20',26)$, $(20',26'')$, $(20'',26)$, $(20'',26')$, $(20,30'')$, $(20'',30')$, $(22,22')$, $(22,24')$, $(22,24'')$, $(22',24')$, $(22',24'')$, $(22'',24)$, $(22,26'')$, $(22',26)$, $(22,28'')$, $(22',28'')$, $(22'',28)$, $(22'',28')$, $(22'',32')$, $(24',24'')$, $(24',26)$, $(24',26')$, $(24',26'')$, $(24'',26)$, $(24'',26')$, $(24'',26'')$, $(24',30'')$, $(24'',30')$	$(4',8'')$, $\boxed{(4,12')}$, ... $(4,16'')$, $(4',16'')$, ... $(4,22'')$, $(4',22'')$, ... $(6,14'')$, $(6,18')$, $(6,1$ ... $(8',8'')$, $(8'',8'')$, $(8'',10')$, $(8,12')$, $(8,12'')$, $(8',12')$, $(8',12'')$, $(8'',12')$, $(8'',12'')$, $(8,14')$, $(8',14)$, $(8',14'')$, $(8'',14')$, $(8'',14'')$, $(8,16'')$, $(8'',16'')$, $(8,18')$, $(8,18'')$, $(8',18')$, $(8',18'')$, $(8'',18')$, $(8'',18'')$, $(8,22'')$, $(8'',22'')$, $(10,10')$, $(10',10')$, $(10,12')$, $(10',12'')$, $(10,14')$, $(10,14'')$, $(10',14)$, $(10',14')$, $(10,16'')$, $(10',16'')$, $(10,18'')$, $(10',18')$, $(12',12')$, $(12'',12'')$, $(12',14)$, $(12',14')$, $(12',14'')$, $(12'',14)$, $(12'',14')$, $(12'',14'')$, $(12',18')$, $(12'',18'')$, $(14,14'')$, $(14',14')$, $(14,16'')$, $(14'',16'')$

Feruglio, Strumia, Titov
[2305.08908]

Petcov, Tanimoto
[2404.00858]

• • •

listed all 462 one can get with Γ'N

The minimal & next-to-minimal landscapes

• • •

S_4' only

$(\hat{7}',12), (\hat{7}',18), (\hat{9}',12), (\hat{9}',16), (\hat{9}',20),$
$(10',12), (10,\widehat{15}'), (10',\widehat{15}'), (10,18'),$
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$(12',\widehat{15}'), (12,16'), (12',16), (12,\widehat{17}),$
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$(14',16), (14,18'), (14',18), (14,\widehat{19}'),$
$(14',\widehat{19}'), (14,\widehat{21}), (14',\widehat{21}'), (14,22'),$
$(14',22), (14,\widehat{25}), (14',\widehat{25}'), (\widehat{15},\widehat{15}'),$
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$(22',24'), (22,\widehat{25}), (22',\widehat{25}'), (22,26'),$
$(22',26), (22,\widehat{27}), (22',\widehat{27}), (22,\widehat{29}),$
$(22',\widehat{29}'), (\widehat{23},24), (\widehat{23}',24'), (\widehat{23}',\widehat{25}),$
$(\widehat{23}',\widehat{25}'), (\widehat{23}',26), (\widehat{23},\widehat{27}'), (\widehat{23}',\widehat{27}),$
$(\widehat{23},28), (\widehat{23}',28'), (\widehat{23},30'), (\widehat{23},\widehat{31}'),$
$(\widehat{23}',\widehat{31}), (\widehat{23},34'), (24',\widehat{25}), (24',\widehat{25}'),$
$(24',26), (24',26'), (24',\widehat{29}), (24',\widehat{29}'),$
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$(6',10'), (6,12'), (6',12'), (6,\widehat{13}'),$
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$(8,16'), (8,\widehat{17}), (8,\widehat{17}'), (8,\widehat{19}), (8,20'),$
$(8,\widehat{23}), (\hat{9},\hat{9}), (\hat{9},\hat{9}'), (\hat{9}',\hat{9}'), (\hat{9},10'),$
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$(\hat{9},\widehat{13}'), (\hat{9}',\widehat{13}), (\hat{9}',\widehat{13}'), (\hat{9},14'), (\hat{9}',14),$
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$(\hat{9}',\widehat{17}), (\hat{9}',\widehat{17}'), (\hat{9},\widehat{19}), (\hat{9},20'), (\hat{9}',20'),$
$(\hat{9},\widehat{23}), (10',10'), (10,\widehat{11}'), (10',\widehat{11}'),$
$(10,12'), (10',12'), (10,\widehat{13}'), (10',\widehat{13}),$
$(10,14'), (10',14), (10,\widehat{15}), (10',\widehat{15}),$
$(10,16'), (10',16'), (10,\widehat{17}), (10',\widehat{17}'),$
$(10,\widehat{19}), (10',\widehat{19}), (\widehat{11}',\widehat{11}'), (\widehat{11}',\widehat{13}'),$
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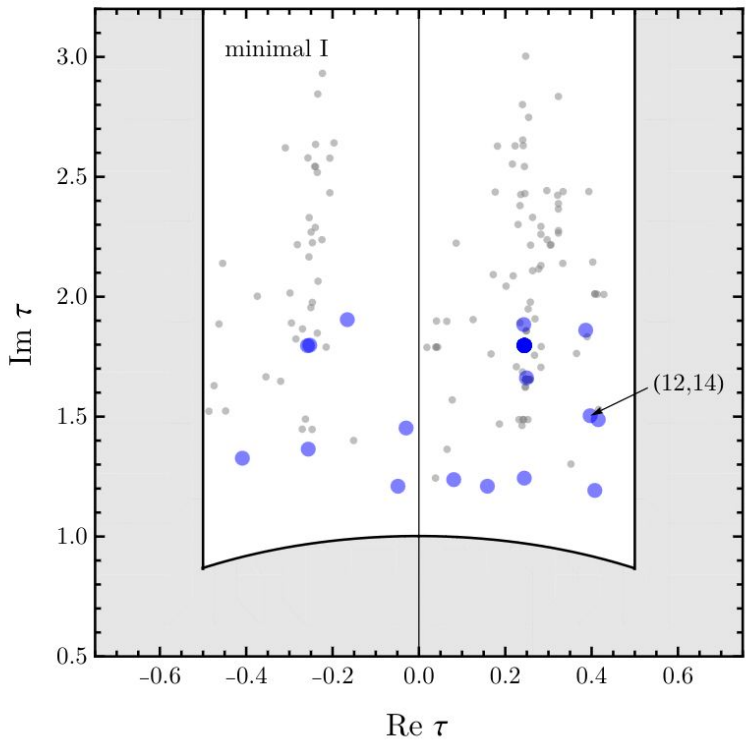
$(\widehat{11}',\widehat{17}'), (\widehat{11}',20'), (12',\widehat{13}), (12',\widehat{13}'),$
$(12',14), (12',14'), (12',\widehat{17}), (12',\widehat{17}'),$
$(\widehat{13},\widehat{13}), (\widehat{13},\widehat{13}'), (\widehat{13}',\widehat{13}'), (\widehat{13},14),$
$(\widehat{13}',14'), (\widehat{13},\widehat{15}), (\widehat{13},16'), (\widehat{13}',16'),$
$(\widehat{13},\widehat{19}), (14,14'), (14,\widehat{15}), (14',\widehat{15}),$
$(14,\widehat{17}), (14',\widehat{17}'), (\widehat{15},\widehat{17})$

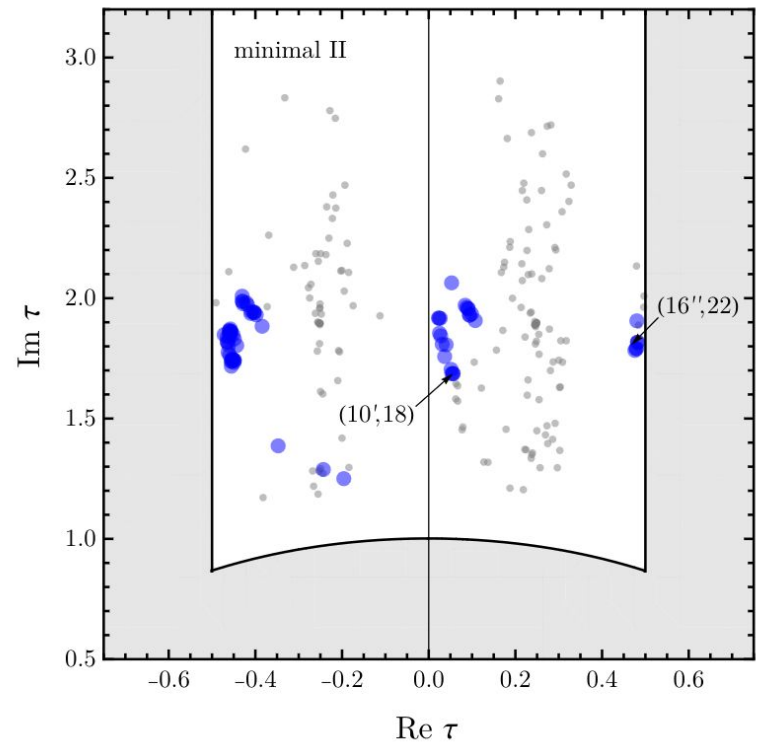
fin

A peek into the minimal model landscape

models without transposition



models with transposition



Moduli selection from the minimization of **parameter hierarchies** (log spread), using the normalization proposal in Petcov [2311.04185]

An S_4' benchmark

- Quark mass hierarchies and mixing angles can be reproduced by **O(1) parameters**

