Gaussian Formalism:

From **Heisenberg's Uncertainty Principle** to **Time-Boundary Effect** and **Lorentz-Covariant Complete Basis** for Spinors

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with

Naoya Ogawa (Independent) [2403.19440]

Kenzo Ishikawa (Hokkaido), **Kenji Nishiwaki** (Shiv Nadar) [PRD, 2023]

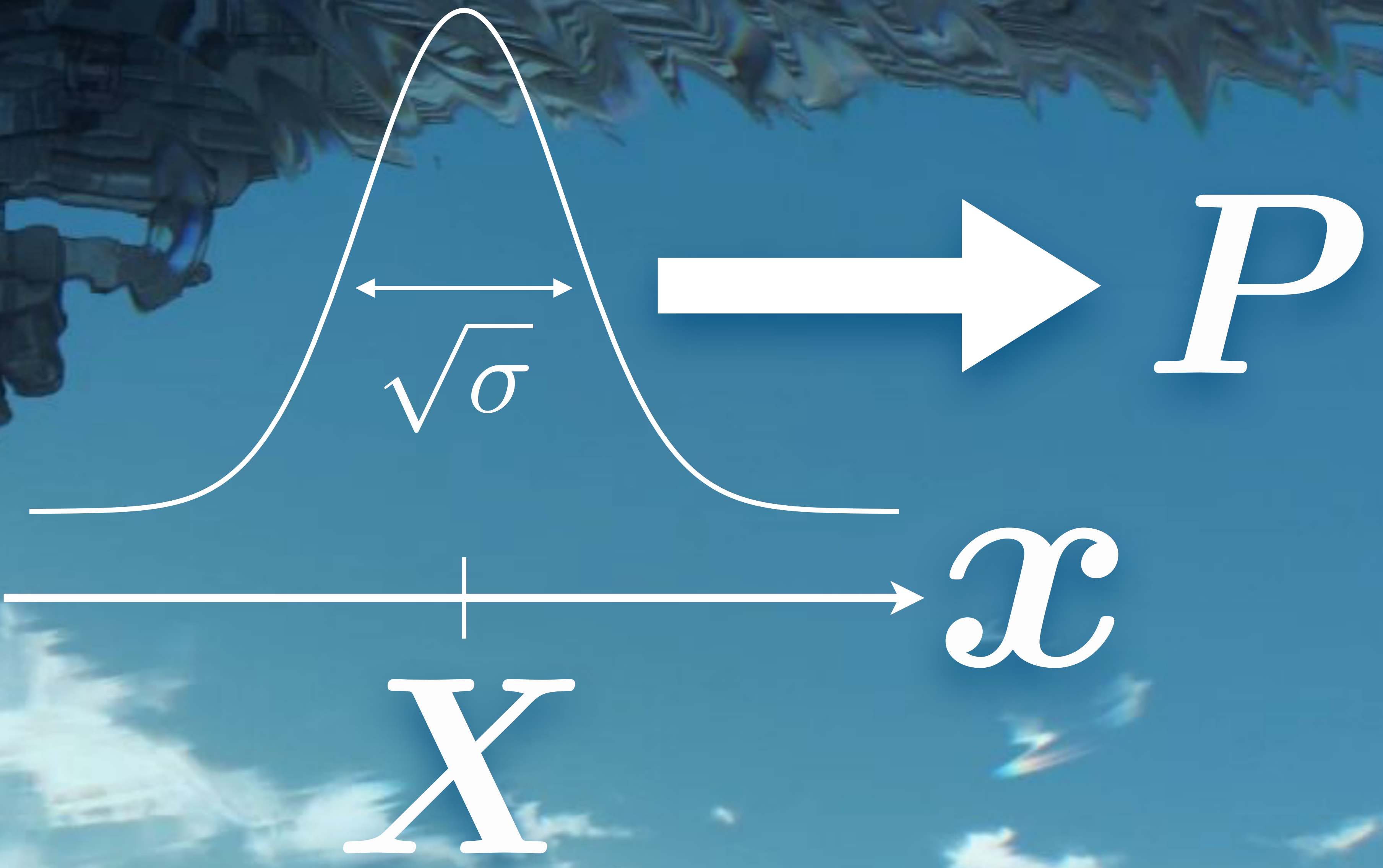
Juntaro Wada (Tokyo) [2307.05932]

Pheno's put aside in this talk

- Gaussian QM: **Neutrino decoherence**
  - With **Haruhi Mitani** (TWCU) [PLB, 2023]
- Gaussian QFT: **Isospin anomalies** (at up to 9.5 σ !) in vector meson decay
  - Explained in region where **time-boundary effect** (discussed below) dominates
  - With **Ishikawa**, **Jinnouchi**, **Nishiwaki** [EPJC, 2023]

Gaussian wave packet

$$\langle x \mid \boldsymbol{X}, \boldsymbol{P}; \sigma \rangle \propto e^{i\boldsymbol{P}\cdot\boldsymbol{x} - \frac{1}{2\sigma}(\boldsymbol{x}-\boldsymbol{X})^2}$$



- So what?
- We've all learnt it in QM kindergarten

[Picture from web]
https://twitter.com/dededeanime/status/1757691159059595359/photo/1

Three theoretical applications

Plan

1. **Joint measurement** of position and momentum (quantum information)
2. **Time-Boundary Effect** proven! (QFT)
3. **Lorentz-invariant/covariant complete basis** for scalar/spinor (QFT)

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Lee-Tsutsui formalism

[Lee, Tsutsui, 2020; Lee, 2022]

- Pro: Claimed to include all preceding **uncertainty relations**:
  - **Kennard-Robertson** and **Schroedinger** (in any QM textbook)
  - **Arthus-Kelly-Goodman** (errors and cost of measurement)
  - **Ozawa** (errors and disturbances)
  - **Yuen-Lux** and **Watanabe-Sagawa-Ueda** (estimation theory)
- Con: Too **abstract**, lacks **concrete implementation**
  - Only simplest **two-level system** so far (before ours)

Errors and uncertainty relation (just to show how it looks)

- Lee-Tsutsui error and uncertainty:

$$\varepsilon_{\hat{\rho}_{\mathrm{in}}}[\hat{A}] := \sqrt{\mathrm{Tr}\left[\hat{A}^2\hat{\rho}_{\mathrm{in}}\right] - \int_{\Omega} \mathrm{d}\omega \left(M_{\rho_{\mathrm{in}}\star}\hat{A}(\omega)\right)^2 p_{\mathrm{in}}(\omega)}$$

$$\varepsilon_{\hat{\rho}_{\mathrm{in}}}[\hat{x}]\,\varepsilon_{\hat{\rho}_{\mathrm{in}}}[\hat{p}] \geq \sqrt{\mathcal{I}^2_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}] + \mathcal{R}^2_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}]}$$

where

$$\mathcal{I}_{\hat{\rho}}\left[\hat{A},\hat{B};\mathsf{M}\right] := \left\langle \frac{\left[\hat{A},\,\hat{B}\right]}{2i} \right\rangle_{\hat{\rho}} - \left\langle \frac{\left[\widehat{M^{\star}M_{\star}\hat{A}},\,\hat{B}\right]}{2i} \right\rangle_{\hat{\rho}} - \left\langle \frac{\left[\hat{A},\,\widehat{M^{\star}M_{\star}\hat{B}}\right]}{2i} \right\rangle_{\hat{\rho}},$$

$$\mathcal{R}_{\hat{\rho}}\left[\hat{A},\hat{B};\mathsf{M}\right] := \left\langle \frac{\left\{\hat{A},\,\hat{B}\right\}}{2} \right\rangle_{\hat{\rho}} - \left\langle M_{\star}\hat{A},\,M_{\star}\hat{B} \right\rangle_{M\hat{\rho}}.$$

[Lee, Tsutsui, 2020]

- Lee error and uncertainty:

$$\widetilde{\varepsilon}_{\hat{\rho}_{\mathrm{in}}}[\hat{A}] := \sqrt{\int_{\Omega} \mathrm{d}\omega \left(M^{\star-1}_{\hat{\rho}_{\mathrm{in}}}\hat{A}(\omega)\right)^2 p_{\mathrm{in}}(\omega) - \mathrm{Tr}\left[\hat{A}^2\hat{\rho}_{\mathrm{in}}\right]}$$

$$\widetilde{\varepsilon}_{\hat{\rho}_{\mathrm{in}}}[\hat{x}]\,\widetilde{\varepsilon}_{\hat{\rho}_{\mathrm{in}}}[\hat{p}] \geq \sqrt{\mathcal{I}^2_{0\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}] + \widetilde{\mathcal{R}}^2_{\hat{\rho}_{\mathrm{in}}}[\hat{x},\hat{p}]}$$

where

$$\mathcal{I}_{0\hat{\rho}}\left[\hat{A},\hat{B}\right] := \left\langle \frac{\left[\hat{A},\,\hat{B}\right]}{2i} \right\rangle_{\hat{\rho}},$$

$$\widetilde{\mathcal{R}}_{\hat{\rho}}\left[\hat{A},\hat{B};\mathsf{M}\right] := \left\langle \frac{\left\{\hat{A},\,\hat{B}\right\}}{2} \right\rangle_{\hat{\rho}} - \left\langle M^{\star-1}\hat{A},\,M^{\star-1}\hat{B} \right\rangle_{M\hat{\rho}}.$$

