Quarkonium Productions in e^+e^- Collider with their QCD calculations up to NNLO

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1. Introduction

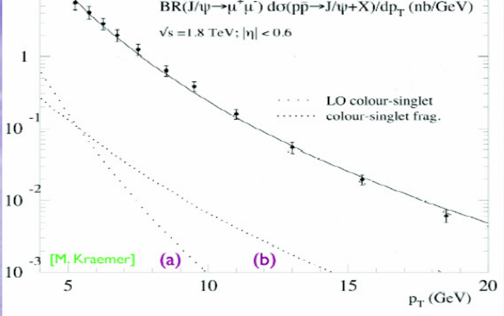
2. J/ψ production at the B factories (with NLO QCD)
   - double charmonium production
   - Inclusive J/ψ production

3. Progress in NNLO QCD Correction to J/ψ production at B factories
   - $e^+e^- \to J/\psi + \eta_c$
   - $e^+e^- \to J/\psi + J/\psi$

4. Summary

Introduction

- Perturbative and non-perturbative QCD, hadronization, factorization
- Color-singlet and Color-octet mechanism was proposed based on NRQCD for heavy quarkonium
- Why so serious to on the test: Clear signal to detect J/ψ, very limited number of nonperturbative parameters, double perturbative expansions on α_s and v (the vilocity of heavy quark in quarkonium) are better since b and c-quark is heavy.
- J/ψ production at the B factories
- J/ψ production and polarization at the Tevatron,HERA and LHC
- LO theoretical predication were given before more than 20 years
- NLO theoretical predications were given within last 15 years.
- The QCD NLO calculations can adequately describe the experimental data?
- How about QCD NNLO resutls? (In last five years)



PRL 99, 132001 (2007) PHYSICAL REVIEW LETTERS week ending 28 SEPTEMBER 2007

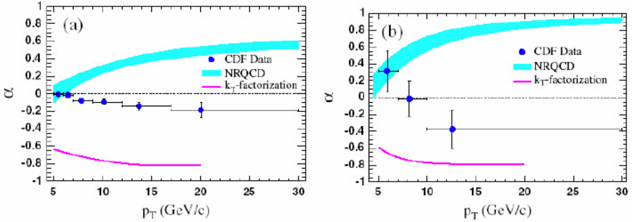


FIG. 4 (color online). Prompt polarizations as functions of p_T: (a) J/ψ and (b) $\psi(2S)$. The band (line) is the prediction from NRQCD [4] (the k_T-factorization model [9]).

$e^+e^- \to J/\psi + \eta_c$

Experimantal Data

BELLE: $\sigma[J/\psi + \eta_c] \times B^{\eta_c}[\geq 2] = (25.6 \pm 2.8 \pm 3.4)$ fb
BARAR: $\sigma[J/\psi + \eta_c] \times B^{\eta_c}[\geq 2] = (17.6 \pm 2.8^{+1.5}_{-2.1})$ fb
[?, ?, ?]

LO NRQCD Predictions

$2.3 \sim 5.5$ fb
[?, ?, ?]

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LO NRQCD Predictions

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[?, ?, ?]

NLO QCD corrections

$K \equiv \sigma^{NLO}/\sigma^{LO} \sim 2$

First given in PRL96, (2006) Y. J. Zhang, Y. J. Gao and K. T. Chao
Confirmed by the analytic result in PRD77, (2008), B. Gong and J. X. Wang

Relativistic corrections

$K \sim 2$

PRD67, (2007) E. Braaten and J. Lee
AIP Conf. Proc. (2007), G.T. Bodwin, D. Kang, T. Kim, J. Lee and C. Yu
PRD75, (2007), Z. G. He, Y. Fan and K. T. Chao
PRD77,(2008),G.T. Bodwin, J. Lee and C. Yu

$e^+e^- \to J/\psi + J/\psi$

Problem

LO NRQCD prediction indicates that the cross section of this process is large than that of $J/\psi + \eta_c$ production by a factor of 1.8, but no evidence for this process was found at the B factories.
PRL90, (2003) G. T. Bodwin, E. Braaten and J. Lee
PRD70, (2004), K. Abe, et al

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PRD70, (2004), K. Abe, et al

NLO QCD corrections

- Greatly decreased, with a K factor ranging from $-0.31 \sim 0.25$ depending on the renormalization scale.
- Might explain the situation.

PRL100, (2008) B. Gong and J. X. Wang

LO NRQCD Predictions:

$e^+e^- \to J/\psi + c\bar{c}$	$0.07 \sim 0.20$pb
$e^+e^- \to J/\psi + gg$	$0.15 \sim 0.3$pb
$e^+e^- \to J/\psi({}^3P_J^8, {}^1S_0^8) + g$	$0.3 \sim 0.8$pb

PRL76,(1996), E. Braaten and Y. C. Chen, PLB577,(2003), K.Y. Liu, Z.G. He and K.T. chao, ...

Experimental Data:

BARAR	$\sigma[e^+e^- \to J/\psi + X] = (2.54 \pm 0.21 \pm 0.21)$ pb
CLEO	$\sigma[e^+e^- \to J/\psi + X] = (1.9 \pm 0.20)$ pb
BELLE	$\sigma[e^+e^- \to J/\psi + X] = (1.45 \pm 0.10 \pm 0.13)$ pb
	$\sigma[e^+e^- \to J/\psi + c\bar{c} + X] = (0.87^{+0.21}_{-0.19} \pm 0.17)$ pb

[?, ?, ?, ?, ?]

New BELLE Data

$$\begin{aligned} \sigma[e^+e^- \to J/\psi + X] &= (1.17 \pm 0.02 \pm 0.07) \text{ pb} \\ \sigma[e^+e^- \to J/\psi + c\bar{c}] &= (0.74 \pm 0.08^{+0.09}_{-0.08}) \text{ pb} \\ \sigma[e^+e^- \to J/\psi + X_{\text{non}-c\bar{c}}] &= (0.43 \pm 0.09 \pm 0.09) \text{ pb} \end{aligned}$$

[?]

