

# Hamiltonian Dynamics

## Lecture 1

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# Hamiltonian dynamics introduction

- In Hamiltonian mechanics, the equations of motion follow from the Hamiltonian,  $H$ , which represents the total energy of a conservative system (the sum of the kinetic energy  $T$  and potential energy  $V$ ).

The Hamiltonian (conservative system)

$$H = T + V$$

Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

where  $(q,p)$  are canonical coordinates.

- Phase space
- Liouville's Theorem
- Action-angle coordinates
- Hamiltonian flow
- Canonical coordinates and transformations
- Symplecticity
- Integrability
- Poisson Brackets
- Lie Algebra

*Key concepts related to Hamiltonian dynamics*

# Pendulum example

## Newtonian approach

$$F = ma = m\ddot{\theta} = -mg\sin\theta$$

## Hamiltonian approach

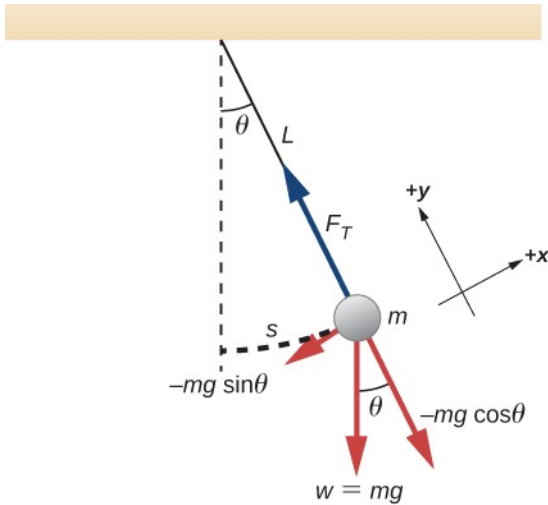
Write the Hamiltonian for the system

$$H(\theta, p_\theta) = T + V = \frac{p_\theta^2}{2mL^2} + mgL(1 - \cos\theta)$$

Hamilton's equations follow

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mL^2}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgL \sin\theta$$





# Historical Approach

Three approaches to Classical Mechanics

Newtonian

Lagrangian

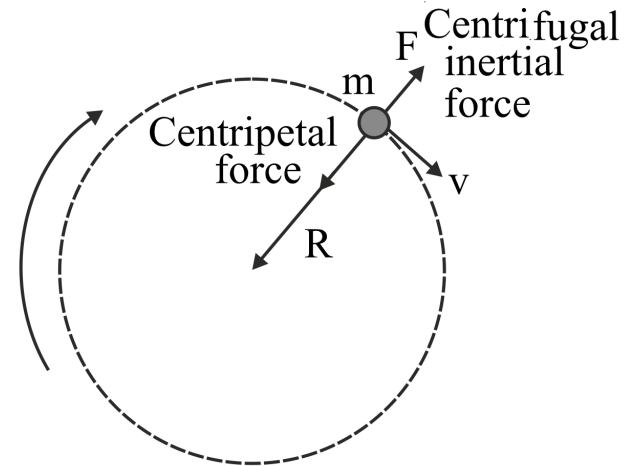
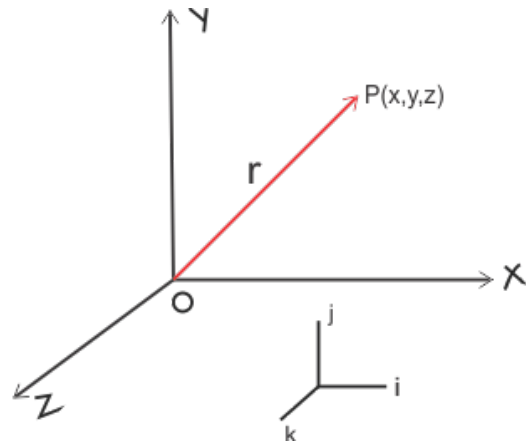
Hamiltonian