[Lee, 2022]

Set of Gaussian packets

- Non-orthogonal:

$$\left\langle \boldsymbol{X}', \boldsymbol{P}'; \sigma \,\middle|\, \boldsymbol{X}, \boldsymbol{P}; \sigma \right\rangle = e^{i \frac{\boldsymbol{P}+\boldsymbol{P}'}{2} \cdot \left(\boldsymbol{X}' - \boldsymbol{X}\right) - \frac{1}{4\sigma}\left(\boldsymbol{X}' - \boldsymbol{X}\right)^2 - \frac{\sigma}{4}\left(\boldsymbol{P}' - \boldsymbol{P}\right)^2}$$

Details irrelevant

- Over-**completeness** in free one-particle subspace: For any fixed σ,

$$\int \frac{\mathrm{d}^d \boldsymbol{X}\, \mathrm{d}^d \boldsymbol{P}}{(2\pi)^d} \left| \boldsymbol{X}, \boldsymbol{P}; \sigma \right\rangle \left\langle \boldsymbol{X}, \boldsymbol{P}; \sigma \right| = \hat{1}$$

- Naturally leads to **positive-operator-valued measure (POVM)**:

[**KO**, Ogawa, 2024]

$$\left\{ \left| \boldsymbol{X}, \boldsymbol{P}; \sigma \right\rangle \left\langle \boldsymbol{X}, \boldsymbol{P}; \sigma \right| \right\}_{(\boldsymbol{X}, \boldsymbol{P}) \in \mathbb{R}^{2d}}$$

POVM measurement [KO, Ogawa, 2024]

- We propose **POVM measurement**

$$M : \hat{\rho} \mapsto p$$

with

$$p(\boldsymbol{X}, \boldsymbol{P}) = \mathrm{Tr}\left[\hat{\rho}\, |\boldsymbol{X}, \boldsymbol{P}; \sigma\rangle \langle \boldsymbol{X}, \boldsymbol{P}; \sigma|\right]$$

from any quantum state (density operator) ρ to classical state (PDF) p in phase space

- Resultant function p properly satisfies condition as PDF:

$$\int \frac{\mathrm{d}^d \boldsymbol{X}\, \mathrm{d}^d \boldsymbol{P}}{(2\pi)^d} p(\boldsymbol{X}, \boldsymbol{P}) = 1$$

Position and momentum operators in phase space

[**KO**, Ogawa, 2024]

- We obtain natural expressions analogous to those in plane-wave basis:

$$\hat{x}_i = \int \frac{\mathrm{d}^d \boldsymbol{X}\, \mathrm{d}^d \boldsymbol{P}}{(2\pi)^d} X_i \left| \boldsymbol{X}, \boldsymbol{P}; \sigma \right\rangle \left\langle \boldsymbol{X}, \boldsymbol{P}; \sigma \right|$$

$$\hat{p}_i = \int \frac{\mathrm{d}^d \boldsymbol{X}\, \mathrm{d}^d \boldsymbol{P}}{(2\pi)^d} P_i \left| \boldsymbol{X}, \boldsymbol{P}; \sigma \right\rangle \left\langle \boldsymbol{X}, \boldsymbol{P}; \sigma \right|$$

$$(i = 1, \dots, d)$$

How about some concrete initial state?

[**KO**, Ogawa, 2024]

Setup

- Initial state: Gaussian pure state

$$\hat{\rho}_{\mathrm{in}} = \left|\boldsymbol{X}_{\mathrm{in}}, \boldsymbol{P}_{\mathrm{in}}; \sigma_{\mathrm{in}}\right\rangle \left\langle \boldsymbol{X}_{\mathrm{in}}, \boldsymbol{P}_{\mathrm{in}}; \sigma_{\mathrm{in}}\right|$$

- Measurement by POVM:

$$\left\{ \left|\boldsymbol{X}, \boldsymbol{P}; \sigma\right\rangle \left\langle \boldsymbol{X}, \boldsymbol{P}; \sigma\right| \right\}_{(\boldsymbol{X}, \boldsymbol{P}) \in \mathbb{R}^{2d}}$$

Basic parameters: spatial width-squared

[**KO**, Ogawa, 2024]

- Input:
  - σ_{in} for **initial** state uncertainty
  - σ for **detector resolution**
- Output:
  - **Summed** σ_{sum} for **postion** errors
  - **Reduced** σ_{red} for **momentum** errors

$$\sigma_{\text{sum}} = \sigma_{\text{in}} + \sigma$$

$$\sigma_{\text{red}} = \frac{\sigma_{\text{in}}\sigma}{\sigma_{\text{in}} + \sigma}$$

Intermediate steps (just to show how it looks)

[KO, Ogawa, 2024]

$$
\begin{aligned}
\langle X\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= X_{\mathrm{in}}, & \langle P\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= P_{\mathrm{in}}, & \langle XP\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= X_{\mathrm{in}}P_{\mathrm{in}},\\
\langle \overline{X}\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= X_{\mathrm{in}}, & \langle \overline{P}\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= P_{\mathrm{in}}, & \langle \overline{X}\,\overline{P}\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= X_{\mathrm{in}}P_{\mathrm{in}},\\
\langle X^2\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{sum}}}{2}, & \langle P^2\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= P_{\mathrm{in}}^2 + \frac{1}{2\sigma_{\mathrm{red}}}, & &\\
\left\langle \overline{X}^2\right\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{in}}^2}{2\sigma_{\mathrm{sum}}}, & \left\langle \overline{P}^2\right\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} &= P_{\mathrm{in}}^2 + \frac{\sigma}{2\sigma_{\mathrm{in}}\sigma_{\mathrm{sum}}}, & &
\end{aligned}
$$

$$[M_{\mathrm{ph}\star}\widehat{x}](X,P) = \overline{X}, \qquad [M_{\mathrm{ph}\star}\widehat{p}](X,P) = \overline{P}, \qquad \left[M_{\mathrm{ph}}^{\star-1}\widehat{x}\right](X,P) = X, \qquad \left[M_{\mathrm{ph}}^{\star-1}\widehat{p}\right](X,P) = P.$$

$$\|M_{\mathrm{ph}\star}\widehat{x}\|^2_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{in}}^2}{2\sigma_{\mathrm{sum}}}, \qquad \|M_{\mathrm{ph}\star}\widehat{p}\|^2_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = P_{\mathrm{in}}^2 + \frac{\sigma}{2\sigma_{\mathrm{in}}\sigma_{\mathrm{sum}}}, \qquad \left\|M_{\mathrm{ph}}^{\star-1}\widehat{x}\right\|^2_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}^2 + \frac{\sigma_{\mathrm{sum}}}{2}, \qquad \left\|M_{\mathrm{ph}}^{\star-1}\widehat{p}\right\|^2_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = P_{\mathrm{in}}^2 + \frac{1}{2\sigma_{\mathrm{red}}},$$

$$\langle M_{\mathrm{ph}\star}\widehat{x},\, M_{\mathrm{ph}\star}\widehat{p}\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}P_{\mathrm{in}}. \qquad \left\langle M_{\mathrm{ph}}^{\star-1}\widehat{x},\, M_{\mathrm{ph}}^{\star-1}\widehat{p}\right\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = X_{\mathrm{in}}P_{\mathrm{in}}.$$

$$
\begin{aligned}
\widehat{M_{\mathrm{ph}}^{\star} M_{\mathrm{ph}\star}\widehat{x}} &= \int_{\mathbb{R}^2} \frac{\mathrm{d}X\,\mathrm{d}P}{2\pi}\overline{X}\,|X,P;\sigma\rangle\langle X,P;\sigma|, \\
\widehat{M_{\mathrm{ph}}^{\star} M_{\mathrm{ph}\star}\widehat{p}} &= \int_{\mathbb{R}^2} \frac{\mathrm{d}X\,\mathrm{d}P}{2\pi}\overline{P}\,|X,P;\sigma\rangle\langle X,P;\sigma|.
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \frac{\left[\widehat{M_{\mathrm{ph}}^{\star} M_{\mathrm{ph}\star}\widehat{x}},\, \widehat{p}\right]}{2i}\right\rangle_{\widehat{\rho}_{\mathrm{in}}} &= \Im\left\langle \overline{X}\left(\overline{P} + i\frac{X - X_{\mathrm{in}}}{\sigma_{\mathrm{sum}}}\right)\right\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = \frac{\sigma_{\mathrm{in}}}{2\sigma_{\mathrm{sum}}}, \\
\left\langle \frac{\left[\widehat{x},\, \widehat{M_{\mathrm{ph}}^{\star} M_{\mathrm{ph}\star}\widehat{p}}\right]}{2i}\right\rangle_{\widehat{\rho}_{\mathrm{in}}} &= \Im\left\langle \left(\overline{X} - i\sigma_{\mathrm{red}}\left(P - P_{\mathrm{in}}\right)\right)^{*}\overline{P}\right\rangle_{M_{\mathrm{ph}}\widehat{\rho}_{\mathrm{in}}} = \frac{\sigma}{2\sigma_{\mathrm{sum}}},
\end{aligned}
$$

LT error and inequality

Lee error and inequality

Just to show

Result

[**KO**, Ogawa, 2024]

Errors:

LT errors:

$$\varepsilon^2_{\hat{\rho}_{\mathrm{in}}}[\hat{x}] = \frac{\sigma\sigma_{\mathrm{in}}}{2(\sigma+\sigma_{\mathrm{in}})}$$

$$\varepsilon^2_{\hat{\rho}_{\mathrm{in}}}[\hat{p}] = \frac{1}{2(\sigma+\sigma_{\mathrm{in}})}$$

Errors, in opposite way,
can be made **infinitely small**

Lee errors:

$$\widetilde{\varepsilon}^2_{\hat{\rho}_{\mathrm{in}}}[\hat{x}] = \frac{\sigma}{2}$$

$$\widetilde{\varepsilon}^2_{\hat{\rho}_{\mathrm{in}}}[\hat{p}] = \frac{1}{2\sigma}$$

Errors governed by **detector resolution**

Uncertainty:

[LT lower bound] = 0

Trivial?