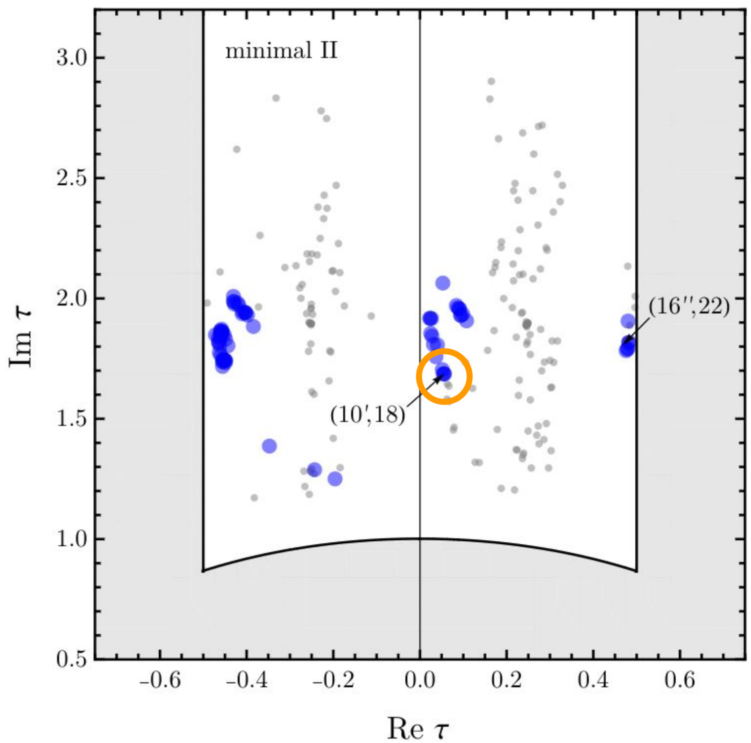
  Also in FST [2305.08908]

- Overall tendency: parametric hierarchies mostly in downs

$$\alpha_2^u \simeq 1.00\,\alpha_1^u\,, \quad \alpha_3^u \simeq 1.65\,\alpha_1^u\,, \quad \alpha_{12}^u \simeq 1.27\,\alpha_1^u\,,$$
$$\alpha_{13,1}^u \simeq 1.47\,\alpha_1^u\,, \quad \alpha_{13,2}^u \simeq 1.61\,\alpha_1^u\,,$$

$$\alpha_2^d \simeq 0.48\,\alpha_1^d\,, \quad \alpha_3^d \simeq 0.05\,\alpha_1^d\,, \quad \alpha_{12}^d \simeq 7.92\,\alpha_1^d\,,$$
$$\alpha_{13,1}^d \simeq -7.90\,\alpha_1^d\,, \quad \alpha_{13,2}^d \simeq 0.05\,\alpha_1^d\,,$$

models with transposition



$$\tau \simeq 0.06 + 1.69\,i$$

Extra! Extra! Read all about it!

Feruglio, Parriciatu, Strumia, Titov [2406.01689] 

- can use **lower weights** if we **forego minimality**
- can get **non-Abelian irreps** if we **add vector-like quarks**
- consider models with level N > 1, either using Γ_N or $\Gamma(N)$ instead of Γ

Γ_2 :

	SM quarks			Extra vector-like quarks			
	Q	D^c	U^c	D'^c	D'	U'^c	U'
$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$	$2_{1/6}$	$1_{1/3}$	$1_{-2/3}$	$1_{1/3}$	$1_{-1/3}$	$1_{-2/3}$	$1_{2/3}$
Flavour symmetry Γ_2	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_1$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_1$	$\mathbf{2} \oplus \mathbf{1}_0$	$\mathbf{2} \oplus \mathbf{1}_0$
Modular weights k_Φ	-2	-2	-2	$+2$	$+2$	$+2$	$+2$

Γ_3 :

	SM quarks			Extra vector-like quarks			
	Q	D^c	U^c	D'^c	D'	U'^c	U'
$\mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y$	$2_{1/6}$	$1_{1/3}$	$1_{-2/3}$	$1_{1/3}$	$1_{-1/3}$	$1_{-2/3}$	$1_{2/3}$
Flavour symmetry Γ_3	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
Modular weights k_Φ	-1	±1	±1	$+1$	∓1	$+1$	∓1

+ modifications to the minimal Kähler

To close, I hope I have convinced you that...

Modular symmetries can…

…offer a **predictive framework** for flavour

…provide an origin for **CP violation** (CPV)

…help solve the **strong CP** problem

Modular symmetries can...

...offer a **predictive framework** for flavour

...provide an origin for **CP violation** (CPV)

...help solve the **strong CP** problem

they can arise from stringy constructions...

Parting words (i.e. what next?)

- modular symmetry breaking as the only source of CPV?
- natural origin of mass hierarchies?
- hints of universality?
- use TD to fix Kähler, irreps, weights?
- pheno beyond masses and mixing?

Parting words (i.e. what next?)

- modular symmetry breaking as the only source of CPV?
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- use TD to fix Kähler, irreps, weights!
- pheno beyond masses and mixing?

State of the art

See next talk, by H. P. Nilles


Obrigado!

Backup slides

Modular-invariant SUSY actions

Ferrara et al, '89

$$W(\psi;\tau) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1\ldots i_n,s} \left(Y_{i_1\ldots i_n,s}(\tau)\,\psi_{i_1}\ldots\psi_{i_n}\right)_{\mathbf{1},s}$$

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\; K(\psi,\overline{\psi};\tau,\overline{\tau}) + \int d^4x\, d^2\theta\; W(\psi;\tau) + \text{h.c.}$$

τ is a dimensionless spurion: once its value is fixed, it **parameterizes all** modular sym. breaking

One may argue that *Y*'s play the role of flavons, but structures are **completely fixed** given the modulus VEV

SUSY breaking effects?

- **RGEs & threshold corrections** need to be considered, depend on tan β and unknown SUSY spectrum
- **SUSY-breaking** corrections can be made negligible via separation of scales (power counting argument)
- Under reasonable conditions, predictions may be unaffected

Feruglio and Criado [1807.01125]

What about... the Kähler?

- **Not holomorphic**: unconstrained by the symmetry!
- Under a modular transformation, invariant up to:

  $$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) \;\rightarrow\; K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) + f(\chi_i; \tau) + f(\overline{\chi}_i; \overline{\tau})$$

- Minimal choice:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) = -h\,\Lambda_0^2\,\log(-i(\tau - \overline{\tau})) + \sum_i \frac{|\chi_i|^2}{(-i(\tau - \overline{\tau}))^{k_i}}$$

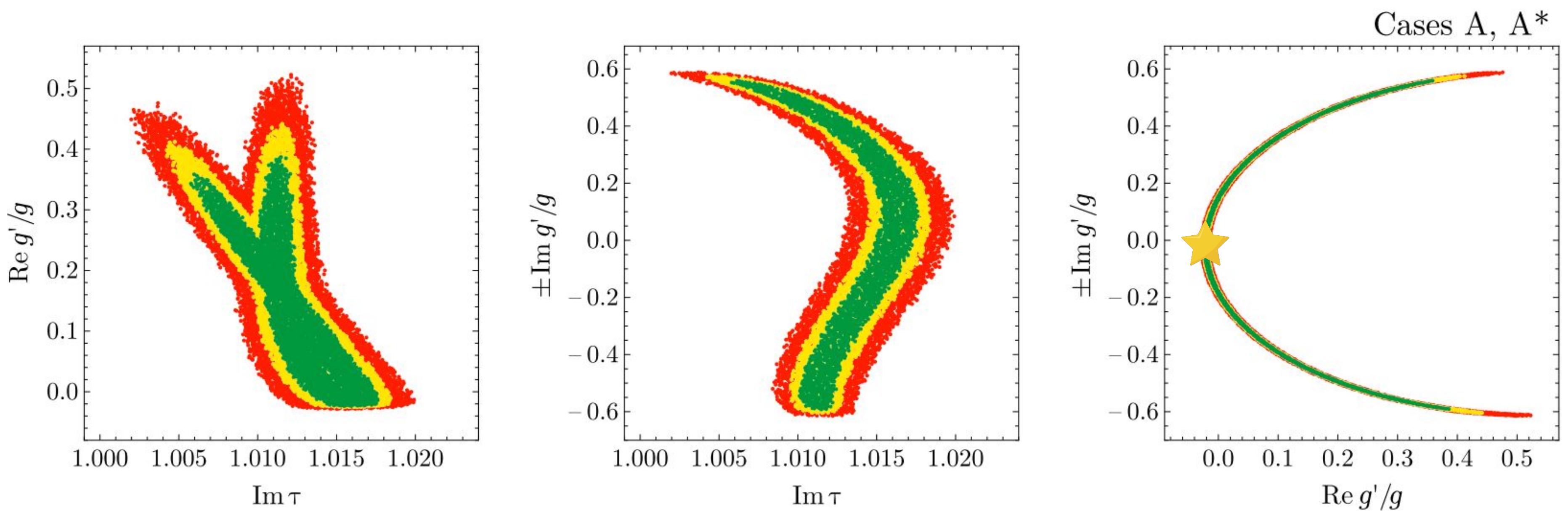
impacts pheno → should be justified from the top-down

Chen, Ramos-Sánchez and Ratz [1909.06910]

- Further constraints may arise from the (unavoidable...) combination of modular + traditional flavour symmetries

Nilles, Ramos-Sanchez, Vaudrevange [2004.05200]

Correlations between parameters in the first S4 example model



Novichkov, JP, Petcov, Titov [1811.04933]

The QCD angle is holomorphic

Furthermore, extra non-minimal kinetic terms are possible, because the 3×3 kinetic matrices $Z_f(\tau, \tau^\dagger)$ of fermions $f = \{u_R, d_R, Q\}$ are not holomorphic in τ, and modular invariance allows them to depend on the CP-violating parameters $\tau, \tau^\dagger$ in new ways. These non-minimal kinetic terms reduce the predictive power of flavour models based on modular symmetries [28, 41–43] and are often assumed to be negligible.

Such extra complex terms are not a problem for our proposed interpretation of the QCD problem, $\bar\theta = 0$. Indeed each kinetic matrix Z_f can be brought to canonical form via a general linear transformation of the three generations of $f_{1,2,3}$ quarks: a linear transformation affects both $\arg\det M_q$ and $\theta_{\rm QCD}$ (via the anomaly) but leaves the physical combination $\bar\theta$ invariant. Furthermore, these linear transformations can be chosen in ways that leave $\arg\det M_q$ and $\theta_{\rm QCD}$ separately invariant, by decomposing each kinetic matrix Z_f either as $Z_f = H_f^\dagger H_f$ (where H_f is an hermitian matrix, see e.g. [44]) or as $Z_f = V_f^\dagger \Delta_f^2 V_f$ (where Δ_f is a diagonal matrix with real positive entries and V_f is a product of 3 complex rotations with unit determinant). The consequent linear transformation of quark fields affects their masses and mixings (including the CKM phase) without affecting $\arg\det M_q$.

