Hamiltonian Dynamics Problem Sheet

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Problem 1. Write Hamilton's equations for the following Hamiltonians

$$H(\theta, p; t) = \frac{p^2}{2mR^2} - \frac{1}{2}mR^2\omega^2 \sin^2\theta - mgR\cos\theta$$
$$H(x, z, p_x, p_z; t) = \frac{1}{m}\left(\frac{p_x^2}{2} - p_x p_z + p_z^2\right) + \frac{1}{2}kz^2$$

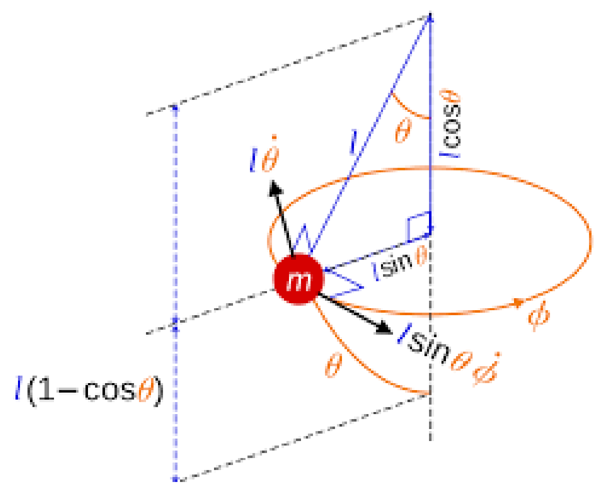

Figure 1: Spherical pendulum

Problem 2. A spherical pendulum is a pendulum that is suspended from a pivot mounting, which enables it to swing in any of an infinite number of vertical planes through the point of suspension. The kinetic (T) and potential (V) energies in in the coordinate system shown in the figure are given by

$$T = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) \tag{1}$$

and

$$V = mgl(1-\cos\theta) \tag{2}$$

where l is the length of the pendulum. Write down the Lagrangian then calculate the canonical momenta and hence write the Hamiltonian. Finally write Hamilton's equations.

Problem 3. The Hamiltonian for a particle travelling through an ideal solenoid with constant axial field B_z may be written

$$H(x, y, P_x, P_y; z) = \frac{\left(P_x + \frac{1}{2}qB_z y\right)^2}{2m\gamma} + \frac{\left(P_y - \frac{1}{2}qB_z x\right)^2}{2m\gamma}$$

where x, y are the tranverse coordinates and P_x, P_y the transverse canonical momenta. It is evident that the Hamiltonian contains the cross term $(yP_x - xP_y)$, i.e. the system is coupled. Make use of the following generating function

$$F_2(x, \hat{P}_x, y, \hat{P}_y, t) = \left(x\hat{P}_x + y\hat{P}_y\right)\cos\theta(z) + \left(x\hat{P}_y - y\hat{P}_x\right)\sin\theta(z)$$

to make a canonical transformation to new variables $\hat{x}, \hat{y}, \hat{P}_x, \hat{P}_y$ where

$$\theta = \int \omega_L dz$$

is the *Larmor angle* and $\omega_L = \frac{qB_z}{2m\gamma}$ is the *Larmor frequency.* Show that the resulting Hamiltonian $\hat{H}(\hat{x}, \hat{y}, \hat{P}_x, \hat{P}_y; z)$ is decoupled. Note, that in this case the relation

$$\hat{H} = H + \frac{\partial F_2}{\partial z}$$

is non-trivial.

Problem 4. The potential term in the Hamiltonian of a normal sextupole of normalised strength k_2 and length L is given by

$$H = \frac{k_2 L}{3!}\left(x^3 - 3xy^2\right) \tag{3}$$

where (x,y) are the transverse coordinates. Transform to action-angle coordinates (J, ϕ) making use of the relation

$$x = \sqrt{J_x\beta_x}\cos\phi_x$$
$$y = \sqrt{J_y\beta_y}\cos\phi_x$$

where $\beta_{x,y}$ are the transverse betatron functions. Considering only the terms that involve J_x (i.e. setting $J_y = 0$), identify the resonance driving terms in this case (given by the value of n in $\cos n\phi$).

Problem 5. In the Hénon map the coordinates are updated as follows

$$\begin{pmatrix} x \\ p \end{pmatrix} = R(2\pi q)\begin{pmatrix} x \\ p + x^2 \end{pmatrix} \tag{4}$$

where $R(\theta)$ is the rotation matrix, q is the tune and (x,p) are canonical coordinates. This map is analogous to the case of a linear lattice with a single thin sextupole (in one transverse dimension).

Write a code (e.g. in Python) to iterate this map a few hundred times starting with a set of a few hundred starting coordinates. Plot all the coordinates after each iteration on a single phase space figure. Produce phase plots at tunes close to 1/5, 1/4 and 1/3.

You should observe a region of bounded motion, chaos, islands of stability, fixed points etc. The example below was produced with q = 0.21. A set of 300 starting coordinates were distributed uniformly along the x-axis in the range -1 to +1 and tracked for 1000 iterations (the initial p coordinate was set to zero in all cases).

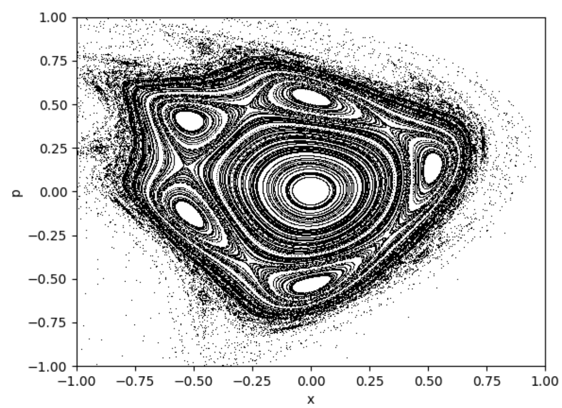

Figure 2: Example phase space in case of Hénon map (q=0.21).