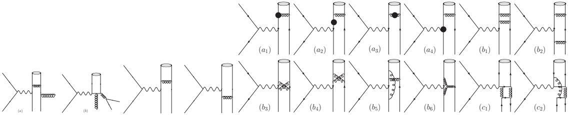
$$\sigma^{(1)} = \sigma^{(0)} \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left[a(\hat{s}) + \beta_0 \ln\left(\frac{\mu}{2m_c}\right) \right] \right\}$$

m_c(GeV)	$\alpha_s(\mu)$	$\sigma^{(0)}$(pb)	$a(\hat{s})$	$\sigma^{(1)}$(pb)	$\sigma^{(1)}/\sigma^{(0)}$
1.4	0.267	0.341	2.35	0.409	1.20
1.5	0.259	0.308	2.57	0.373	1.21
1.6	0.252	0.279	2.89	0.344	1.23

Consistent results from two group:
PRL102, (2009) Y. Q. Ma, Y. J. Zhang and K. T. Chao
PRL102, (2009) B. Gong and J. X. Wang

Relativistic Correction enchance results about a factor 1.3 from two group:
PRD81, (2010) Z. G. He, Y. Fan and K. T. Chao
PRD82, (2010). Y. Jia

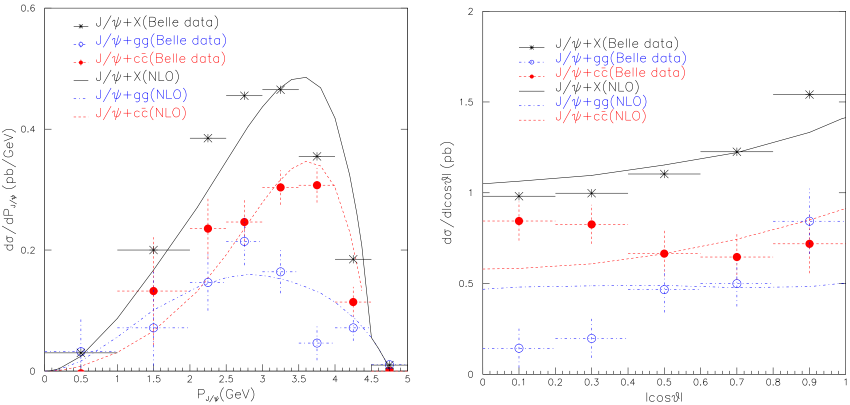
$e^+e^- \to J/\psi + c\bar{c}$



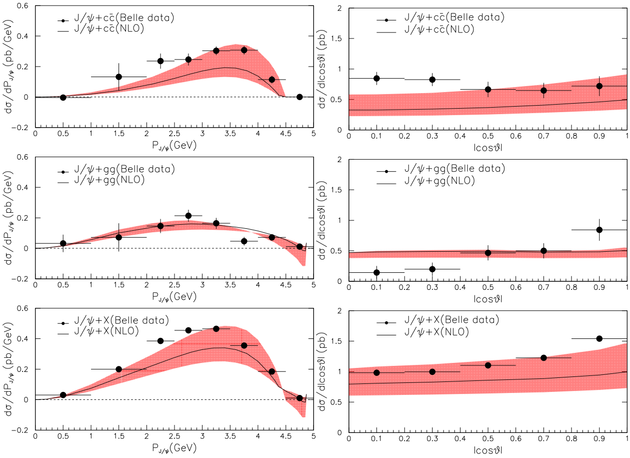
$$\sigma^{(1)} = \sigma^{(0)} \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left[a(\hat{s}) + \beta_0 \ln\left(\frac{\mu}{2m_c}\right) \right] \right\}$$

m_c(GeV)	$\alpha_s(\mu)$	$\sigma^{(0)}$(pb)	$a(\hat{s})$	$\sigma^{(1)}$(pb)	$\sigma^{(1)}/\sigma^{(0)}$
1.4	0.267	0.224	8.19	0.380	1.70
1.5	0.259	0.171	8.94	0.298	1.74
1.6	0.252	0.129	9.74	0.230	1.78

Cross sections with different charm quark mass m_c with the renormalization scale $\mu = 2m_c$ and $\sqrt{s} = 10.6$ GeV. The former result given by PRL98, (2007) Y. J. Zhang and K. T. Chao confirmed by PRD80, (2009) B. Gong and J. X. Wang



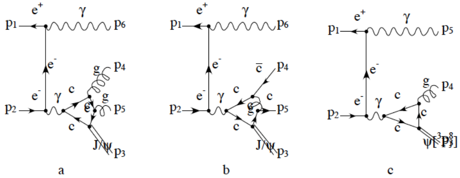
Momentum distribution of inclusive J/ψ production with $\mu = \mu^*$ and $m_c = 1.4\ \mathrm{GeV}$ is taken for the $J/\psi cc$ channel. The contribution from the feed-down of ψ' has been added to all curves by multiplying a factor of 1.29.



Momentum and angular distributions of inclusive J/ψ production.

The contribution from the feed-down of ψ' has been added to all curves by multiplying a factor of 1.29.