This discussion shows that, unlike fermion masses and mixing angles, the physical $\bar\theta$ angle is a holomorphic quantity completely insensitive to the Kähler potential and can be effectively constrained by modular invariance alone, at least in the limit of unbroken supersymmetry.

Feruglio, Strumia, Titov [2305.08908]

Modular anomalies

Canonical normalisation

$$K \supset \frac{\Phi^\dagger \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}} = \Phi^\dagger_{\text{can}} \Phi_{\text{can}} \qquad \Phi_{\text{can}} = \{\phi_{\text{can}}, \psi_{\text{can}}\}$$

$$\psi_{\text{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_\Phi}{2}} \psi_{\text{can}} = e^{-ik_\Phi \alpha(\tau)} \psi_{\text{can}} \qquad \alpha(\tau) = \arg(c\tau + d)$$

Modular symmetry acts on canonically normalised fields as a τ-dependent phase rotation (with $\tau = \tau(x)$)

Conditions for modular-gauge anomaly cancellation

$$\mathrm{SU}(3)_C : \quad A \equiv \sum_{i=1}^{3} \left(2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}}\right) = 0$$

$$\mathrm{SU}(2)_L : \quad \sum_{i=1}^{3} \left(3k_{Q_i} + k_{L_i}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y : \quad \sum_{i=1}^{3} \left(k_{Q_i} + 8k_{u_{Ri}} + 2k_{d_{Ri}} + 3k_{L_i} + 6k_{e_{Ri}}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

from Arsenii Titov's slides at Cosmic WISPers 2024

tests of modulus couplings

G-J. Ding, FF, 2003.13448

non standard neutrino interactions

$$\mathcal{L} = i \sum_{f=e,e^c,\nu} \overline{f}\,\overline{\sigma}^\mu \partial_\mu f + \frac{1}{2}\partial_\mu\varphi_\alpha\partial^\mu\varphi_\alpha - \frac{1}{2}M_\alpha^2\varphi_\alpha^2$$
$$- (m_e + \mathcal{Z}_\alpha^e\varphi_\alpha)e^c e - \frac{1}{2}\nu(m_\nu + \mathcal{Z}_\alpha^\nu\varphi_\alpha)\nu + h.c. + \ldots .$$

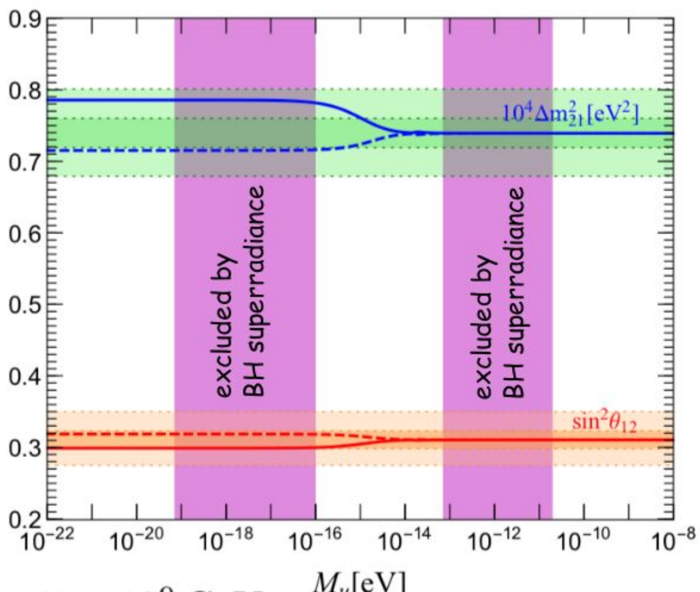
$$\tau = \langle\tau\rangle + \frac{\varphi_u + i\,\varphi_v}{\sqrt{2}}$$

in medium with non-zero electron number density

small, unless the modulus is very light

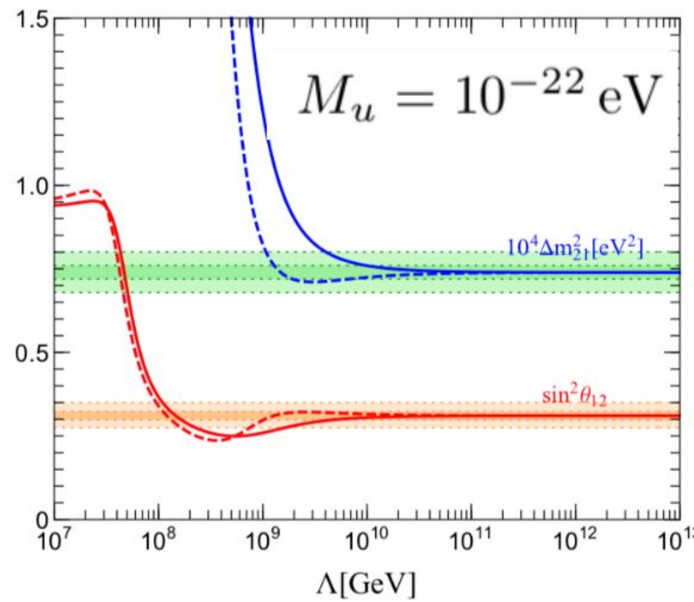
$$\delta m_\nu(0) = -n_e^0 \frac{\mathrm{Re}(\mathcal{Z}^e)\mathcal{Z}^\nu}{M^2(R)}\,,$$

in the sun:



$\Lambda = 5 \times 10^9$ GeV

[modulus VEV]



from Feruglio's slides at Bethe Workshop, citing [2003.13448]

But what if I normalize *this* way?

A comment on normalizations

mod. form of weight K

$$W \supset \sum_i \left(\psi\, \boxed{\alpha_i Y_i^{(K)}}\, \psi^c\right)_{\mathbf{1}} H_q$$

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$$W \supset \sum_i \left(\psi\, \alpha_i Y_i^{(K)}\, \psi^c\right)_1 H_q$$

$\lvert\alpha\rvert$	$\lVert Y \rVert_E$
1	**100**
10	**10**
100	**1**

Same model predictions!

how can we discuss natural α's?

how do we interpret a hierarchy between α's?

how can we claim modular symmetries are responsible for hierarchies, not α's?

(norms of Y's are not fixed by group theory)

But what if I normalize *this* way?

A comment on normalizations

can trust these:

$$Y_{\mathbf{3}} \sim \begin{pmatrix} 1 \\ \epsilon \\ \epsilon^2 \end{pmatrix}$$

cannot trust these:

$$M \sim \left(\begin{array}{cc|c} 1 & \epsilon & 1 \\ \epsilon & \epsilon^2 & \epsilon \\ \hline \epsilon & \epsilon^2 & \epsilon \end{array}\right)$$

how can we claim modular symmetries are responsible for hierarchies, not α's?

discussion in Varzielas, Levy, JP, Petcov [2307.14410]

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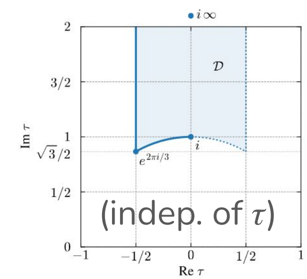
A normalization proposal / choice

Petcov [2311.04185], D. Zagier (1981)

A "global" normalisation based on the Petersson inner product

$$\mathrm{N}\left[Y^{(K)}\right]^2 \equiv \iint_{\mathcal{D}} \sum_i \left|Y_i^{(K)}(x+iy)\right|^2 (2y)^K \frac{dx\,dy}{y^2} \overset{!}{=} 1$$



(indep. of τ)

different prescription for non-cusp forms (yet another if K=1)

- Is there a general top-down recipe?
- Basis ambiguity if there are several forms of the same weight and irrep

Lessons from eclectic flavor symmetries

see [1901.03251, 1908.00805, 2001.01736, **2004.05200**, 2006.03059, 2112.06940]
and Nilles, Ramos-Sánchez [2404.16933] for a recent summary

- There is no possible scheme with just modular flavor symmetries
- Also discrete R-symmetries seem unavoidable
- A limited type of groups appear (e.g. *T'*)

<table>
<tr><th colspan="2">nature of symmetry</th><th>outer automorphism of Narain space group</th><th colspan="5">flavor groups</th></tr>
<tr><td rowspan="6">eclectic</td><td rowspan="2">modular</td><td>rotation S $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="3" rowspan="2">T'</td><td rowspan="6">$\Omega(2)$</td></tr>
<tr><td>rotation T $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td rowspan="4">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td><td rowspan="4">$\Delta'(54,2,1)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation C $= \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
<tr><td>rotation R $= \gamma_{(3)} \in \mathrm{SL}(2,\mathbb{Z})_U$</td><td colspan="3">$\mathbb{Z}_9^R$</td></tr>
</table>

Nilles, Ramos-Sanchez, Vaudrevange [2006.03059]

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- Weights (typically fractional) correlated with irreps

sector	matter fields	eclectic flavor group $\Omega(2)$								
		modular T' subgroup				traditional $\Delta(54)$ subgroup				$\mathbb{Z}_9^R$
	Φ_n	irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

Nilles, Ramos-Sanchez, Vaudrevange [2004.05200]

Lessons from eclectic flavor symmetries

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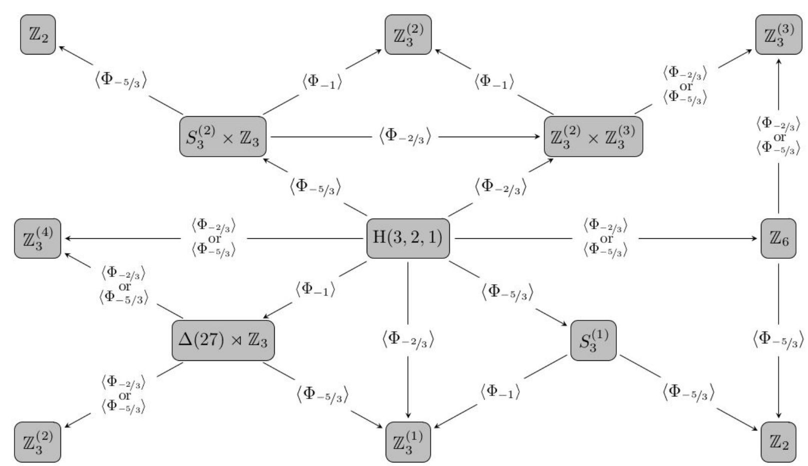
Baur et al. [2112.06940]



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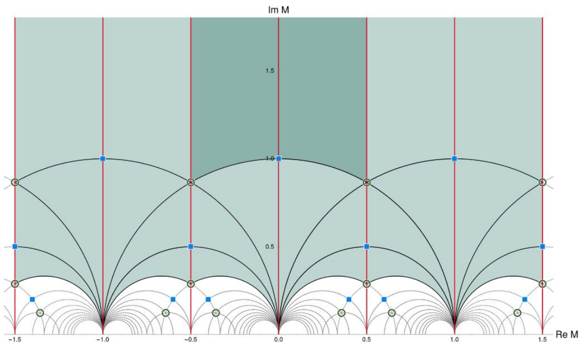
- Larger fundamental domains (Γ(N) instead of Γ?)