[Lee lower bound] = 1/2

Recovers Heisenberg's!

Summary of this part

- Gaussian **POVM** as **joint measurement** in phase space
- First realization of **Lee-Tsutsui formalism** in infinite-dimensional space
- **Lee errors** and **inequality** seem more plausible than LT ones.

Plan

1. **Joint measurement** of position and momentum (quantum information)
2. **Time-Boundary Effect** proven! (QFT)
3. **Lorentz-invariant/covariant complete basis** for scalar/spinor (QFT)

Gaussian basis applicable to QFT

- Gaussian basis can expand any quantum field by

$$\widehat{A}^\dagger(X, \boldsymbol{P}; \sigma)\,|0\rangle = |X, \boldsymbol{P}; \sigma\rangle$$

with commutator

$$\left[\widehat{A}(X, \boldsymbol{P}; \sigma)\,,\, \widehat{A}^\dagger\left(X', \boldsymbol{P}'; \sigma'\right)\right] = \left\langle X, \boldsymbol{P}; \sigma \,\middle|\, X', \boldsymbol{P}'; \sigma' \right\rangle$$

such that

$$\widehat{\phi}(x) = \int \frac{\mathrm{d}^d \boldsymbol{X}\, \mathrm{d}^d \boldsymbol{P}}{\left(2\pi\right)^d} \left[f_{X, \boldsymbol{P}; \sigma}(x)\, \widehat{A}(X, \boldsymbol{P}; \sigma) + \mathrm{h.c.}\right]$$

and similarly for complex scalar, spinor and vector fields.

Backup

- Expansion coefficient is known function:

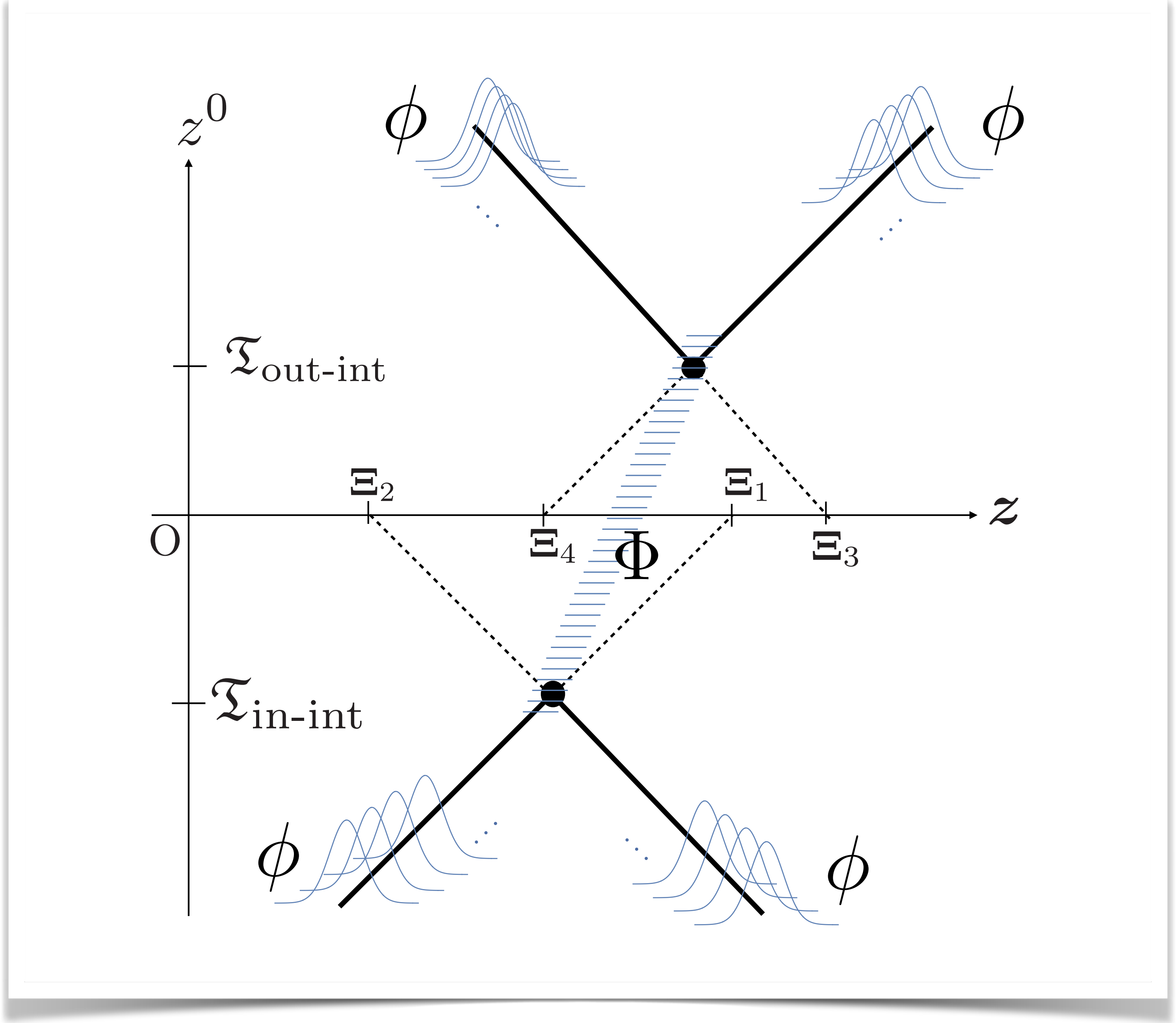
$$f_{\varphi,\sigma;X,\boldsymbol{P}}(x) := \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2E_\varphi(\boldsymbol{p})}} \langle \varphi; x \,|\, \varphi; \boldsymbol{p} \rangle \langle \varphi; \boldsymbol{p} \,|\, \varphi, \sigma; \Pi \rangle$$

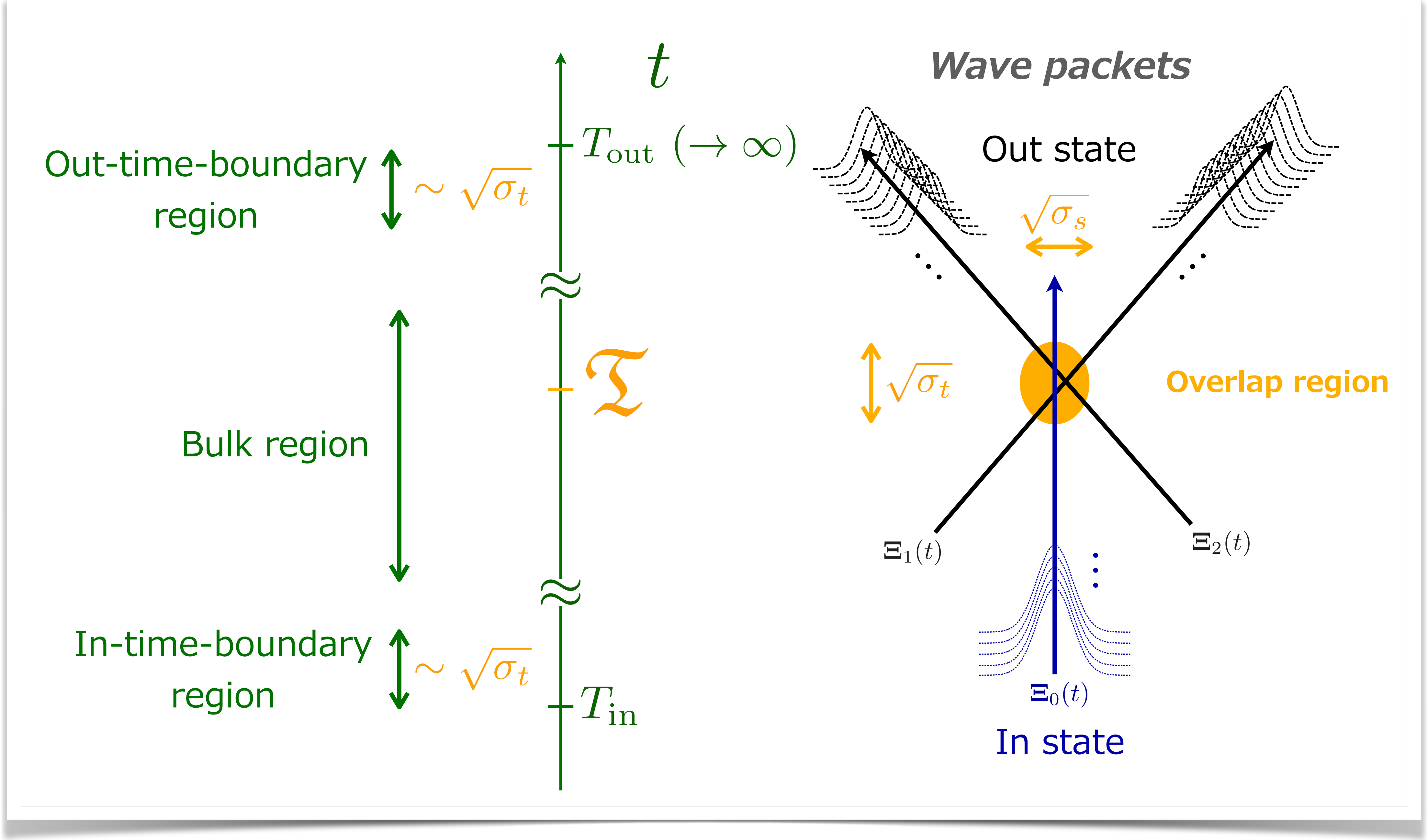
$$= \left(\frac{\sigma}{\pi}\right)^{3/4} \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2p^0}\,(2\pi)^{3/2}} e^{ip\cdot(x-X)-\frac{\sigma}{2}(\boldsymbol{p}-\boldsymbol{P})^2}\bigg|_{p^0=E_\varphi(\boldsymbol{p})}$$