Constraint for color-octet matrix element of $c\bar{c}(^1S_0^8, 3P_J^8)$



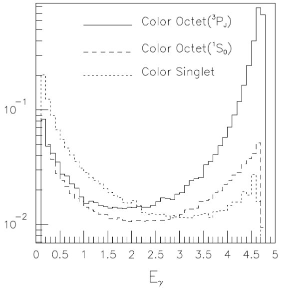


FIG. 3: The differential cross section distribution vs the energy of emitted hard photon

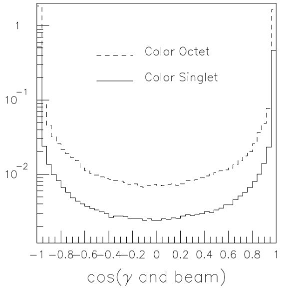


FIG. 4: The differential cross section distribution vs the $cos(\theta)$ of the emitted hard photon and the beam

From the contribution of $e^+e^- \to \gamma + J/\psi(^1S_0^8, ^3P_J^8) + g$ at NLO

hep-ph/0311292 (AIP Conf.Proc. 1092 (2009) 1) , J. X. Wang

Constraint for color-octect matrix element of $c\bar{c}(^1S_0^8, 3P_J^8)$

$$\sigma[e^+e^- \to J/\psi + X_{\mathrm{non}-c\bar{c}}] = (0.43 \pm 0.09 \pm 0.09)\ \mathrm{pb}$$

$$\sigma[e^+e^- \to J/\psi + X_{\mathrm{non}-c\bar{c}}]^{color-singleTh} > (0.43)\ \mathrm{pb}$$

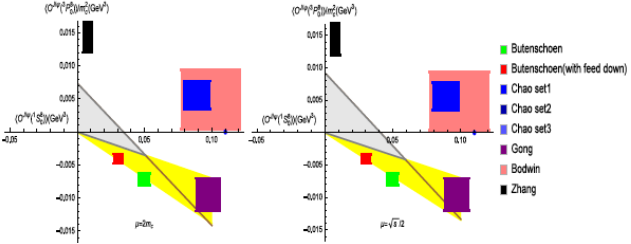
$$\sigma[e^+e^- \to J/\psi + X_{\mathrm{non}-c\bar{c}}]^{color-octetTh} > (0.6)\ \mathrm{pb}$$

From the contribution of $e^+e^- \to J/\psi(^1S_0^8, ^3P_J^8) + g$ at NLO

PRD81, (2010) Y. J. Zhang, Y. Q. Ma, K. Wang and K. T. Chao

Constraint for color-octect matrix element of $c\bar{c}({}^1S_0^8, 3P_J^8)$

$\sqrt{s}$ (GeV)	$\hat{\sigma}({}^3S_1^1)$ pb/GeV^3	$\hat{\sigma}({}^1S_0^8)$ pb/GeV^3	$m_c^2\hat{\sigma}({}^3P_J^8)$ pb/GeV^3
4.6	2.0 (2.7)	$386.5^{+90.8}_{-79.6}$ $(452.3^{+86.1}_{-80.2})$	$7037.1^{+659.1}_{-1034.5}$ $(8186.8^{+1034.1}_{-1522.7})$
4.8	1.9 (2.4)	$344.5^{+73.9}_{-66.9}$ $(391.7^{+67.4}_{-65.2})$	$5191.5^{+387.9}_{-575.4}$ $(5880.0^{+635.5}_{-868.8})$
5.2	1.7 (2.0)	$269.0^{+53.0}_{-45.3}$ $(290.8^{+45.8}_{-41.3})$	$3021.3^{+114.4}_{-216.1}$ $(3261.6^{+232.7}_{-348.4})$
5.4	1.6 (1.8)	$238.2^{+45.1}_{-39.2}$ $(251.9^{+38.0}_{-35.0})$	$2380.6^{+63.4}_{-122.7}$ $(2516.1^{+148.4}_{-214.1})$
5.6	1.5 (1.6)	$211.0^{+52.0}_{-33.5}$ $(218.8^{+44.8}_{-29.2})$	$1906.3^{+86.3}_{-73.6}$ $(1975.5^{+34.7}_{-138.9})$



From the contribution of $e^+e^- \to J/\psi({}^1S_0^8, {}^3P_J^8) + g$ at NLO

Eur.Phys.J. C77 (2017) no.9, 597; Y.J. Li, G.Z. Xu, P.P. Zhang, Y.J. Zhang, K.Y. Liu

Motivation

Theoretical Calculation

- The joint NLO QCD and relativistic correction has been investigated.
  H.-R. Dong, F. Feng and Y. Jia, PRD 2012、X.-H. Li and J.-X. Wang, Chin. Phys. C 2014
- The improved NLO prediction by using PMC shows excellent agreement with the experimental measurements.
  Z. Sun, X.-G. Wu, Y. Ma and S.J. Brodsky, PRD 2018
- The challenging NNLO correction of this process has been calculated.
  F. Feng, Y. Jia, Z. Mo, W.-L. Sang and J.-Y. Zhang, arXiv:1901.08447
- The light-cone sum rules has also been suggested to solve this discrepancy.
  L. Zeng, H.-B. Fu, D.-D. Hu, L.-L. Chen, W. Cheng and X.-G. Wu, PRD 2021

Motivation

1、In 2019, the challenging NNLO correction of this process was calculated in arXiv:1901.08447 , however the precision of master integrals is not satisfied.

2、In 2022, a powerful algorithm named Auxiliary Mass Flow has been pioneered by Liu and Ma, which can be used to compute the Feynman integrals with very high precision.

Calculation of the NNLO SDCs

- The SDCs can be derived by the perturbative matching procedure.
- In the lowest-order nonrelativistic approximation, only the color-singlet contribution need to be considered.
- Nearly 2000 two-loop diagrams for the processes $\gamma^* \to (c\bar{c})[{}^3S_1^{[1]}] + (c\bar{c})[{}^1S_0^{[1]}]$ (FeynArts) T. Hahn, CPC 2001

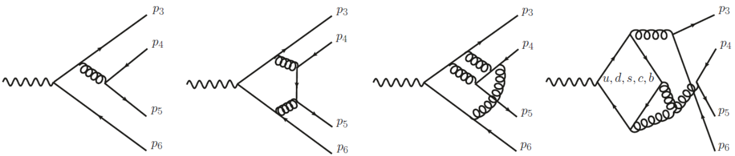


Figure 1. Some representative Feynman diagrams for $\gamma^* \to (c\bar{c})[{}^3S_1^{[1]}] + (c\bar{c})[{}^1S_0^{[1]}]$