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- Larger fundamental domains ($\Gamma(N)$ instead of Γ?)
- Top-down and bottom-up do **not yet** meet

but there are a few BU attempts:
Chen et al. [2108.02240]; Ding et al. [2303.02071];
Li, Ding [2308.16901]; Li, Lu, Ding [2405.13460]

Moduli stabilization

early attempts: [1909.05139, 1910.11553]

Simplest modular-invariant potentials?

- Studied by Cvetič, Font, Ibáñez, Lüst and Quevedo (1991) $\mathcal{N}=1$ SUGRA

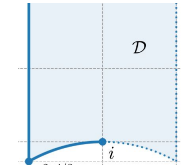
$$V(\tau,\overline{\tau}) = \frac{\Lambda_V^4}{8(\mathrm{Im}\,\tau)^3|\eta|^{12}}\left[\frac{4}{3}\left|iH' + \frac{3}{2\pi}H\hat{G}_2\right|^2 (\mathrm{Im}\,\tau)^2 - 3|H|^2\right]$$

$$H(\tau) = (j(\tau) - 1728)^{m/2}\, j(\tau)^{n/3} \qquad W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^6}$$

$m, n = 0, 1, 2, \ldots$

- This potential is **modular-** and **CP-invariant**
- Simplified model, independent of the level *N*

q- and u-expansions of η



$|q| \leq e^{-\sqrt{3}\pi} \simeq 0.004$

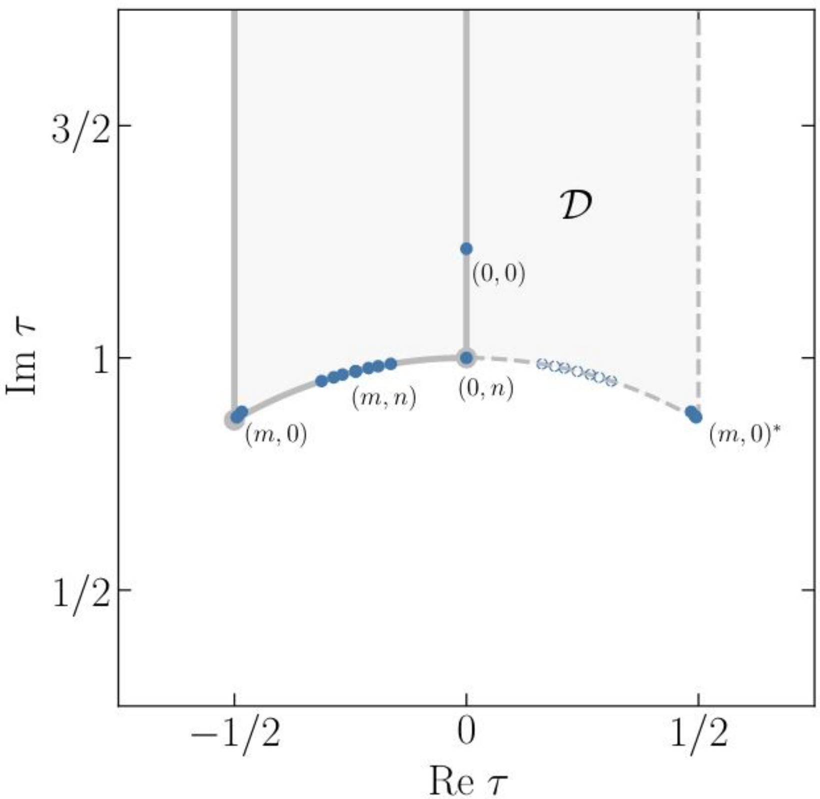
$$\eta = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}} = q^{1/24}\left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22})\right)$$

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2} \qquad \tilde{\eta}(u) \equiv \frac{\eta(u)}{\sqrt{1-u}}$$

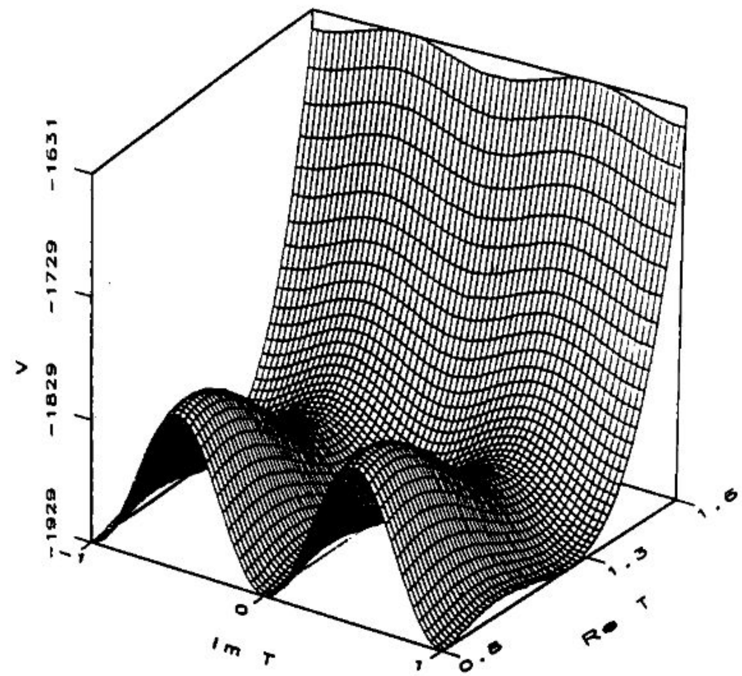
$$u \xrightarrow{ST} \omega^2 u \qquad \tilde{\eta}(u) \xrightarrow{ST} \tilde{\eta}(u)$$

$$\begin{aligned} \tilde{\eta}(u) &\simeq e^{-i\pi/24}\left(0.800579 - 0.573569u^3 - 0.780766u^6 - 0.150007u^9\right) + \mathcal{O}(u^{12}) \\ &\equiv e^{-i\pi/24}\left(\tilde{\eta}_0 + \tilde{\eta}_3 u^3 + \tilde{\eta}_6 u^6 + \tilde{\eta}_9 u^9\right) + \mathcal{O}(u^{12}), \end{aligned}$$

Global minima for (*m*,*n*)-potentials



(*0*,*0*) ✓



“(...) we conjecture that all extrema of V entirely lie on [the boundary].” — Cvetič et al.

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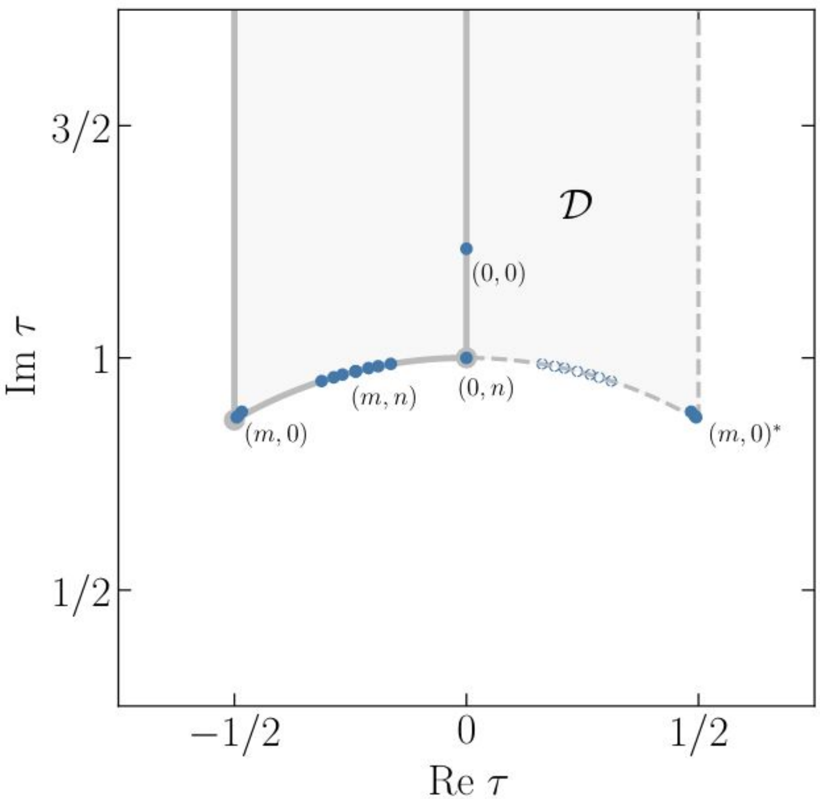
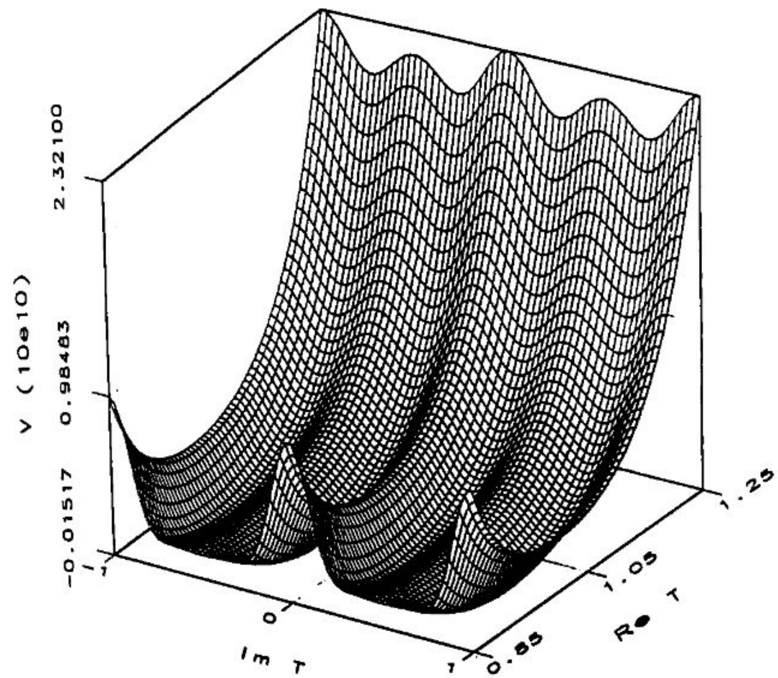