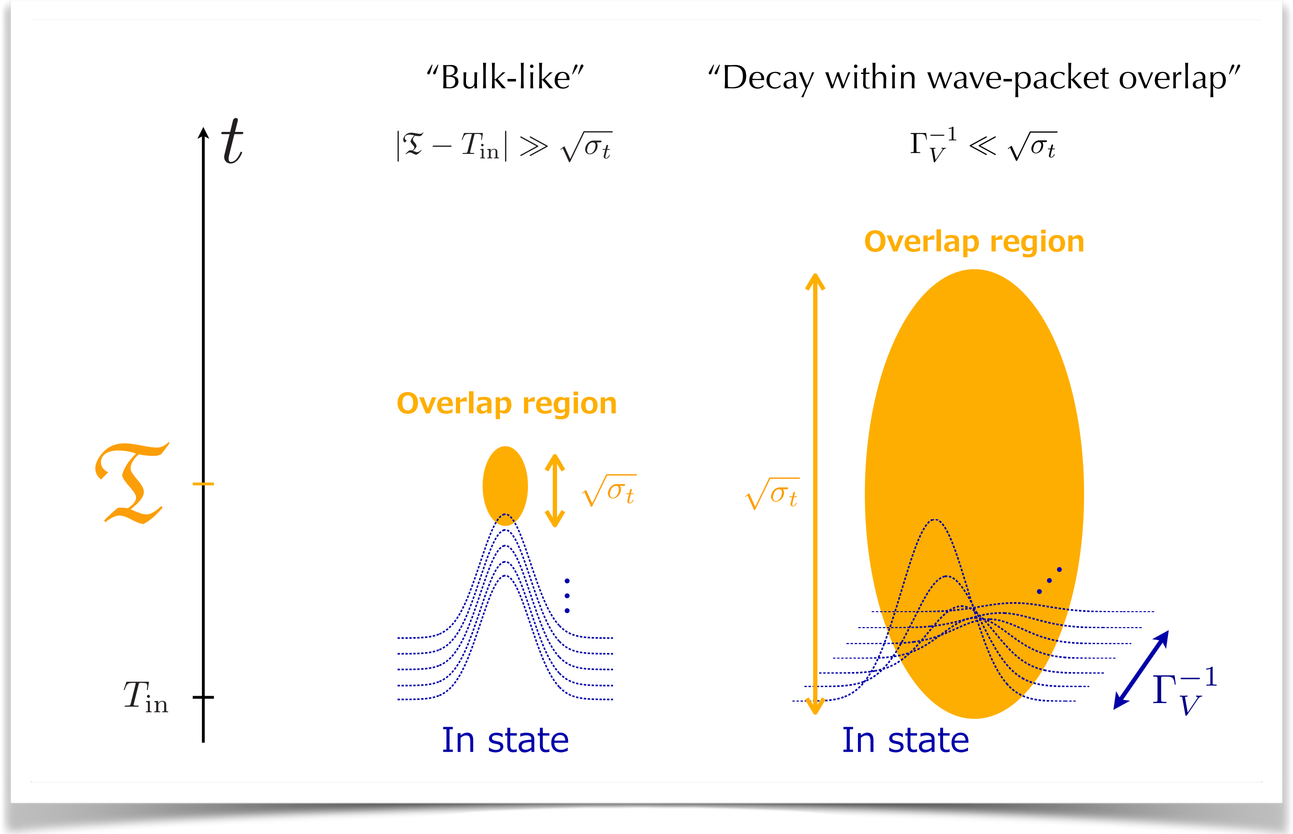
Setup

[Ishikawa, Jinnouchi, Nishiwaki, **KO**, 2023]

- Tree-level two-to-two **scattering**: $\phi\ \phi \to \Phi \to \phi\ \phi$
  - External states $\phi\ \phi$ are treated by **Gaussian wave packets**
- See if initial **time-boundary effect** for **$\Phi \to \phi\ \phi$ decay** emerges
  - In some configurations in final-state integral