# Newtonian Mechanics

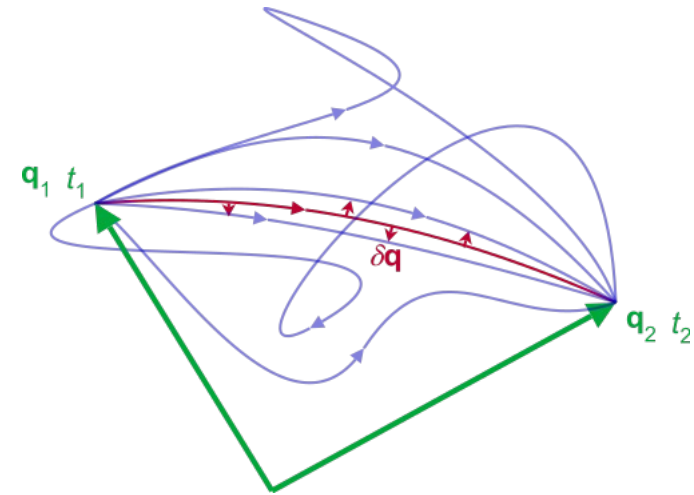
- The key function is the force  $\mathbf{F}(r, \dot{r}, t)$  where  $r$  is the position,  $\dot{r}$  is the velocity and  $t$  is time.
- The equation of motion, in an inertial frame, is

$$\frac{d}{dt} (m\dot{\mathbf{r}}) = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) \quad (2^{\text{nd}} \text{ order differential equation})$$

- In a non-inertial frame fictitious forces may need to be considered. In a non-Cartesian coordinate system, the analysis can get more complicated.



# Lagrangian Mechanics



- Mechanics can be reformulated in way that avoids specifying a force directly.
- Let us define the action  $S$ .

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

- $L(q, \dot{q}, t)$  is the Lagrangian, a function of generalized coordinates, velocities and time.
- Hamilton's principle (often misleadingly called the "principle of least action") holds that the system evolves such that  $S$  is stationary,

$$\delta S = 0$$

- The equation of motion (the *Euler-Lagrange* equation) follows

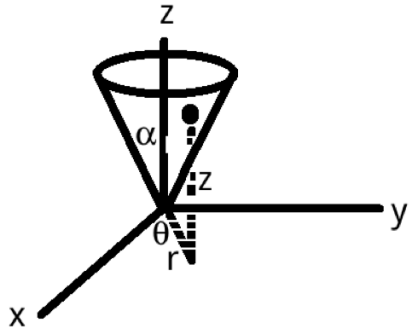
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad (2^{\text{nd}} \text{ order differential equation})$$

- In the case of a conservative force (depends on  $q$  only)

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

- Applies in any coordinate system including non-inertial ones.
- Constraints can be incorporated naturally.

# Lagrangian example – particle on a cone



- Consider a particle rolling due to gravity in a frictionless cone. The cone's opening angle  $\alpha$  places a constraint on the coordinates  $\tan \alpha = r/z$ . We may write the Lagrangian in cylindrical coordinates

$$L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - mgz$$

- Reduce the number of coordinates by eliminating  $z$  via  $z = \frac{r}{\tan \alpha}$ ,  $\dot{z} = \frac{\dot{r}}{\tan \alpha}$

$$L = \frac{m}{2} \left( (1 + \cot^2 \alpha) \dot{r}^2 + r^2 \dot{\theta}^2 \right) - mgr \cot \alpha$$

- The Euler-Lagrange equation for each coordinate...

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

...can be solved to obtain the equations of motion

$$(1 + \cot^2 \alpha) \ddot{r} - r \dot{\theta}^2 + g \cot \alpha = 0$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

# From the Lagrangian to the Hamiltonian

- Perform a *Legendre transformation* to get from the  $L(q_i, \dot{q}_i, t)$  to  $H(q_i, p_i, t)$ .
- Defining the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

- The definition of the Hamiltonian follows

$$H = \sum_{i=1}^n p_i \dot{q}_i - L$$

Can also write  $L = \sum p_i \dot{q}_i - H$

- By comparing the differential of the Hamiltonian and Lagrangian, Hamilton's equations of motion can be found

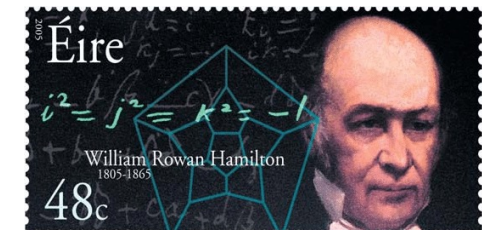
$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Note – in this case we have a pair of first order differential equations for the phase space coordinates.