Renormalization

- The amplitudes are renormalized according to

$$\mathcal{A}(\alpha_s, m_Q) = Z_{2,c}^2 \Big[\mathcal{A}_{\rm bare}^{0l}(\alpha_{s,\rm bare}, m_{Q,\rm bare}) + \mathcal{A}_{\rm bare}^{1l}(\alpha_{s,\rm bare}, m_{Q,\rm bare}) + \mathcal{A}_{\rm bare}^{2l}(\alpha_{s,\rm bare}, m_{Q,\rm bare}) \Big],$$

where $m_{Q,\rm bare} = Z_{m,Q} m_Q \quad \alpha_{s,\rm bare} = \left(\frac{e^{\gamma_E}}{4\pi}\right)^{\epsilon} \mu_R^{2\epsilon} Z_{\alpha_s}^{\overline{\rm MS}} \alpha_s(\mu_R),$

P. Bärnreuther, M. Czakon and P. Fiedler, JHEP 2014
W. Tao, R. Zhu and Z.-J. Xiao, PRD 2022

$$Z_{\alpha_s} = 1 - \left(\frac{\alpha_s^{(n_f)}}{2\pi}\right) \frac{b_0}{2\epsilon} + \left(\frac{\alpha_s^{(n_f)}}{2\pi}\right)^2 \left(\frac{b_0^2}{4\epsilon^2} - \frac{b_1}{8\epsilon}\right)$$

- The renormalized $\mathcal{A}(\alpha_s, m_Q)$ can be obtained by expanding the r.h.s. of such equation over renormalized quantities to $\mathcal{O}(\alpha_s^4)$,

$$\mathcal{A}(\alpha_s, m_Q) = \mathcal{A}^{0l}(\alpha_s, m_Q) + \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-\epsilon} \mathcal{A}^{1l}(\alpha_s, m_Q) + \left(\frac{e^{\gamma_E}}{4\pi}\right)^{-2\epsilon} \mathcal{A}^{2l}(\alpha_s, m_Q) + \mathcal{O}(\alpha_s^4).$$

Phenomenological results

- Input parameters:

PDG, PTEP 2022
G.T. Bodwin, J. Lee and C. Yu, PRD 2008

$\sqrt{s} = 10.58\text{GeV}$， $m_b = 4.78\text{GeV}$， $\alpha(\sqrt{s}) = 1/130.9$， $\alpha_s(M_z) = 0.1179$

$$\langle \mathcal{O}^{J/\psi} \rangle = 0.440\text{GeV}^3,\ \langle \mathcal{O}^{\eta_c} \rangle = 0.437\text{GeV}^3$$

- The numerical results of the NNLO QCD corrections to production at the B factories

$$\begin{aligned}\sigma|_{m_c=1.5\,\text{GeV}} = {} & 115.599\alpha_s^2(\mu_R) + \Big[(177.849 - 12.2654 n_l)\ln\frac{\mu_R^2}{m_c^2} + 10.4752 n_l \\ & +215.393\Big]\alpha_s^3(\mu_R) + \Big[(205.215 - 28.3055 n_l + 0.976053 n_l^2)\left(\ln\frac{\mu_R^2}{m_c^2}\right)^2 \\ & +(609.319 - 28.6518 n_l - 1.66718 n_l^2)\ln\frac{\mu_R^2}{m_c^2} - 736.409\ln\frac{\mu_\Lambda^2}{m_c^2} \\ & -109.15 - 74.7989 n_l - 2.49917 n_l^2\Big]\alpha_s^4(\mu_R),\end{aligned}$$

NNLO cross section

- The NNLO cross section (in fb) of $e^+e^- \to J/\psi + \eta_c$ with three typical charm quark mass m_c under two renormalization scale μ_R choices.

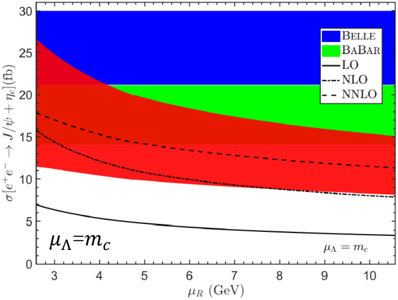
		α_s^2-terms	α_s^3-terms	α_s^4-terms	Total($\mu_\Lambda = m_c$)
$m_c = 1.3$ GeV	$\mu_R = 2m_c$	9.80	11.10	5.70	26.60
	$\mu_R = \sqrt{s}/2$	5.98	7.46	5.77	19.21
$m_c = 1.5$ GeV	$\mu_R = 2m_c$	7.40	7.17	2.45	17.02
	$\mu_R = \sqrt{s}/2$	5.06	5.52	3.35	13.93
$m_c = 1.7$ GeV	$\mu_R = 2m_c$	5.42	4.58	0.88	10.88
	$\mu_R = \sqrt{s}/2$	4.07	3.90	1.74	9.71

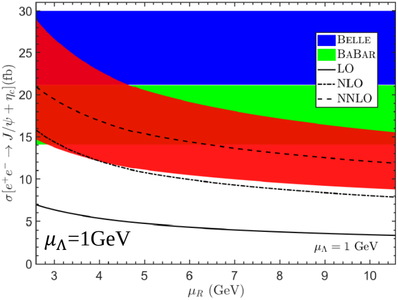
1, 1 : 97% : 33% for $\mu_R = 2m_c$, 1 : 109% : 66% for $\mu_R = \sqrt{s}/2$ (exhibit convergence)
Scale uncertainty of NLO (NNLO) is ~ 27% (18%) (improved)

2, Uncertainties caused by charm quark mass:
59%, 91%, and 197% for $\mu_R = 2m_c$, 38%, 64%, and 120% for $\mu_R = \sqrt{s}/2$

NNLO cross section

- The μ_R dependence of the predicted cross sections at LO, NLO and NNLO levels (central value for m_c=1.5GeV, bound for $m_c \in [1.3\text{GeV}, 1.7\text{GeV}]$)