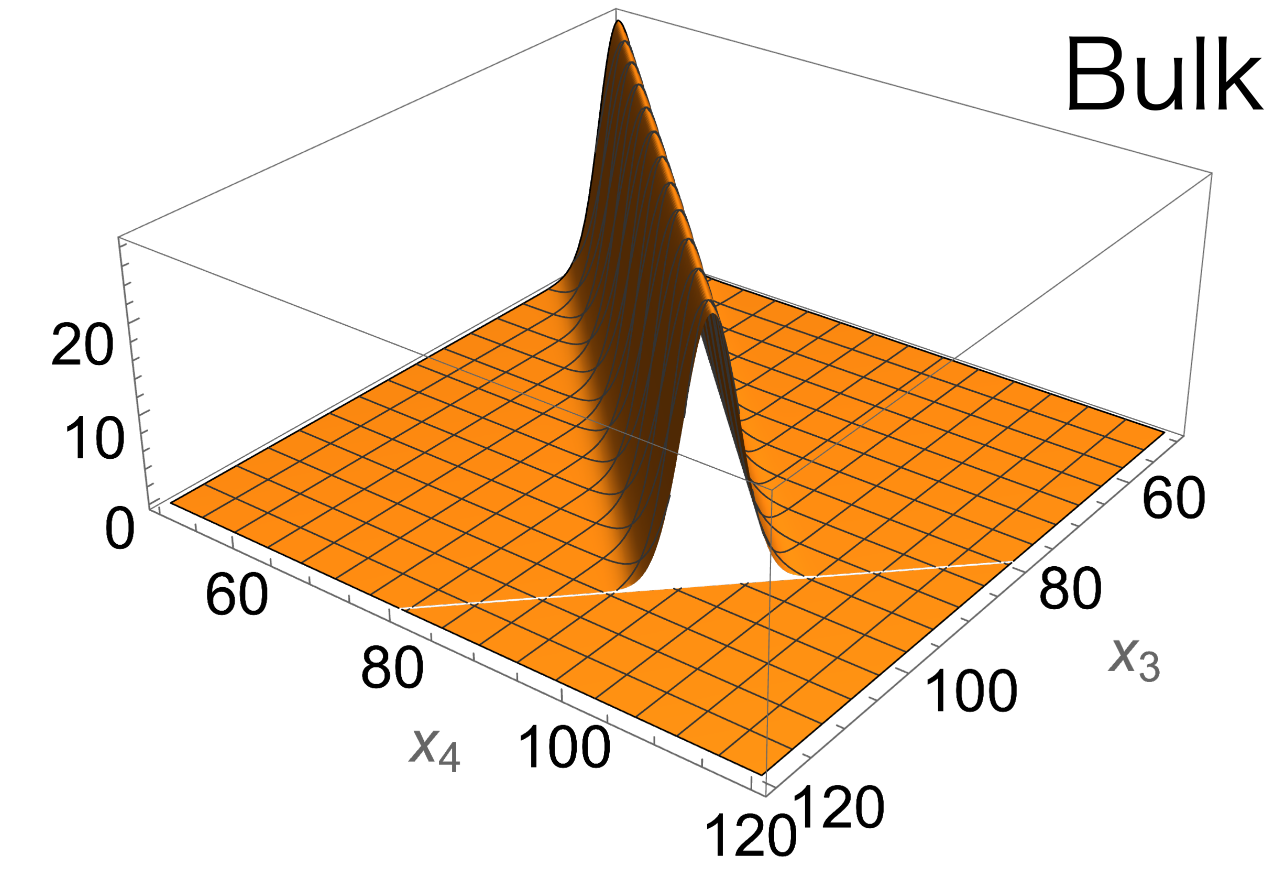




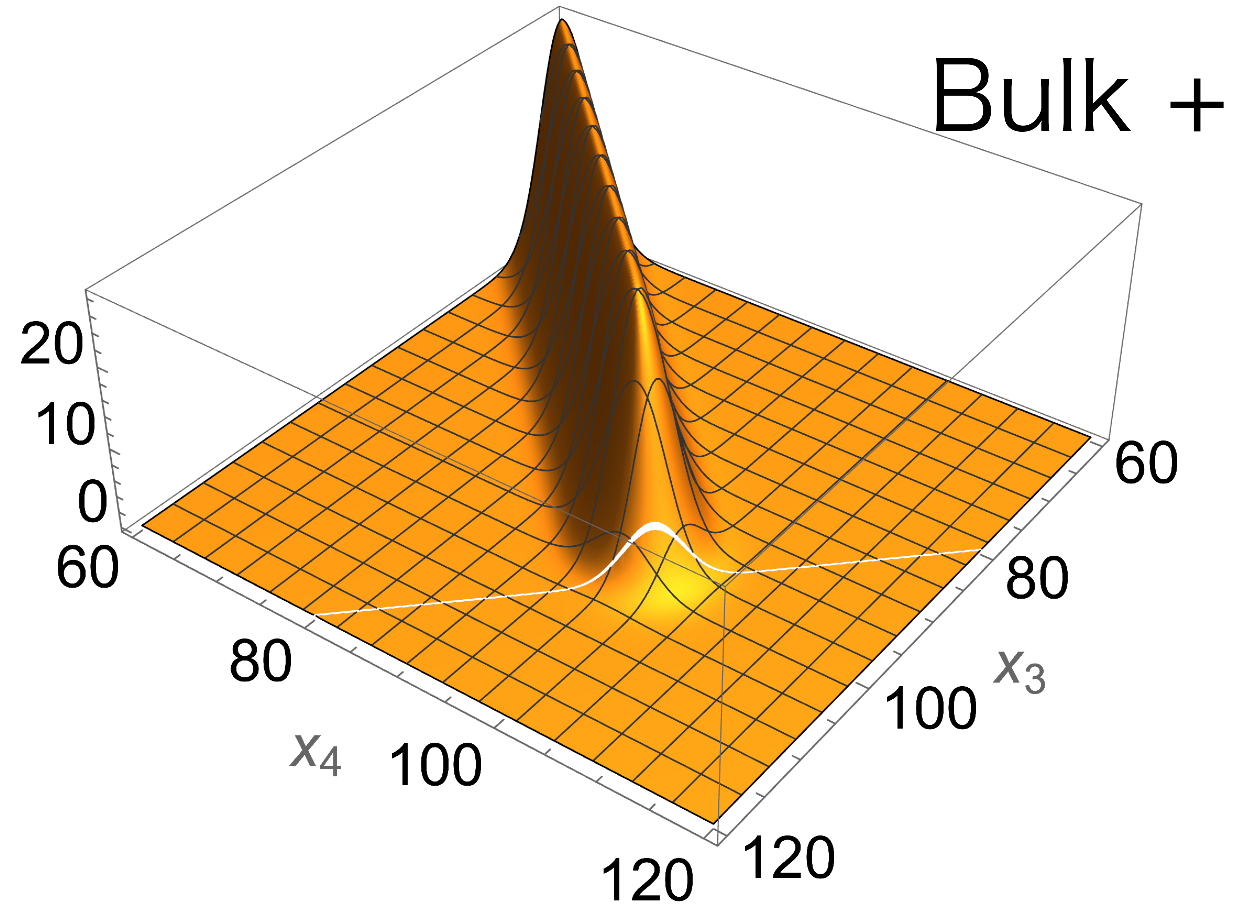
Results for overlap regions in position space

[Ishikawa, Jinnouchi, Nishiwaki, **KO**, 2023]

- Momenta **on resonance**:

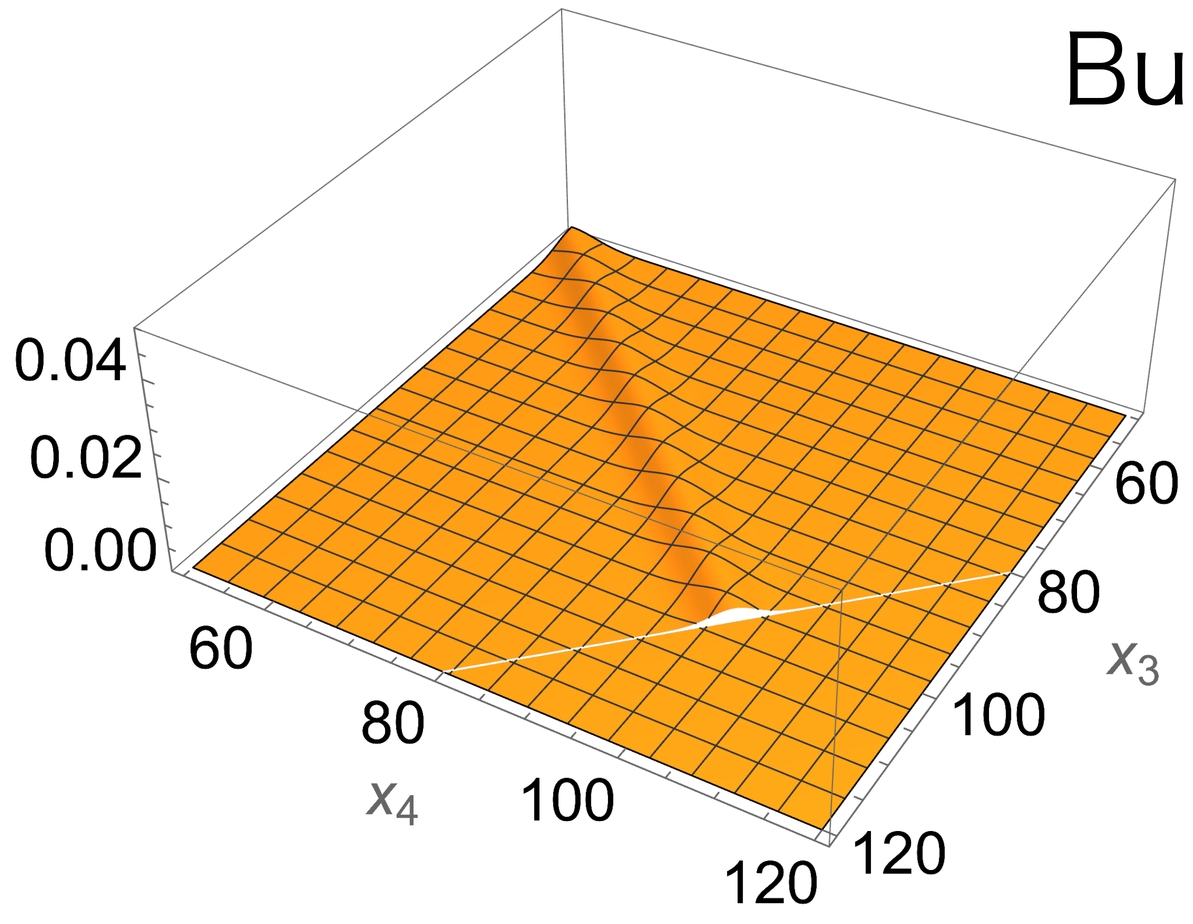


Bulk only

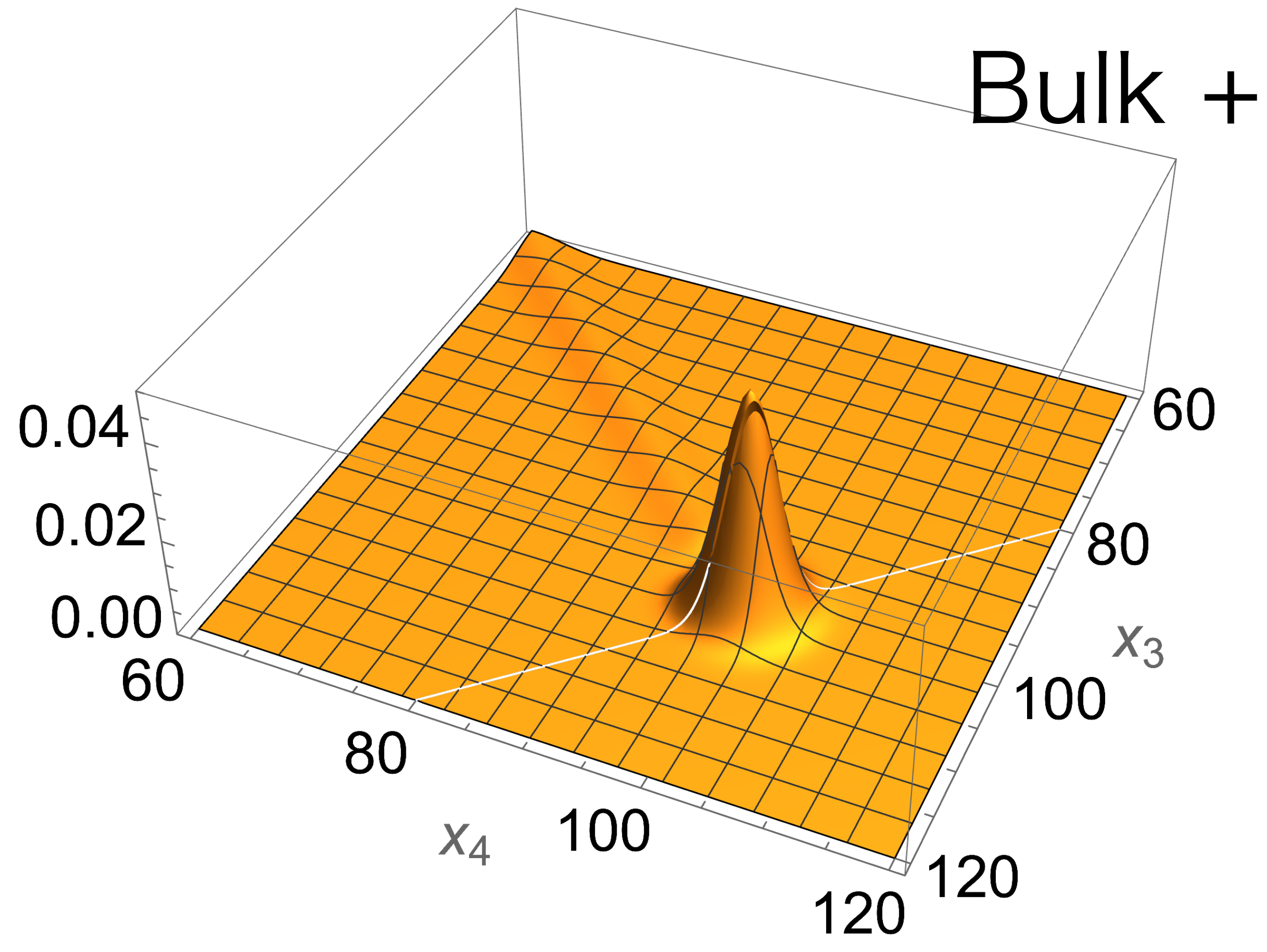


Bulk + Boundary

- Momenta **off resonance**:



Bulk only



Bulk + Boundary

- On resonance, **bulk contribution** dominates
- **Boundary contribution** smoothes out time boundary
- Off resonance, **boundary contribution** dominates over **bulk contribution**
- Though overall size is smaller than on resonance

Summary of this part

- Existence of **time-boundary effect** proven!

Plan

1. **Joint measurement** of position and momentum (quantum information)
2. **Time-Boundary Effect** proven! (QFT)
3. **Lorentz-invariant/covariant complete basis** for scalar/spinor (QFT)

Lorentz-invariant generalization

- Lorentz invariant wave packet [Kaiser, 1977, 1978; Naumov, Naumov, 2010]

$$\langle\!\langle p \,|\, \Pi \rangle\!\rangle \propto e^{-ip\cdot(X + i\sigma P)}$$

where

$$\Pi := (X, P)$$

- This reduces to Gaussian basis in non-relativistic limit.

Manifestly Lorentz-invariant completeness

[**KO**, Wada, 2021]

- We have obtained manifestly **Lorentz-invariant completeness** relation:
  In free one-particle subspace,

$$\int \mathrm{d}^{2d}\Pi_\phi \, |\Pi\rangle\!\rangle \, \langle\!\langle \Pi| = \hat{1}$$

$$\Pi := (X, P)$$

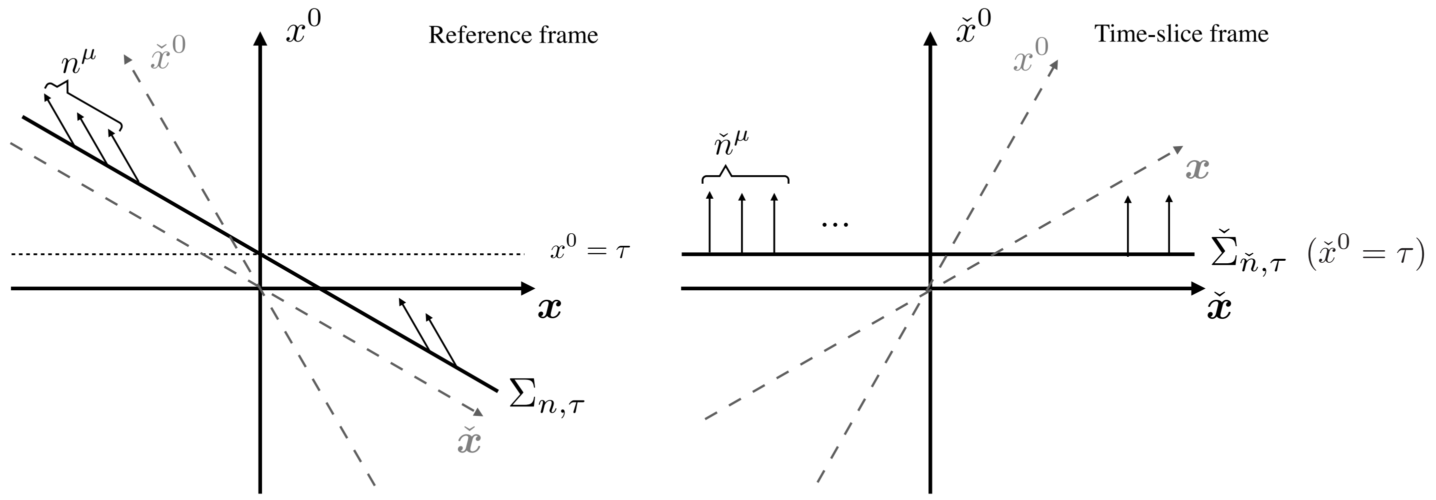
where (whole mess is hidden inside)

$$\int \mathrm{d}^{2d}\Pi_\phi := \frac{1}{\mathcal{M}_\phi} \int \frac{\mathrm{d}^d\Sigma^\mu_X}{(2\pi)^d} (-2P_\mu) \frac{\mathrm{d}^d \boldsymbol{P}}{2P^0} \quad \text{with} \quad \mathrm{d}^d\Sigma^\mu_X := \mathrm{d}^{d+1}X \, \delta(N \cdot X + T) \, N^\mu$$

Don't worry about details

- **Volume element** reduces to ordinary Gaussian phase space integral

$$\propto \int \frac{\mathrm{d}^d\boldsymbol{X} \, \mathrm{d}^d\boldsymbol{P}}{(2\pi)^d}$$ on **equal-time hyperplane**



- Can be used to expand **scalar fields**.

Manifestly Lorentz-covariant packet

- **Spin-diagonal** representation in literature [Naumov, Naumov, 2010; Ishikawa, **KO**, 2018]

$$\langle\!\langle p, s \,|\, \Pi, S \rangle\!\rangle_{\mathrm{D}} := \langle\!\langle p \,|\, \Pi \rangle\!\rangle \, \delta_{sS}$$

$\Pi := (X, P)$

$\langle\!\langle p \,|\, \Pi \rangle\!\rangle \propto e^{-ip\cdot(X+i\sigma P)}$

- Difficulty: Its Lorentz transformation **mixes** wave-packet states with **different positions and momenta**

- We propose **phase-space-diagonal** representation: [**KO**, Wada, 2023]

$$\langle\!\langle p, s \,|\, \Pi, S \rangle\!\rangle \propto \langle\!\langle p \,|\, \Pi \rangle\!\rangle \, \overline{u}(p, s) \, u(P, S)$$

and similarly for anti-particles.

Manifestly Lorentz-covariant completeness

[**KO**, Wada, 2023]

- We now have, in free one-particle subspace,

$$\sum_S \int \mathrm{d}^{2d}\Pi_\psi \, |\Pi, S\rangle\!\rangle \, \langle\!\langle \Pi, S| = \hat{1}$$

where (whole mess is hidden inside)

$$\int \mathrm{d}^{2d}\Pi_\psi := \frac{1}{\mathcal{M}_\psi} \int_{\Sigma_{N,T}} \frac{\mathrm{d}^d\Sigma_X^\mu}{(2\pi)^d} (-2P_\mu) \frac{\mathrm{d}^d \boldsymbol{P}}{2P^0}$$
$$= \frac{\mathcal{M}_\phi}{\mathcal{M}_\psi} \int \mathrm{d}^{2d}\Pi_\phi,$$

Don't worry about details

Summary of this part

- Found **Lorentz-invariant/covariant** wave packets that can expand **scalar/spinor** fields

Summary: Gaussian Formalism

1. Furnishes **POVM** for joint measurement of position and momentum
   - **Heisenberg's uncertainty principle** in new perspective
2. Proves **Time-Boundary Effect** in QFT
3. Generalized to **Lorentz-invariant/covariant complete basis** for scalar/spinor

Future directions

1. **Time-energy** uncertainty. **Bell**-CHSH inequality.
2. Interference with **no-scattering**?! On-shell t-channel scattering? (Thanks B. Grzadkowski)
   (And of course more concrete phenomenology involving neutrinos etc.)
3. **Massless** packets including **vector** fields


Thank you!

Backup

Phenomenological applications

- Tend to be accepted quicker
- Even **before** their **theoretical basis**
- E.g. **time-boundary effect** took 2.5 years to be published
    - Even though being immediately recommended by Referees A & C (B & D were nasty)

Better have scientist editor

[Picture from web]

Referee A: All remarks of previous reports have been fully addressed and the revised manuscript has been tremendously improved by streamlining the discussion. In this way the manuscript is accessible to the larger community. As the overlooked effect has a vast impact on various branches of physics such as neutrino physics, astrophysics and biophysics I can fully recommend this article for publication in Physical Review Letters.