(*1*,*1*) ✓



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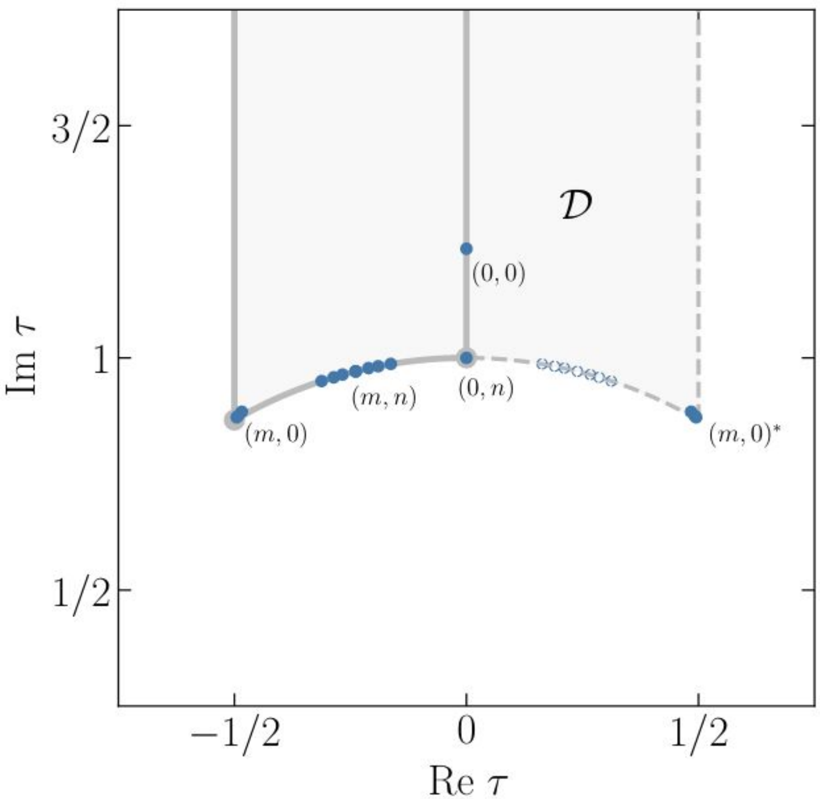
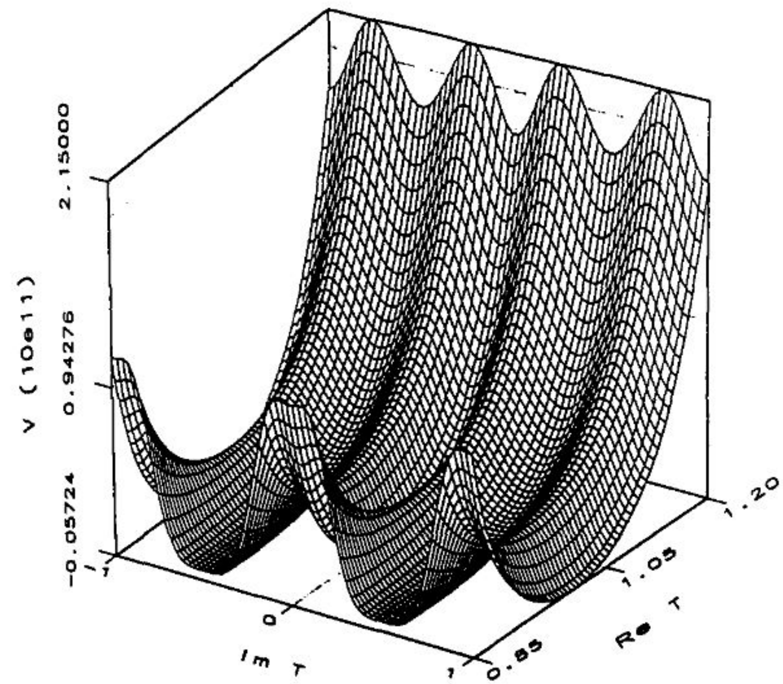


(0,3) ✓



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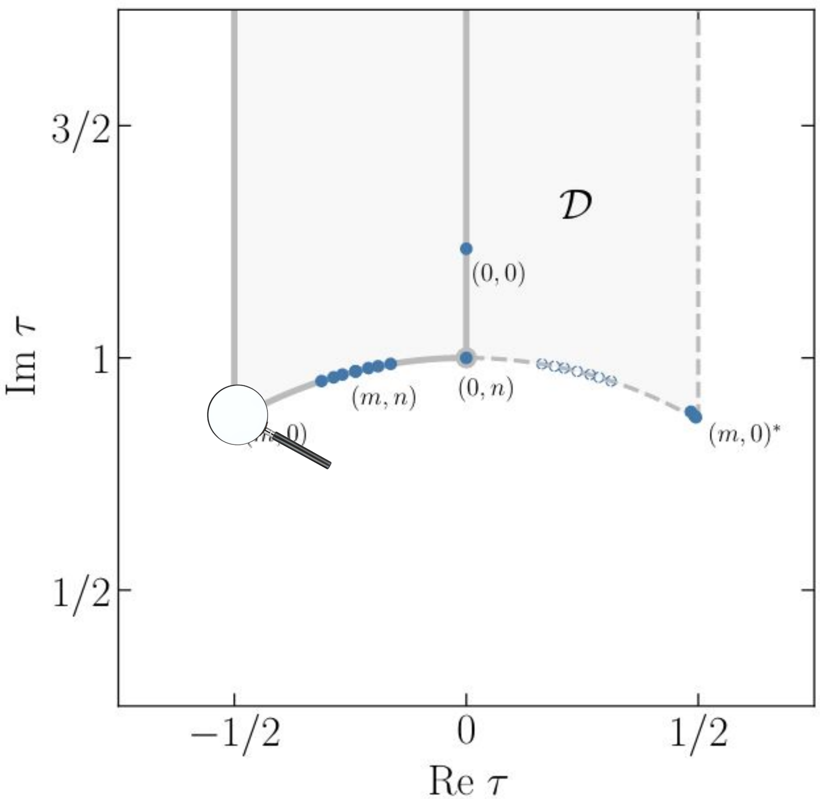