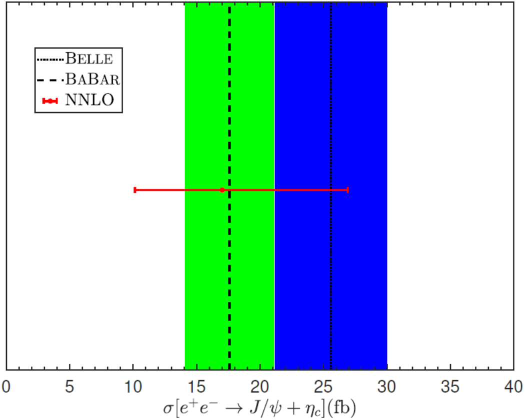




1, NNLO has a milder μ_R dependence than NLO in μ_Λ=m_c
2, NNLO is much closer to experimental value in μ_Λ=1GeV
3, Theoretical prediction near $\mu_R = 2m_c$ agree with the experimental results better

NNLO cross section

- The comparison between the NNLO QCD correction to $e^+e^- \to J/\psi + \eta_c$ and the experimental measurements.



NNLO QCD corrections to $J/\psi + \eta_c$ production at the B factories :

$$\sigma_{\rm NNLO} = 17.02^{+9.58+0+2.44}_{-6.14-3.09-0} = 17.02^{+9.89}_{-6.87}\,(\text{fb})$$

Uncertainties caused by:

$$m_c \in [1.3\text{GeV}, 1.7\text{GeV}]$$
$$\mu_R \in [2m_c, \sqrt{s}/2]$$
$$\mu_\Lambda \in [1\text{GeV}, m_c]$$

Exp:

	BELLE$(\sigma \times \mathcal{B}_{>2})$	BABAR$(\sigma \times \mathcal{B}_{>2}$
$\sigma_{J/\psi+\eta_c}$	$25.6 \pm 2.8 \pm 3.4$	$17.6 \pm 2.8^{+1.5}_{-2.1}$

Motivation

Theoretical Calculation

- The NLO NRQCD predictions, the combined NLO perturbative and relativistic corrections
  B. Gong and J. X. Wang, PRL 2008 -3.4~2.3fb
  Y. Fan, J. Lee and C. Yu, PRD 2013 -12~-0.43fb
- Following the recipe practised in PRD 74, 074014 (2006), splitting the amplitude into the photon-fragmentation and non-fragmentation parts
  Y. Fan, J. Lee and C. Yu, PRD 2013 1~1.5fb
- Following PRD 74, 074014 (2006), the interference and the non-fragmentation parts are then computed through NNLO within NRQCD
  W. L.Sang, F. Feng, Y. Jia, Z. Mo, J. Pan and J. Y. Zhang, PRL 2023 $2.13^{+0.30}_{-0.06}$fb

Motivation

1、The NLO perturbative correction turns out to be negative and significant, the NNLO correction in the standard NRQCD?

2、How to obtain an positive, physical cross section in the standard NRQCD?

Calculation of the NNLO SDCs

- The SDCs can be derived by the perturbative matching procedure.
- In the lowest-order nonrelativistic approximation, only the color-singlet contribution need to be considered.
- Nearly 600 two-loop diagrams for the processes $e^+e^- \to (c\bar{c})[\,^3S_1^{[1]}] + (c\bar{c})[\,^3S_1^{[1]}]$(FeynArts) T. Hahn, CPC 2001

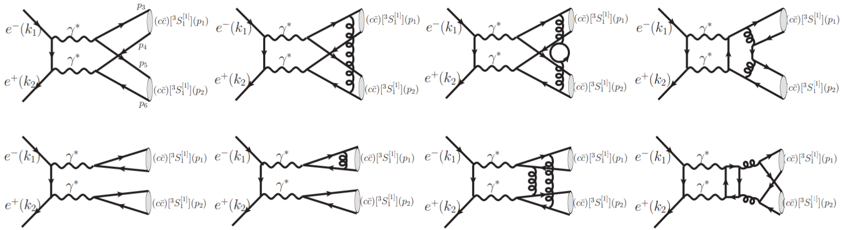


Figure 1. Several representative Feynman diagrams for $e^+e^- \to (c\bar{c})[^3S_1^{[1]}] + (c\bar{c})[^3S_1^{[1]}]$.

Calculating amplitudes

- Complete-basis space

$$\begin{pmatrix} |e_1\rangle \\ |e_2\rangle \\ |e_3\rangle \\ |e_4\rangle \\ |e_5\rangle \\ |e_6\rangle \\ |e_7\rangle \\ |e_8\rangle \\ |e_9\rangle \\ |e_{10}\rangle \end{pmatrix} = \begin{pmatrix} g^{\rho_1\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) \\ k_1^{\rho_1}k_1^{\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) + k_2^{\rho_1}k_2^{\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) \\ k_1^{\rho_1}k_1^{\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) - k_2^{\rho_1}k_2^{\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) \\ k_1^{\rho_1}k_2^{\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) \\ k_2^{\rho_1}k_1^{\rho_2}\bar{v}_{m_e}(k_2).\not{p}_2.u_{m_e}(k_1) \\ k_1^{\rho_2}\bar{v}_{m_e}(k_2).\gamma^{\rho_1}.u_{m_e}(k_1) + k_2^{\rho_1}\bar{v}_{m_e}(k_2).\gamma^{\rho_2}.u_{m_e}(k_1) \\ k_1^{\rho_2}\bar{v}_{m_e}(k_2).\gamma^{\rho_1}.u_{m_e}(k_1) - k_2^{\rho_1}\bar{v}_{m_e}(k_2).\gamma^{\rho_2}.u_{m_e}(k_1) \\ k_1^{\rho_1}\bar{v}_{m_e}(k_2).\gamma^{\rho_2}.u_{m_e}(k_1) + k_2^{\rho_2}\bar{v}_{m_e}(k_2).\gamma^{\rho_1}.u_{m_e}(k_1) \\ k_1^{\rho_1}\bar{v}_{m_e}(k_2).\gamma^{\rho_2}.u_{m_e}(k_1) - k_2^{\rho_2}\bar{v}_{m_e}(k_2).\gamma^{\rho_1}.u_{m_e}(k_1) \\ \bar{v}_{m_e}(k_2).\not{p}_2.\gamma^{\rho_1}.\gamma^{\rho_2}.u_{m_e}(k_1) \end{pmatrix}$$