Referee C: In my understanding, this result sheds new light on several problems of quantum field theory and the related topics. In particular, the interpretation in terms of Lefschetz thimble decomposition is what the physicists should have done in old days. The paper is well written and organized so as to be readable for the readers from other fields. I recommend the editor to publish the paper in this journal.

Referees B says, *I find it hard to conclude whether this result has the relevance and depth that would make me say it ⟪"should be" as opposed to "could be"⟫ published in PRL,* and finishes the letter by the following sentence: *But of course I leave to the Editor the final decision based also on the other referees' observations.* Referee D did not recommend, the whole report being as follows:

Referee D: I would agree with the assessment of Referee B. This is a very old subject and its not clear to me that the intricacies the authors are concerned with are of sufficient interest to a wide enough audience. Nor do I think will it have any overwhelming impact in a sub-field that would overcome the aforementioned shortcoming.

Uncertainty relations for quantum fluctuations

- Kennard-Robertson and Schrödinger inequalities:

Putting $\psi_A := \left(\widehat{A} - \langle \widehat{A} \rangle\right) |\psi\rangle$

Into ordinary Cauchy-Schwarz inequality $\|\psi_A\|^2 \, \|\psi_B\|^2 \geq |\langle \psi_A \,|\, \psi_B \rangle|^2$

$$\sigma_\psi^2\left[\widehat{A}\right] \sigma_\psi^2\left[\widehat{B}\right] \geq \left|\left\langle \frac{[\widehat{A}, \widehat{B}]}{2i} \right\rangle_\psi\right|^2 + \left|\left\langle \frac{\{\widehat{A}, \widehat{B}\}}{2} \right\rangle_\psi - \left\langle \widehat{A} \right\rangle_\psi \left\langle \widehat{B} \right\rangle_\psi\right|^2$$

KR (1927, 1929) — Purely quantum correlation

Schrödinger (1930) — Semi-classical correlation

Uncertainty relations for errors

- Arthurs-Kelly-Goodman (AKG 1965, 1988) takes into account measurement errors: $$\varepsilon_\rho\left[\mathsf{M};\widehat{A}\right]\varepsilon_\rho\left[\mathsf{M};\widehat{B}\right]\geq\left|\left\langle\frac{[\widehat{A},\widehat{B}]}{2i}\right\rangle_\rho\right|$$

- Ozawa (2004):

$$\varepsilon_\rho\left[\mathsf{M};\widehat{A}\right]\varepsilon_\rho\left[\mathsf{M};\widehat{B}\right]+\varepsilon_\rho\left[\mathsf{M};\widehat{A}\right]\sigma_\rho\left[\widehat{B}\right]+\sigma_\rho\left[\widehat{A}\right]\varepsilon_\rho\left[\mathsf{M};\widehat{B}\right]\geq\left|\left\langle\frac{[\widehat{A},\widehat{B}]}{2i}\right\rangle_\rho\right|$$

- Watanabe-Sagawa-Ueda (2011): AGK from quantum estimation theory

Lee-Tsutsui (or Lee) inequality, claimed to include all of them!

E.g. Ozawa inequality [from Lee's slide 2021]

小澤不等式　　本不等式　　小澤不等式

$$\underline{\varepsilon(A)\varepsilon(B)} \geq \underline{\varepsilon_\rho(A)\varepsilon_\rho(B) \geq \sqrt{\mathcal{R}^2 + \mathcal{I}^2}} \geq |\mathcal{I}| \geq \underline{|\langle[A, B]\rangle_\rho|/2 - \varepsilon(A)\sigma(B) - \sigma(A)\varepsilon(B)}$$

Observable and state spaces

	Observables			States		
Quantum	$S(\mathscr{H})$	$\ni \hat{A}$	(self-adjoint operator)	$\supset Z(\mathscr{H})$	$\ni \hat{\rho}$	(density operator)
Classical	$R(\Omega)$	$\ni f$	(real function)	$\supset W(\Omega)$	$\ni p$	(PDF)

$$\hat{A}^\dagger = \hat{A}$$

$$\mathrm{Tr}[\hat{\rho}] = 1 \qquad \hat{\rho} \geq 0$$

Typically pure state $\hat{\rho} = |\psi\rangle\langle\psi|$

$$f^*(\omega) = f(\omega)$$

$$\int_\Omega \mathrm{d}\omega\, p(\omega) = 1 \qquad p(\omega) \geq 0$$

Quantum measurement

- Affine map **M**: $\hat{\rho} \mapsto p$ that allows probability mixture: $\forall \lambda \in [0,1]$,

$$\mathrm{M}\left(\lambda \hat{\rho}_1 + (1-\lambda)\,\hat{\rho}_2\right) = \lambda \mathrm{M}\hat{\rho}_1 + (1-\lambda)\,\mathrm{M}\hat{\rho}_2$$

Hereafter, we write $M\hat{\rho} := \mathrm{M}\hat{\rho}$

Density operators		Self-adjoint operators
$Z(\mathscr{H}) \ni \hat{\rho}$	$\subset$	$S(\mathscr{H}) \ni \hat{A}$
$\downarrow$ **M**		**M*** $\uparrow$ LT adjoint/pullback; $\downarrow$ **M**$_\star$, **M**$^{\star -1}$ Pushforward, Inverse
$W(\Omega) \ni p$	$\subset$	$R(\Omega) \ni f$
PDFs		Real functions

Backup: Measurement in Lee-Tsutsui formalism

[Lee, Tsutsui, 2020]

- Quantum state = density operator ρ, where Tr[ρ]=1
  - (For pure state, it is just $\rho = |\psi\rangle\langle\psi|$.)
- Classical state = PDF p(ω), where ω=(**X**,**P**) ∈ [phase space Ω] = $\mathbb{R}^{2d}$
- Measurement = Affine map M: $\rho \to$ p (= M(ρ) =: Mρ)
  - That naturally admits **probability mixture**:
  - $M(\lambda \rho_1 + (1-\lambda)\rho_2) = \lambda M\rho_1 + (1-\lambda) M\rho_2$ $\quad (0 \leqq \lambda \leqq 1)$

Backup: Pullback of classical observable f

[Lee, Tsutsui, 2020]

- Quantum observable = Self-adjoint operator A $(=A^\dagger)$
- Classical observable = Real function f(ω), where ω=(**X**,**P**) ∊ [phase space Ω] = $\mathbb{R}^{2d}$
- Pullback of f: For any ρ,

$$\left\langle \widehat{M^\star f} \right\rangle_{\hat\rho} = \langle f \rangle_{M\hat\rho}$$

  - M*f gives the same quantum expectation value (under ρ) as classical expectation value of f (under Mρ)

LT adjoint/pullback

- Lee-Tsutsui adjoint **M★**: $\forall \hat{\rho} \in Z(\mathscr{H}),$

$$\left\langle \widehat{M^\star f} \right\rangle_{\hat{\rho}} = \langle f \rangle_{M\hat{\rho}}$$

Given **measurement M**,
LT adjoint of a real function is
the operator that gives the same expectation values
for all quantum states.

$$\langle f \rangle_p := \int_\Omega \mathrm{d}\omega\, f(\omega) p(\omega)$$

Pullback and pushforward

- **Pullback** is defined for equivalence classes (crucial for LT but irrelevant for us)

$$\widehat{A} \overset{\widehat{\rho}}{\sim} \widehat{B} \Longleftrightarrow \left\|\widehat{A} - \widehat{B}\right\|_{\widehat{\rho}} = 0,$$

$$f \overset{\rho}{\sim} g \Longleftrightarrow \|f - g\|_{\rho} = 0.$$

- **Pushforward M★** is defined by,

$$\left\langle \widehat{A}, \widehat{M^{\star} f} \right\rangle_{\widehat{\rho}} = \left\langle M_{\star}\widehat{A}, f \right\rangle_{M\widehat{\rho}}$$

$$\mathrm{Tr}\left[\frac{\left\{\widehat{A}, \widehat{M^{\star} f}\right\}}{2}\widehat{\rho}\right] = \int_{\Omega} \mathrm{d}\omega \left[M_{\star}\widehat{A}\right](\omega)\, f(\omega)\, [M\widehat{\rho}](\omega)$$

Explicit form of Lee-Tsutsui inequality

- LT error:

$$\varepsilon_{\widehat{\rho}}\left[\widehat{A};\mathsf{M}\right] := \sqrt{\left\|\widehat{A}\right\|_{\widehat{\rho}}^{2} - \left\|M_{\star}\widehat{A}\right\|_{M\widehat{\rho}}^{2}} = \sqrt{\mathrm{Tr}\left[\rho\widehat{A}^{2}\right] - \int_{\Omega}\mathrm{d}\omega\,\left[M_{\star}\widehat{A}\right]^{2}(\omega)\,[M\widehat{\rho}](\omega)}$$