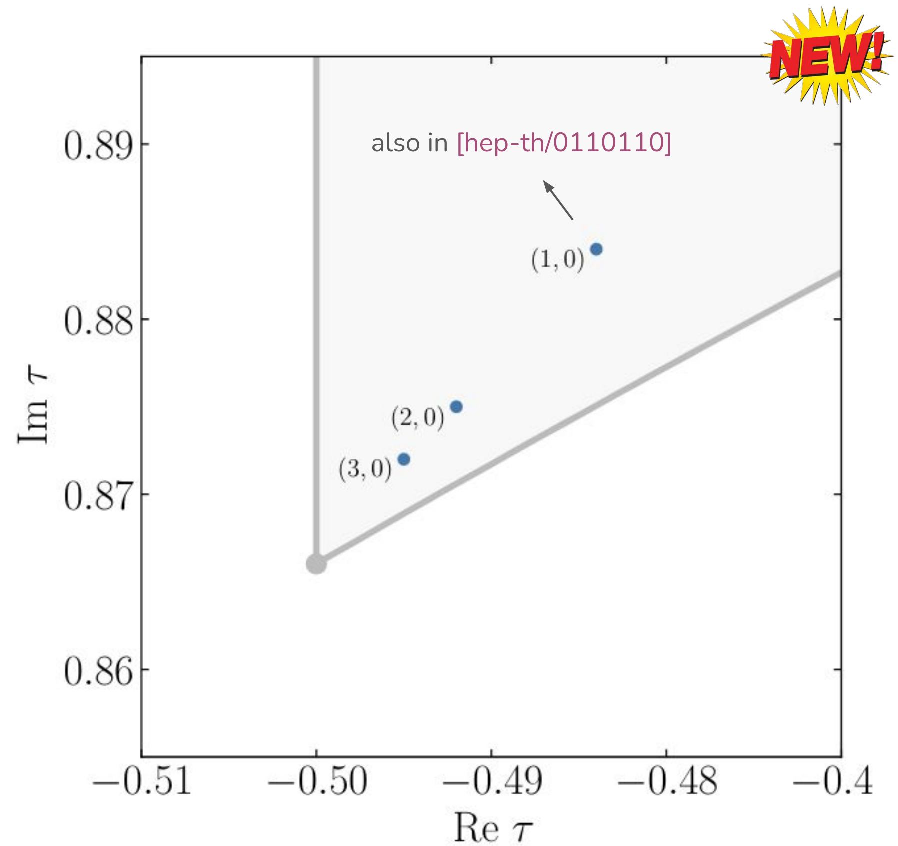

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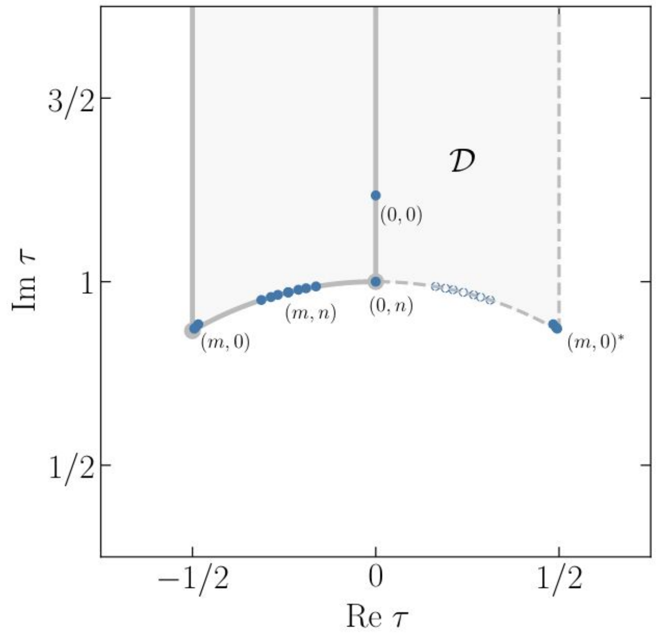
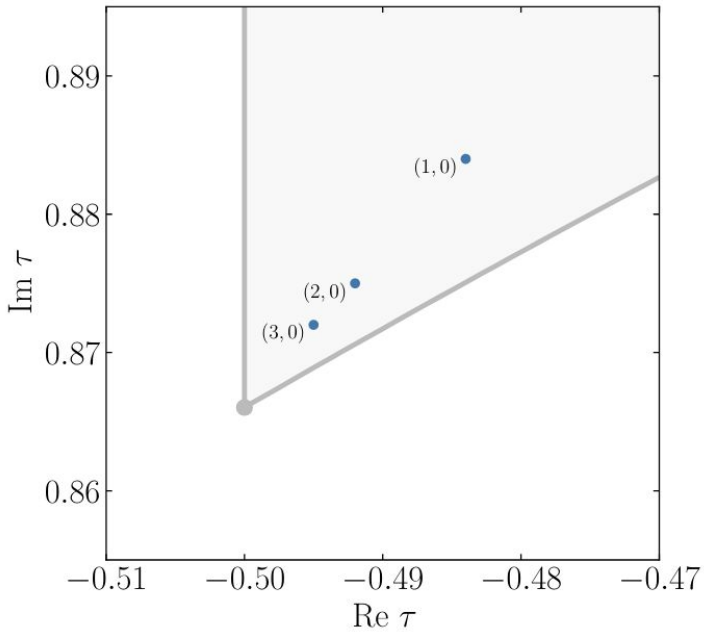
these results later confirmed by Leedom, Righi, Westphal [2212.03876]





$(0, 0)$ is a single minimum at $\tau \simeq 1.2i$ on the imaginary axis, corresponding to the case $m = n = 0$;

$(0, n)$ is a single minimum at the symmetric point $\tau = i$ attained when $m = 0$, $n \neq 0$;

$(m, 0)$ and $(m, 0)^*$ are a pair of degenerate minima for each $m \neq 0$ and $n = 0$: $(m, 0)$ is located in the vicinity of the left cusp $\tau = \omega$, approaching this symmetric point as m increases, while $(m, 0)^*$ is its CP-conjugate;

(m, n) is a series of minima on the unit arc, corresponding to $m \neq 0$, $n \neq 0$; these minima shift towards $\tau = \omega$ ($\tau = i$) along the arc as m (n) grows.

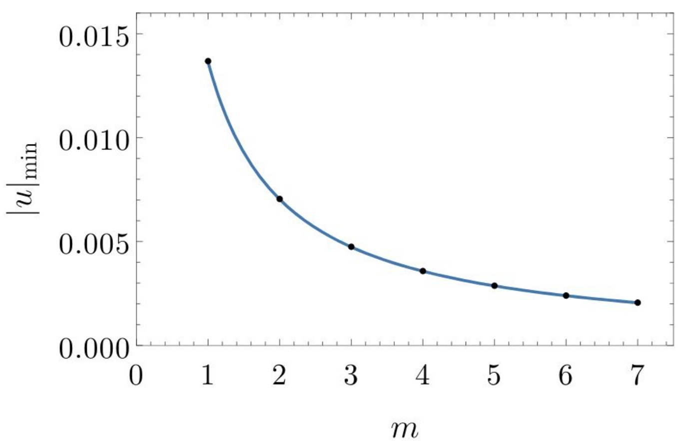
The (*m*,0) family of potentials

- *u*-expand (*m*,0) potentials to analyse them near the left cusp

$$V_{m,0} = \Lambda_V^4 \frac{1728^m}{\sqrt{3}\,\tilde{\eta}_0^{12}} \left\{-1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4\right\} + \mathcal{O}(|u|^6)$$

$$A_m \equiv \frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\, m + \frac{6\,|\tilde{\eta}_3|}{\tilde{\eta}_0} \simeq 68.78\,m + 4.30$$

- "Mexican"-hat potential (cusp is a maximum!)



$$|u|_{\rm min} \simeq \left(A_m^2 - 3\right)^{-1/2} \simeq A_m^{-1} = \frac{0.0145}{m + 0.0625}$$

The (*m*,0) family of potentials

(phase dependence)

$$u = |u|e^{i\phi}$$

- *u*-expanding to higher order shows dependence on $\phi \in [-\pi/3, 0]$

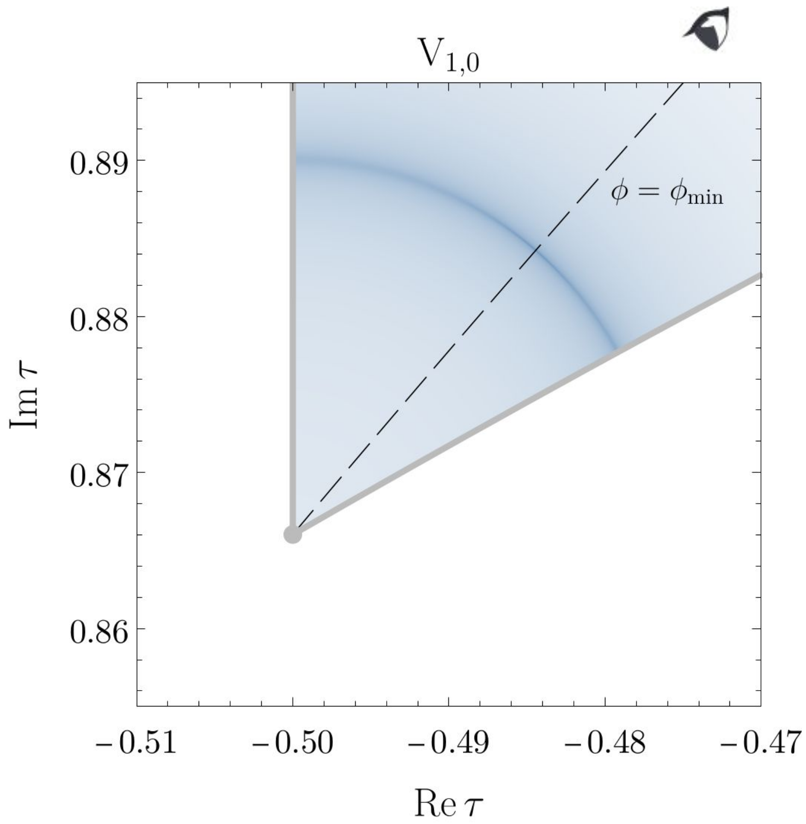
$$V_{m,0} \propto -1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4 + \left(-4 + 2A_m^2 + B_m^2 \cos 6\phi\right)|u|^6 + 2A_m B_m^2 \cos 3\phi\,|u|^7 + \left(-5 + 3A_m^2 + 2B_m^2 \cos 6\phi\right)|u|^8 + \mathcal{O}(|u|^9)$$

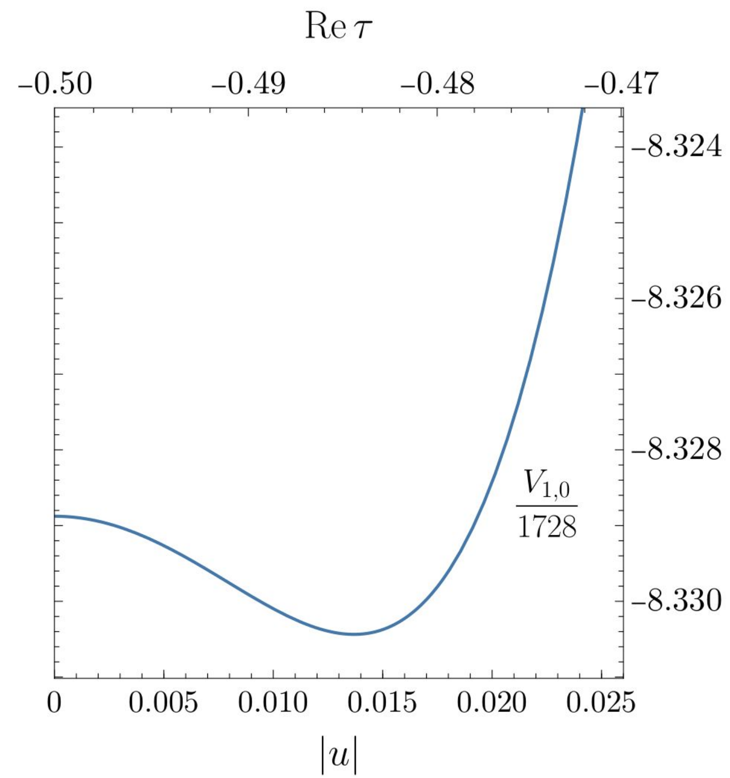
$$B_m^2 \equiv \frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\,m\left[\frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}(m-2) + \frac{3\,(31\,\tilde{\eta}_3^2 - 10\tilde{\eta}_0\tilde{\eta}_6)}{\tilde{\eta}_0|\tilde{\eta}_3|}\right] + \frac{6\,(7\tilde{\eta}_3^2 - 2\tilde{\eta}_0\tilde{\eta}_6)}{\tilde{\eta}_0^2}$$
$$\simeq 4730.60\,m^2 - 2069.73\,m + 33.26\,.$$