- Amplitudes

$$\mathcal{A}^{nl}|_{n=0,1,2} = \sum_{i=1}^{10} c_i^{nl}|e_i\rangle$$

$$\mathcal{A}^{ml}\mathcal{A}^{nl,*} = \sum_{i=1}^{10}\sum_{j=1}^{10} c_i^{ml}G_{i,j}c_j^{nl,*}$$

$$c_i^{nl}|_{n=0,1,2} = \sum_{i=1}^{10} G_{i,j}^{-1}d_j^{nl}$$

$$d_i^{nl}|_{n=0,1,2} = \langle\mathcal{A}^{nl}|e_i\rangle$$

$$G_{i,j} = \langle e_i|e_j\rangle$$

Differential cross section

- Then, the differential cross section can be written as

$$\frac{d\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^3S_1^{[1]}]}}{d|\cos\theta|} = \frac{1}{8s}\frac{\kappa}{16\pi}\left|\mathcal{A}^{0l}+\mathcal{A}^{1l}+\mathcal{A}^{2l}+\mathcal{O}(\alpha_s^3)\right|^2$$
$$= \frac{1}{8s}\frac{\kappa}{16\pi}\Big(|\mathcal{A}^{0l}|^2+2\mathrm{Re}(\mathcal{A}^{1l}\mathcal{A}^{0l,*})+2\mathrm{Re}(\mathcal{A}^{2l}\mathcal{A}^{0l,*})+|\mathcal{A}^{1l}|^2$$
$$+2\mathrm{Re}(\mathcal{A}^{2l}\mathcal{A}^{1l,*})+|\mathcal{A}^{2l}|^2+\cdots\Big),$$

where $\kappa = \sqrt{1-(16m_c^2)/s}$ and θ is the angle between the J/ψ and the beam.

- The square of NNLO amplitude (S-NNLO)

Finite and gauge invariant

$$\boxed{|\mathcal{A}^{0l}|^2}+\boxed{2\mathrm{Re}(\mathcal{A}^{1l}\mathcal{A}^{0l,*})}+\boxed{2\mathrm{Re}(\mathcal{A}^{2l}\mathcal{A}^{0l,*})+|\mathcal{A}^{1l}|^2}+\boxed{2\mathrm{Re}(\mathcal{A}^{2l}\mathcal{A}^{1l,*})+|\mathcal{A}^{2l}|^2}+\cdots$$

LO NLO NNLO

- There still remains IR divergence in $\mathcal{A}^{2l}\mathcal{A}^{0l,*}$, $\mathcal{A}^{2l}\mathcal{A}^{1l,*}$ $|\mathcal{A}^{2l}|^2$.

Differential cross section

- The anomalous dimension for the NRQCD current J

$$\gamma_J = \frac{\mathrm{d}\ln Z_J}{\mathrm{d}\ln\mu} = -C_F\,(2C_F + 3C_A)\,\frac{\pi^2}{6}\left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3).$$

A. Czarnecki and K. Melnikov, PRL 1998、M. Beneke, A. Signer and V.A. Smirnov, PRL 1998
A. Czarnecki and K. Melnikov, PLB 2001

- By including the two-loop corrections to the NRQCD bilinear operators carrying the quantum number of J/ψ in $\overline{\text{MS}}$ scheme

$$\langle \mathcal{O}^{(c\bar{c})[^3S_1^{[1]}]}(^3S_1^{[1]})\rangle|_{\overline{\text{MS}}} = 2N_c\left[1 - \alpha_s^2(\mu_R)\left(\frac{\mu_\Lambda^2 e^{\gamma_E}}{\mu_R^2 4\pi}\right)^{-2\epsilon}\left(\frac{C_F^2}{3} + \frac{C_F C_A}{2}\right)\frac{1}{2\epsilon}\right]$$

H.S. Chung, JHEP 2020

- The differential cross section for $e^+e^- \to J/\psi + J/\psi$ can be written as

$$\frac{d\sigma_{e^+e^-\to J/\psi + J/\psi}}{d|\cos\theta|} = \frac{d\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^3S_1^{[1]}]}}{d|\cos\theta|}\,\frac{\langle \mathcal{O}^{J/\psi}(^3S_1^{[1]})\rangle^2}{\langle \mathcal{O}^{(c\bar{c})[^3S_1^{[1]}]}(^3S_1^{[1]})\rangle^2|_{\overline{\text{MS}}}}$$

$$= (f_0 + f_1\alpha_s + f_2\alpha_s^2 + \boxed{f_3\alpha_s^3 + f_4\alpha_s^4} + \cdots)|R_s^{J/\psi}(0)|^4$$

where $\langle \mathcal{O}^{J/\psi}(^3S_1^{[1]})\rangle \approx N_c|R_s^{J/\psi}(0)|^2/(2\pi)$ incomplete

Phenomenological results

- Input parameters:

PDG, PTEP 2022
G.T. Bodwin, J. Lee and C. Yu, PRD 2008

$$\sqrt{s} = 10.58\text{GeV},\quad m_b = 4.8\text{GeV},\quad m_c = 1.5\text{GeV},\quad \alpha(2m_c) = 1/132.6,$$
$$\alpha_s(M_z) = 0.1179,\ \left|R_s^{J/\psi}(0)\right|^2_{LO} = 0.492\text{GeV}^3,\ \left|R_s^{J/\psi}(0)\right|^2_{NLO} = 0.796\text{GeV}^3,$$
$$\left|R_s^{J/\psi}(0)\right|^2_{NNLO,\mu_\Lambda=1GeV} = 1.810\text{GeV}^3,$$