- LT inequality:

$$\varepsilon_{\widehat{\rho}}\left[\widehat{A};\mathsf{M}\right]\varepsilon_{\widehat{\rho}}\left[\widehat{B};\mathsf{M}\right] \geq \sqrt{\mathcal{I}_{\widehat{\rho}}^{2}\left[\widehat{A},\widehat{B};\mathsf{M}\right] + \mathcal{R}_{\widehat{\rho}}^{2}\left[\widehat{A},\widehat{B};\mathsf{M}\right]},$$

where

$$\mathcal{I}_{\widehat{\rho}}\left[\widehat{A},\widehat{B};\mathsf{M}\right] := \left\langle\frac{\left[\widehat{A},\,\widehat{B}\right]}{2i}\right\rangle_{\widehat{\rho}} - \left\langle\frac{\left[\widehat{M^{\star}M_{\star}\widehat{A}},\,\widehat{B}\right]}{2i}\right\rangle_{\widehat{\rho}} - \left\langle\frac{\left[\widehat{A},\,\widehat{M^{\star}M_{\star}\widehat{B}}\right]}{2i}\right\rangle_{\widehat{\rho}},$$

$$\mathcal{R}_{\widehat{\rho}}\left[\widehat{A},\widehat{B};\mathsf{M}\right] := \left\langle\frac{\left\{\widehat{A},\,\widehat{B}\right\}}{2}\right\rangle_{\widehat{\rho}} - \left\langle M_{\star}\widehat{A},\,M_{\star}\widehat{B}\right\rangle_{M\widehat{\rho}}.$$

Meaning of LT error

$$\varepsilon_{\widehat{\rho}}\left[\widehat{A};\mathsf{M},f\right] := \sqrt{\left\|\widehat{A}-\widehat{M^{\star}f}\right\|_{\widehat{\rho}}^{2}+\left(\sigma^{2}_{M\widehat{\rho}}[f]-\sigma^{2}_{\widehat{\rho}}\left[\widehat{M^{\star}f}\right]\right)}$$

$$= \sqrt{\varepsilon^{2}_{\widehat{\rho}}\left[\widehat{A};\mathsf{M}\right]+\left\|M_{\star}\widehat{A}-f\right\|_{M\widehat{\rho}}^{2}}$$

- Choosing test function f to be pushforward $M_{\star}A$ gives the optimum.

Explicit form of Lee inequality

- Lee error:

$$\widetilde{\varepsilon}_{\widehat{\rho}}\left[\widehat{A};\mathsf{M}\right] := \sqrt{\left\|M^{\star-1}\widehat{A}\right\|_{M\widehat{\rho}}^2 - \left\|\widehat{A}\right\|_{\widehat{\rho}}^2}$$

- Lee inequality:

$$\widetilde{\varepsilon}_{\widehat{\rho}}\left[\widehat{A};\mathsf{M}\right]\widetilde{\varepsilon}_{\widehat{\rho}}\left[\widehat{B};\mathsf{M}\right] \geq \sqrt{\mathcal{I}_{0\widehat{\rho}}^2\left[\widehat{A},\widehat{B}\right] + \widetilde{\mathcal{R}}_{\widehat{\rho}}^2\left[\widehat{A},\widehat{B};\mathsf{M}\right]},$$

where

$$\mathcal{I}_{0\widehat{\rho}}\left[\widehat{A},\widehat{B}\right] := \left\langle \frac{\left[\widehat{A},\widehat{B}\right]}{2i} \right\rangle_{\widehat{\rho}},$$

$$\widetilde{\mathcal{R}}_{\widehat{\rho}}\left[\widehat{A},\widehat{B};\mathsf{M}\right] := \left\langle \frac{\left\{\widehat{A},\widehat{B}\right\}}{2} \right\rangle_{\widehat{\rho}} - \left\langle M^{\star-1}\widehat{A},\, M^{\star-1}\widehat{B}\right\rangle_{M\widehat{\rho}}.$$

Our propsal: POVM measurement for Heisenberg's uncertainty

- We propose (non-projective) POVM:

$$\{ \, |X, P; \sigma\rangle \langle X, P; \sigma| \, \}_{(X,P)\in\mathbb{R}^2}$$

- POVM measurement:

$$[M_{\rm ph}\hat{\rho}](X, P) = \mathrm{Tr}\big[|X, P; \sigma\rangle \langle X, P; \sigma| \, \hat{\rho}\big]$$

- Here (for simplicity), we focus on the pure state:

$$\hat{\rho}_{\rm in} = |X_{\rm in}, P_{\rm in}; \sigma_{\rm in}\rangle \langle X_{\rm in}, P_{\rm in}; \sigma_{\rm in}|$$

Measurement result

$$\sigma_{\text{sum}} := \sigma + \sigma_{\text{in}}, \qquad \sigma_{\text{red}} := \frac{1}{\frac{1}{\sigma} + \frac{1}{\sigma_{\text{in}}}} = \frac{\sigma\sigma_{\text{in}}}{\sigma + \sigma_{\text{in}}}.$$

$$[M_{\text{ph}}\widehat{\rho}_{\text{in}}](X, P) =$$

$$|\langle X, P; \sigma \mid X_{\text{in}}, P_{\text{in}}; \sigma_{\text{in}}\rangle|^2 = 2\pi \left(\frac{1}{\sqrt{\pi\sigma_{\text{sum}}}} e^{-\frac{1}{\sigma_{\text{sum}}}(X - X_{\text{in}})^2} \right) \left(\sqrt{\frac{\sigma_{\text{red}}}{\pi}} e^{-\sigma_{\text{red}}(P - P_{\text{in}})^2} \right)$$

- Limit of **spatially finer** detector than original wave-packet size $\sigma \ll \sigma_{\text{in}}$,
  - Spatial part governed by σ_{in}
  - Momentum part governed by σ (information of original packet lost)
- Vice versa

Concrete realization of pullback and pushforward:

- LT adjoint/pullback:

$$\widehat{M_{\mathrm{ph}}^{\star} f} = \int_{\mathbb{R}^2} \frac{\mathrm{d}X\,\mathrm{d}P}{2\pi} f(X,P) \left|X,P;\sigma\right\rangle \left\langle X,P;\sigma\right|$$

- Decomposition of position and momentum operators:

$$\widehat{x} = \int_{\mathbb{R}^2} \frac{\mathrm{d}X\,\mathrm{d}P}{2\pi} X \left|X,P;\sigma\right\rangle \left\langle X,P;\sigma\right|, \qquad \widehat{p} = \int_{\mathbb{R}^2} \frac{\mathrm{d}X\,\mathrm{d}P}{2\pi} P \left|X,P;\sigma\right\rangle \left\langle X,P;\sigma\right|,$$

  - Giving their inverse: $\left[M_{\mathrm{ph}}^{\star-1}\widehat{x}\right](X,P) = X,$ $\qquad \left[M_{\mathrm{ph}}^{\star-1}\widehat{p}\right](X,P) = P.$

- Pushforward: $[M_{\mathrm{ph}\star}\widehat{x}](X,P) = \overline{X},$ $\qquad [M_{\mathrm{ph}\star}\widehat{p}](X,P) = \overline{P},$

$$\overline{X} := \sigma_{\mathrm{red}}\left(\frac{X}{\sigma} + \frac{X_{\mathrm{in}}}{\sigma_{\mathrm{in}}}\right), \qquad \overline{P} := \frac{\sigma P + \sigma_{\mathrm{in}} P_{\mathrm{in}}}{\sigma + \sigma_{\mathrm{in}}}.$$

Our result for LT errors and inequality

- LT errors:

$$\varepsilon^2_{\widehat{\rho}_{\mathrm{in}}}[\widehat{x};\mathsf{M}_{\mathrm{ph}}] = \frac{\sigma_{\mathrm{red}}}{2}, \qquad \varepsilon^2_{\widehat{\rho}_{\mathrm{in}}}[\widehat{p};\mathsf{M}_{\mathrm{ph}}] = \frac{1}{2\sigma_{\mathrm{sum}}}.$$

- LT inequality:

$$\frac{1}{2}\sqrt{\frac{\sigma_{\mathrm{red}}}{\sigma_{\mathrm{sum}}}} \geq 0,$$

$$\sigma_{\mathrm{sum}} := \sigma + \sigma_{\mathrm{in}}, \qquad \sigma_{\mathrm{red}} := \frac{1}{\frac{1}{\sigma} + \frac{1}{\sigma_{\mathrm{in}}}} = \frac{\sigma\sigma_{\mathrm{in}}}{\sigma + \sigma_{\mathrm{in}}}.$$

- **No lower bound!**
- Product of errors takes mamixum at $\sigma = \sigma_{\mathrm{in}}$.
- For projective measurement of position $\sigma \to 0$, (or momentum $\sigma \to \infty$), it becomes **trivial** 0=0!

Our result for Lee errors and inequality

- Lee errors:

$$\widetilde{\varepsilon}^2_{\widehat{\rho}_{\mathrm{in}}}[\widehat{x};\mathsf{M}_{\mathrm{ph}}] = \frac{\sigma}{2}, \qquad \widetilde{\varepsilon}^2_{\widehat{\rho}_{\mathrm{in}}}[\widehat{p};\mathsf{M}_{\mathrm{ph}}] = \frac{1}{2\sigma}.$$

  - Depends only on detector resolution!
- Lee inequality: 1/2 ≧ 1/2.
  - For initial pure Gaussian state ρ_{in},
  - Lee inequality saturated regardless of detector resolution σ