- Phase of *u* mostly determined by $|u|^6$ and $|u|^7$ terms

$$\phi_{\text{min}} \simeq -\frac{2\pi}{9} = -40^\circ$$

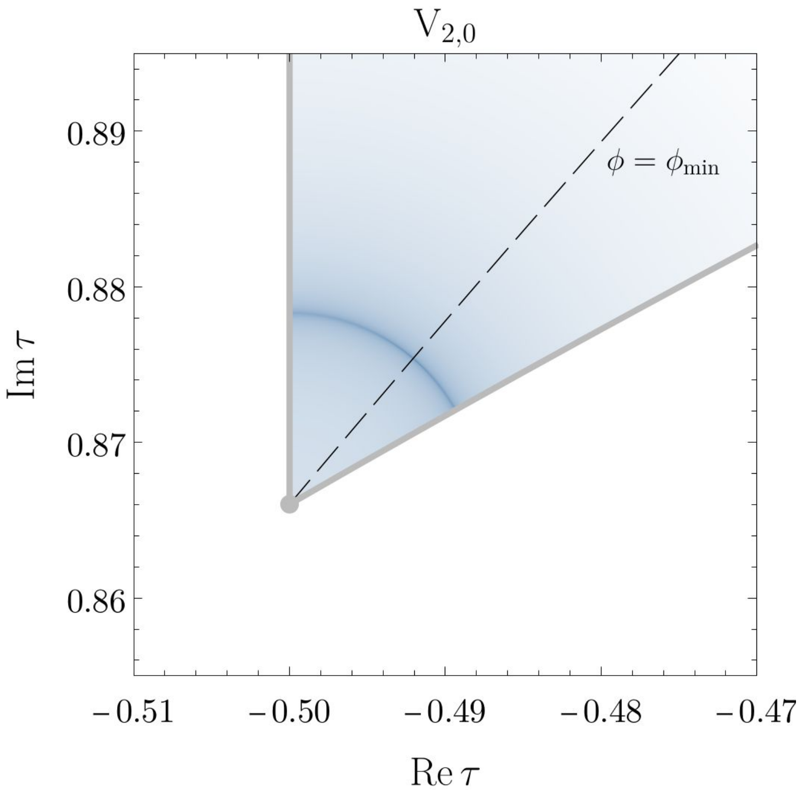
The (m,0) family of potentials (m = 1)

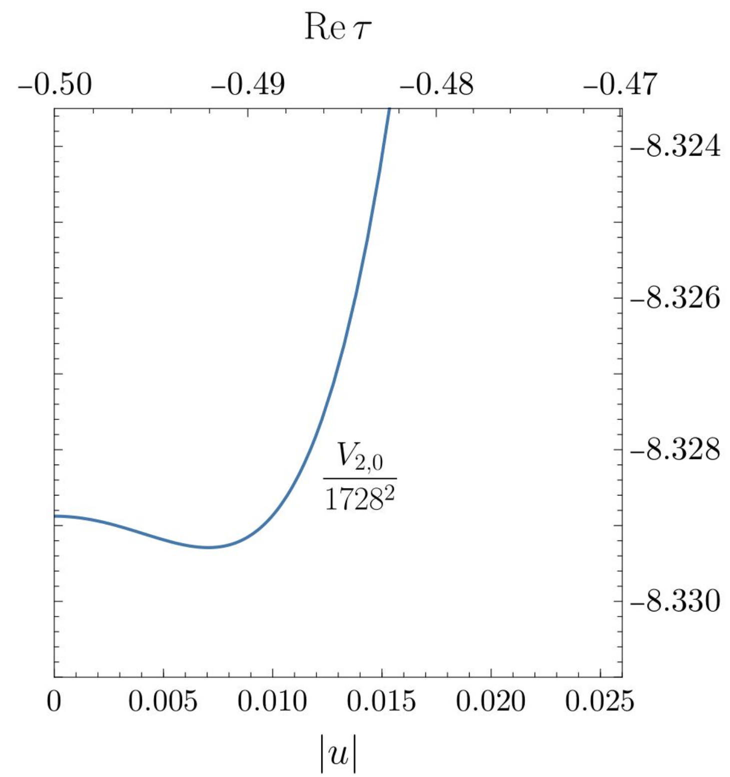




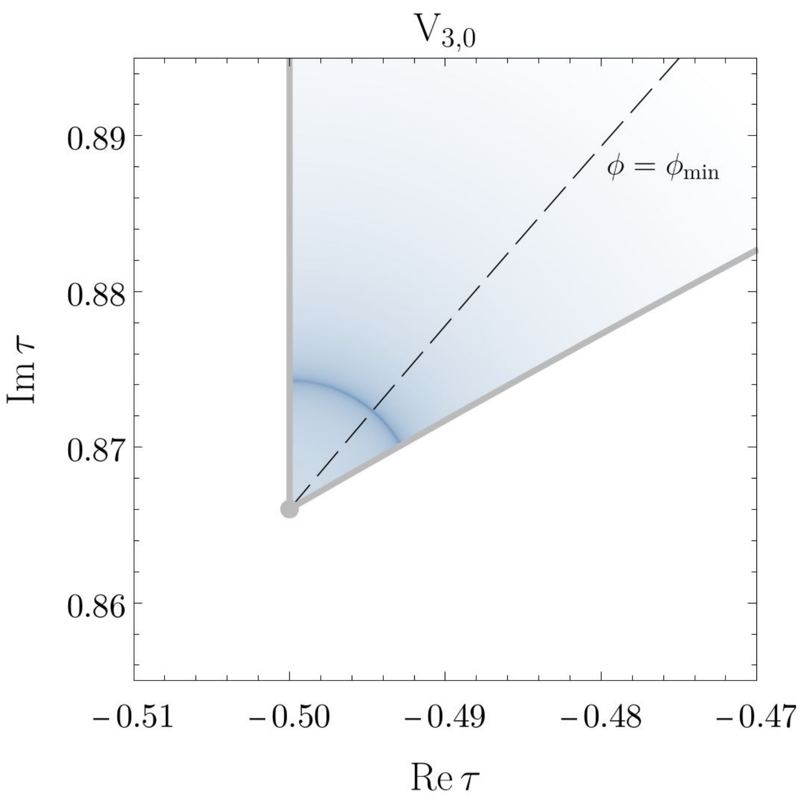
"Mexican"-hat potential

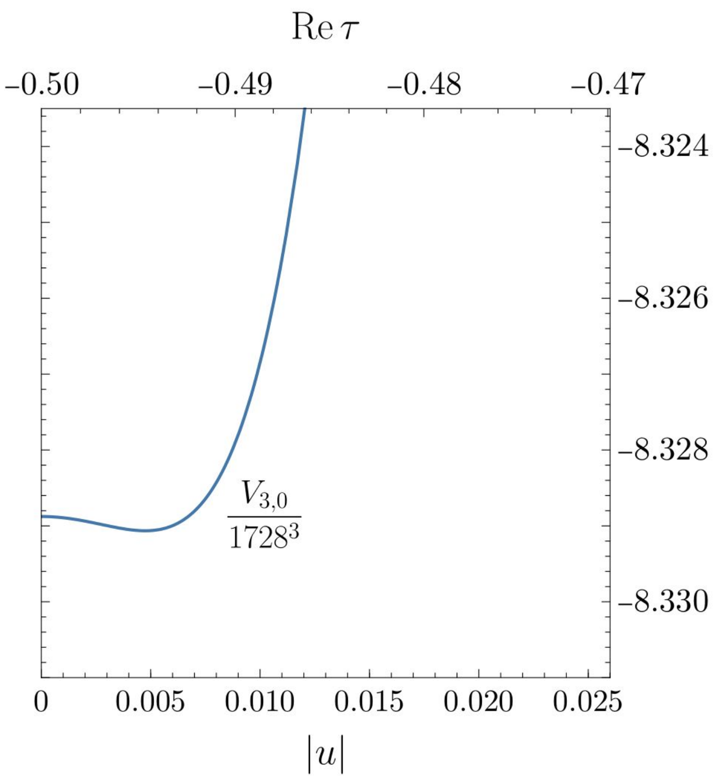
The (m,0) family of potentials (m = 2)





The (*m*,0) family of potentials (*m* = 3)