- The leptonic decay widths : $\Gamma_{J/\psi\to e^+e^-} = 5.53\text{keV}$

$$\Gamma_{J/\psi\to e^+e^-} = \frac{4\alpha^2 e_c^2}{m_{J/\psi}^2}|R_s^{J/\psi}(0)|^2\Bigg\{1 - 2C_F\frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\Bigg[-2C_F\beta_0\ln\frac{\mu_R^2}{m_c^2} - 3\pi^2 C_F\left(\frac{1}{18}C_F + \frac{1}{12}C_A\right)\ln\frac{\mu_\Lambda^2}{m_c^2} + C_AC_F\left(\frac{89\pi^2}{144} - \frac{151}{72} - \frac{5\pi^2}{6}\ln 2 - \frac{13}{4}\zeta_3\right)$$
$$+ C_F^2\left(\frac{23}{8} - \frac{79\pi^2}{36} + \pi^2\ln 2 - \frac{1}{2}\zeta_3\right) + C_FT_Fn_H\left(\frac{22}{9} - \frac{2\pi^2}{9}\right) + \frac{11}{18}C_FT_Fn_L\Bigg]\Bigg\}^2,$$

F. Feng, Y. Jia, Z. Mo, J. Pan, W.-L. Sang, and J.-Y. Zhang, arXiv:2207.14259

Phenomenological results

- The differential cross section for $e^+e^- \to J/\psi + J/\psi$ can be written as

$$\frac{d\sigma_{e^+e^-\to J/\psi+J/\psi}}{d|\cos\theta|} = \frac{d\sigma_{e^+e^-\to(c\bar{c})[^3S_1^{[1]}]+(c\bar{c})[^3S_1^{[1]}]}}{d|\cos\theta|}\frac{\langle\mathcal{O}^{J/\psi}(^3S_1^{[1]})\rangle^2}{\langle\mathcal{O}^{(c\bar{c})[^3S_1^{[1]}]}(^3S_1^{[1]})\rangle^2|_{\overline{\mathrm{MS}}}}$$
$$= (f_0 + f_1\alpha_s + f_2\alpha_s^2 + f_3\alpha_s^3 + f_4\alpha_s^4 + \cdots)|R_s^{J/\psi}(0)|^4$$

$\lvert\cos\theta\rvert$	f_0	f_1	f_2
0.193	3.0687	−11.1472	$-43.3988 + 0.5647n_f - 11.1472\beta_0 L_\mu - 15.9116L_{\mu_\Lambda}$
0.402	3.8973	−14.2469	$-54.8858 + 0.7247n_f - 14.2469\beta_0 L_\mu - 20.2080L_{\mu_\Lambda}$
0.601	5.9069	−21.6244	$-83.0903 + 1.1036n_f - 21.6244\beta_0 L_\mu - 30.6282L_{\mu_\Lambda}$
0.698	7.9392	−28.9429	$-111.9326 + 1.4775n_f - 28.9429\beta_0 L_\mu - 41.1664L_{\mu_\Lambda}$
0.800	12.0746	−43.5649	$-171.1529 + 2.2221n_f - 43.5649\beta_0 L_\mu - 62.6088L_{\mu_\Lambda}$
0.849	15.7238	−56.2870	$-223.7382 + 2.8694n_f - 56.2870\beta_0 L_\mu - 81.5310L_{\mu_\Lambda}$
0.902	22.8893	−80.9980	$-327.4304 + 4.1287n_f - 80.9980\beta_0 L_\mu - 118.6851L_{\mu_\Lambda}$
0.922	27.1569	−95.6123	$-389.3525 + 4.8758n_f - 95.6123\beta_0 L_\mu - 140.8136L_{\mu_\Lambda}$
0.951	37.0190	−129.2083	$-532.7535 + 6.6029n_f - 129.2083\beta_0 L_\mu - 191.9502L_{\mu_\Lambda}$
0.975	50.9428	−176.3416	$-735.9322 + 9.0683n_f - 176.3416\beta_0 L_\mu - 264.1479L_{\mu_\Lambda}$
0.999	54.7376	−187.8744	$-797.7502 + 10.2369n_f - 187.8744\beta_0 L_\mu - 283.8247L_{\mu_\Lambda}$

where $\beta_0 = \frac{1}{\ldots}(11 - \frac{2}{\ldots}n_f)$, $L_\mu = \ln\frac{\mu_R^2}{\ldots}$, $L_{\mu_\Lambda} = \ln\frac{\mu_\Lambda^2}{\ldots}$

Phenomenological results

$\lvert\cos\theta\rvert$	f_3
0.193	$92.0656 + 12.3660 L_\mu + 28.9002 L_{\mu_\Lambda}$
0.402	$117.4510 + 15.9175 L_\mu + 36.9363 L_{\mu_\Lambda}$
0.601	$178.1131 + 24.2128 L_\mu + 56.0634 L_{\mu_\Lambda}$
0.698	$238.4933 + 32.2766 L_\mu + 75.0372 L_{\mu_\Lambda}$
0.800	$359.3808 + 48.0764 L_\mu + 112.9460 L_{\mu_\Lambda}$
0.849	$464.6776 + 61.6160 L_\mu + 145.9294 L_{\mu_\Lambda}$
0.902	$669.2768 + 87.6152 L_\mu + 209.9949 L_{\mu_\Lambda}$
0.922	$790.2901 + 102.8788 L_\mu + 247.8839 L_{\mu_\Lambda}$
0.951	$1068.4986 + 137.7803 L_\mu + 334.9846 L_{\mu_\Lambda}$
0.975	$1458.9501 + 186.4323 L_\mu + 457.1819 L_{\mu_\Lambda}$
0.999	$1557.3600 + 196.9479 L_\mu + 487.0818 L_{\mu_\Lambda}$

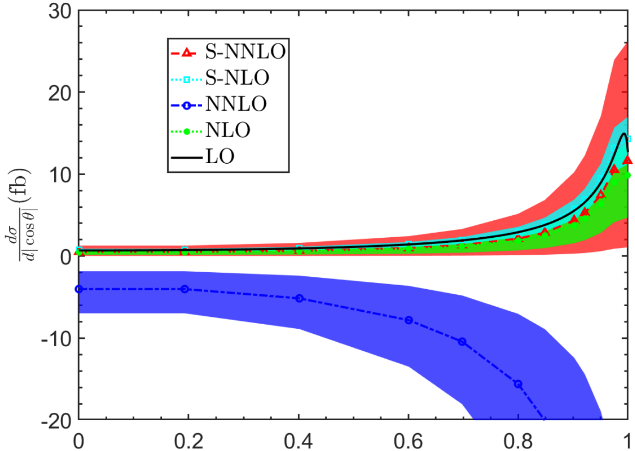
Phenomenological results

$\lvert\cos\theta\rvert$	f_4
0.193	$209.5396 + 56.1687L_\mu + 131.4701L_{\mu_\Lambda} + 17.6318L_\mu L_{\mu_\Lambda} + 3.7722L_\mu^2 + 20.6262L_{\mu_\Lambda}^2$
0.402	$265.3937 + 71.6562L_\mu + 166.7230L_{\mu_\Lambda} + 22.5346L_\mu L_{\mu_\Lambda} + 4.8556L_\mu^2 + 26.1955L_{\mu_\Lambda}^2$
0.601	$401.8986 + 108.6657L_\mu + 252.5597L_{\mu_\Lambda} + 34.2039L_\mu L_{\mu_\Lambda} + 7.3860L_\mu^2 + 39.7032L_{\mu_\Lambda}^2$
0.698	$540.8027 + 145.5020L_\mu + 339.6221L_{\mu_\Lambda} + 45.7797L_\mu L_{\mu_\Lambda} + 9.8459L_\mu^2 + 53.3639L_{\mu_\Lambda}^2$
0.800	$824.0584 + 219.2561L_\mu + 517.0743L_{\mu_\Lambda} + 68.9077L_\mu L_{\mu_\Lambda} + 14.6655L_\mu^2 + 81.1595L_{\mu_\Lambda}^2$
0.849	$1074.4249 + 283.4970L_\mu + 673.7844L_{\mu_\Lambda} + 89.0306L_\mu L_{\mu_\Lambda} + 18.7958L_\mu^2 + 105.6884L_{\mu_\Lambda}^2$
0.902	$1566.1342 + 408.3217L_\mu + 981.5335L_{\mu_\Lambda} + 128.1166L_\mu L_{\mu_\Lambda} + 26.7267L_\mu^2 + 153.8510L_{\mu_\Lambda}^2$
0.922	$1858.9494 + 482.1512L_\mu + 1164.8197L_{\mu_\Lambda} + 151.2324L_\mu L_{\mu_\Lambda} + 31.3829L_\mu^2 + 182.5361L_{\mu_\Lambda}^2$
0.951	$2536.2604 + 651.8733L_\mu + 1588.3677L_{\mu_\Lambda} + 204.3721L_\mu L_{\mu_\Lambda} + 42.0295L_\mu^2 + 248.8244L_{\mu_\Lambda}^2$
0.975	$3491.4667 + 890.0966L_\mu + 2186.5424L_{\mu_\Lambda} + 278.9239L_\mu L_{\mu_\Lambda} + 56.8706L_\mu^2 + 342.4140L_{\mu_\Lambda}^2$
0.999	$3766.2171 + 950.1359L_\mu + 2354.0053L_{\mu_\Lambda} + 297.1657L_\mu L_{\mu_\Lambda} + 60.0783L_\mu^2 + 367.9209L_{\mu_\Lambda}^2$

- They are not changed when we demand 10-digit or 20-digit precision for each Feynman integral family

Differential cross section

- The differential cross section for $e^+e^- \to J/\psi + J/\psi$. (central value for $\mu_R = \sqrt{s}/2$, bound for $\mu_R \in [2m_c, \sqrt{s}]$)



Integrated cross section

- The integrated cross section (in fb) of $e^+e^- \to J/\psi + J/\psi$ at the B factories :

$$\sigma_{\mathrm{S-NNLO}} = 1.76^{+2.41+0.25}_{-1.64-0.25} = 1.76^{+2.42}_{-1.66}\ (\mathrm{fb}),$$

Uncertainties caused by: $\mu_R \in [2m_c, \sqrt{s}]$ and the method for estimating the integrated cross section from the differential cross section.

$$\frac{|f(x_2)-f(x_1)|}{2}(x_2 - x_1)$$

- Results of PRL 131 (2023) 161904

σ (fb)	Fragmentation	LO	NLO	NNLO
Optimized NRQCD	2.52	1.85	$1.93^{+0.05}_{-0.01}$	$2.13^{+0.30}_{-0.06}$
Traditional NRQCD		6.12	$1.56^{+0.73}_{-2.95}$	$-2.38^{+1.27}_{-5.35}$

- Exp: an upper limit is placed, $\sigma(e^+e^- \to J/\psi + J/\psi)\,\mathcal{B}_{>2} < 9.1$ fb at the 90% confidence level,

Summary

- For B-factories: NRQCD at NLO of α_s and v can well described J/ψ production data. strong constraint to the values of color-octect matrix element of $c\bar{c}({}^1S_0^8, 3P_J^8)$ to almost zero, which give dominant contribution to for J/ψ hadronproduction.
- The NNLO QCD Correction calculations shown they can improve the theoretical description on the experimental measurements.
- More NNLO calculations are much more difficult, but may be expected.

Thank you!