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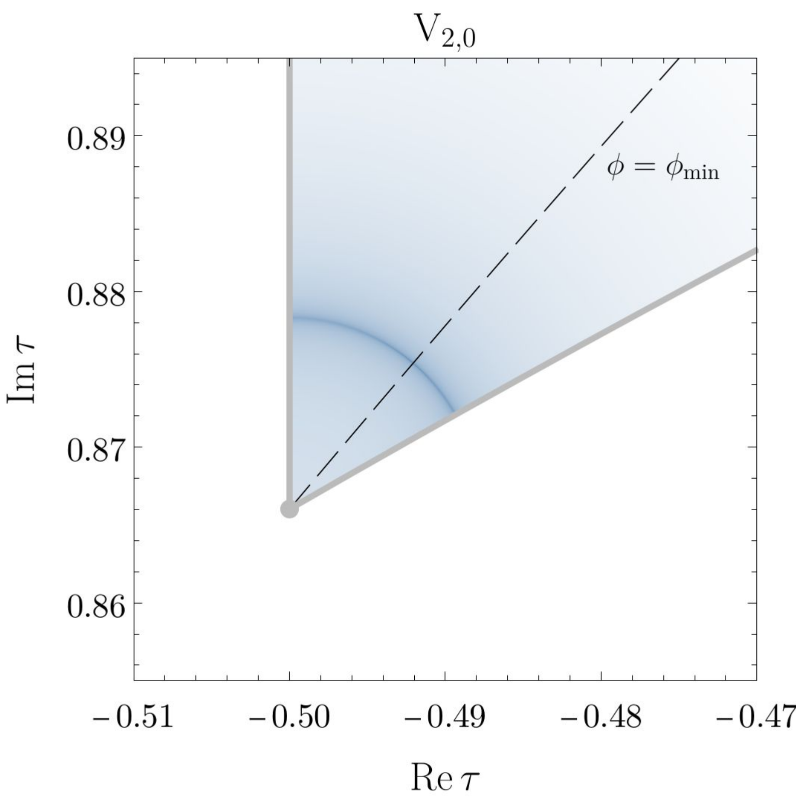
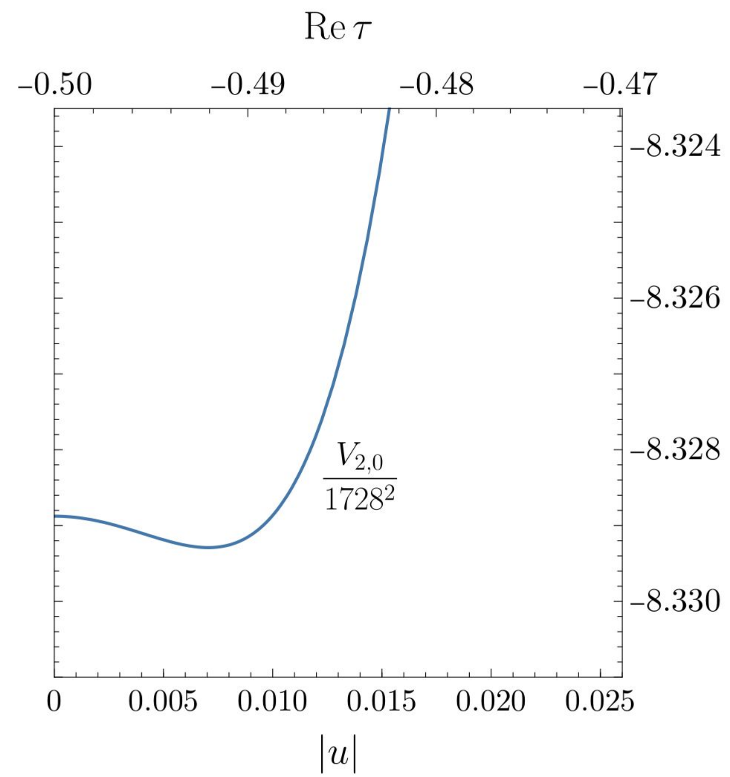
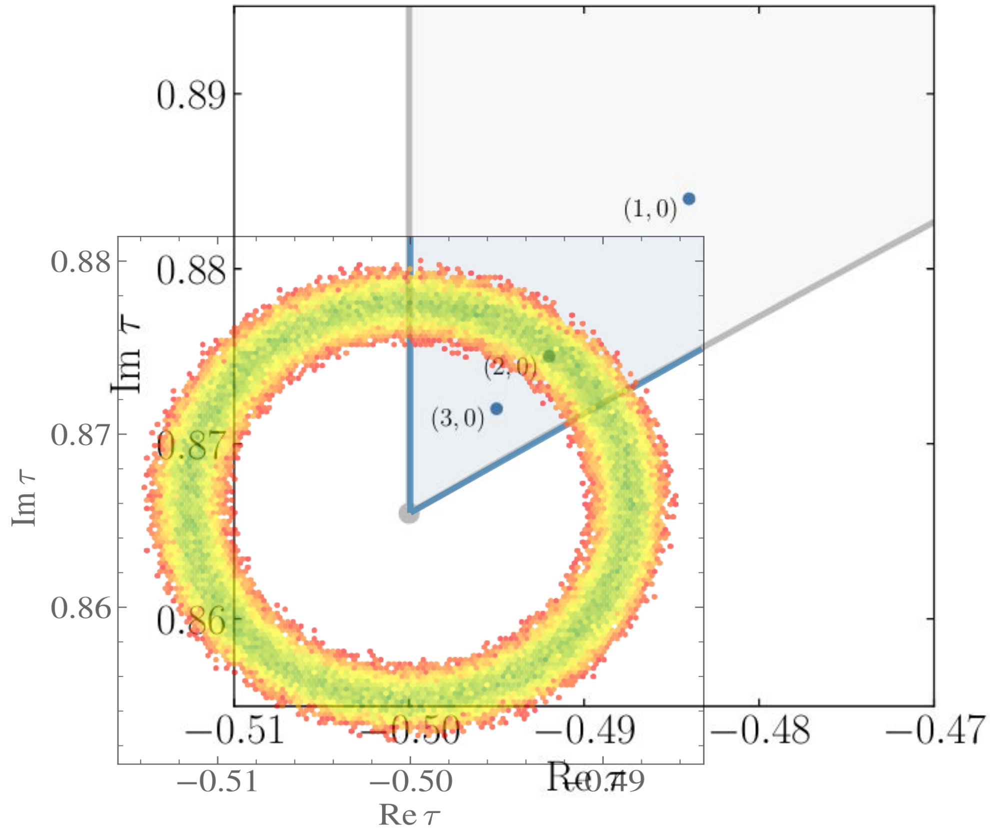




$$|\epsilon| \simeq 0.02 \Leftrightarrow |u| \simeq 0.007$$

Matching puzzle pieces?



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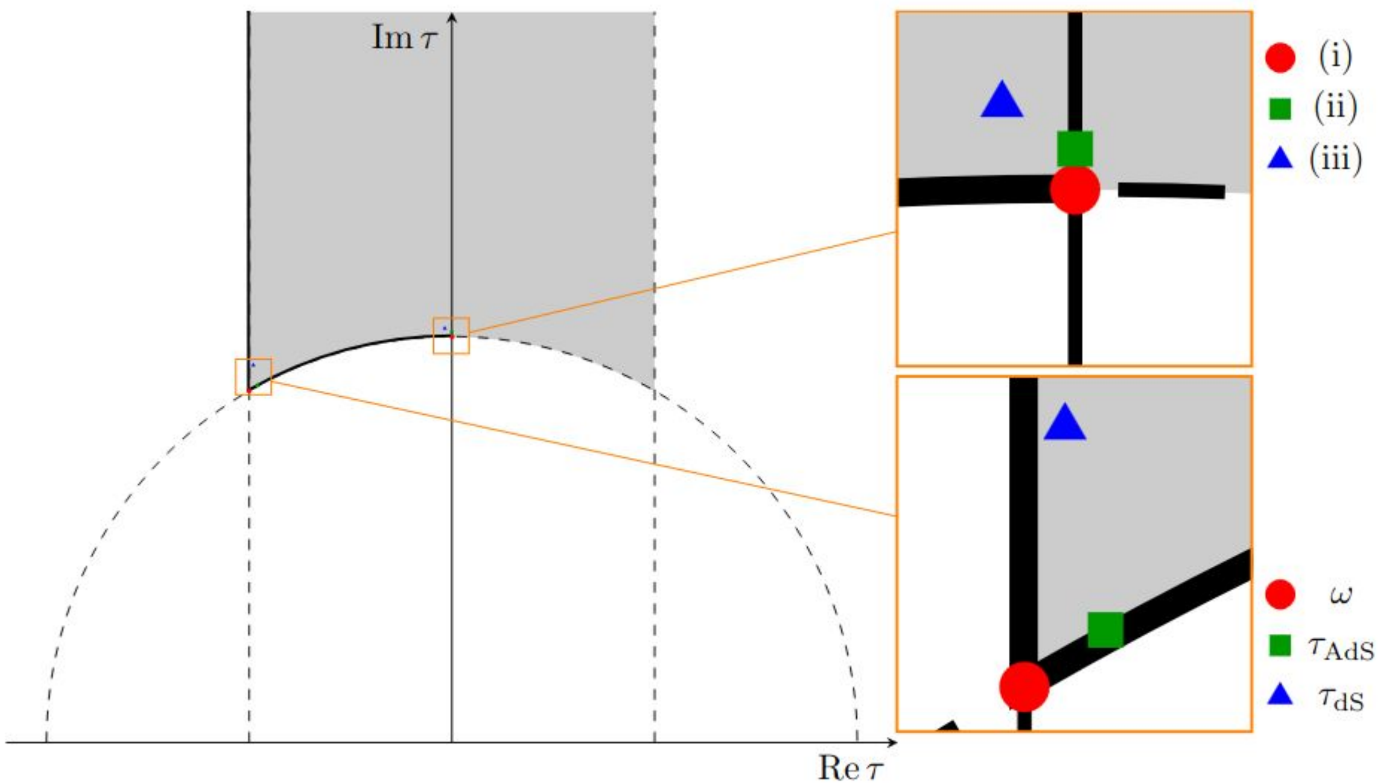

making the model more realistic by **uplifting AdS vacuum**
Knapp-Perez et al. [2304.14437]

No, there is no tuning in choosing this form of the superpotential (arguably)

$$H(\tau) \propto (J(\tau) - 1)^{m/2}$$

Subset of all possible $H(\tau)$ which vanish only at the symmetric point $\tau = i$ (itself distinguished by modular symmetry)

$$J(\tau) \equiv j(\tau)/1728$$

The global SUSY limit (a comment)

$$\mathfrak{n} = \kappa^2 \Lambda_K^2 \to 0$$

$$K(\tau, \overline{\tau}) = -\Lambda_K^2 \log(2 \,\mathrm{Im}\, \tau)$$

$$\kappa^2 = 8\pi / M_P^2$$

$$W(\tau) = \Lambda_W^3 H(\tau)$$

$$H(\tau) = (j(\tau) - 1728)^{m/2} j(\tau)^{n/3} \mathcal{P}(j(\tau))$$

$$V(\tau, \overline{\tau}) = \frac{4\Lambda_W^6}{\Lambda_K^2} (\mathrm{Im}\, \tau)^2 \left| H'(\tau) \right|^2$$

- Global minima are zeros of *H'*
- non-trivial $\mathcal{P}(j)$ can be engineered to produce minima at arbitrary points in the fundamental domain