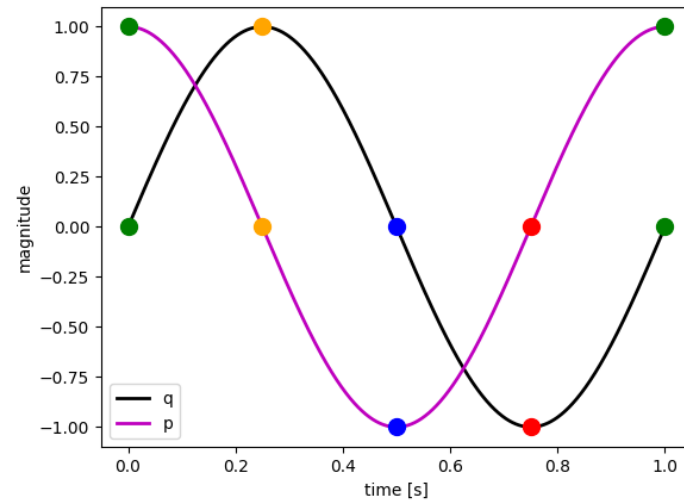
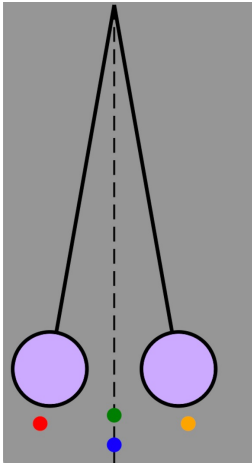
# Summary of approaches

	Newtonian	Lagrangian	Hamiltonian
Key functional	$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$	$L(q_i, \dot{q}_i, t)$	$H(q_i, p_i, t)$
Equation of motion	$\frac{d}{dt} (m\dot{\mathbf{r}}) = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$	$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$	$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$
Strengths	<ul style="list-style-type: none"> <li>Can include dissipative forces</li> </ul>	<ul style="list-style-type: none"> <li>Ease of incorporating constraints</li> <li>Flexibility of coordinate system</li> </ul>	<ul style="list-style-type: none"> <li>First order differential equations</li> <li>Connection to powerful geometric theory that flows from the conservation of energy.</li> </ul>

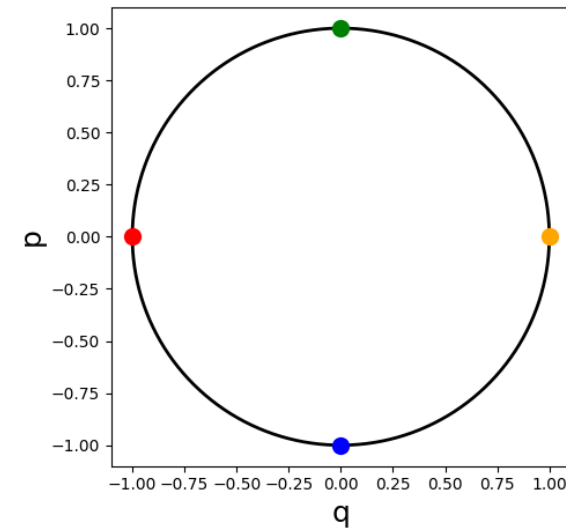


# Phase space (1)

- In Hamiltonian mechanics, the canonical momenta  $p_i$  are promoted to coordinates on equal footing with the generalized coordinates  $q_i$ .
- The coordinates  $(q, p)$  are canonical variables, and the space of canonical variables is known as phase space.



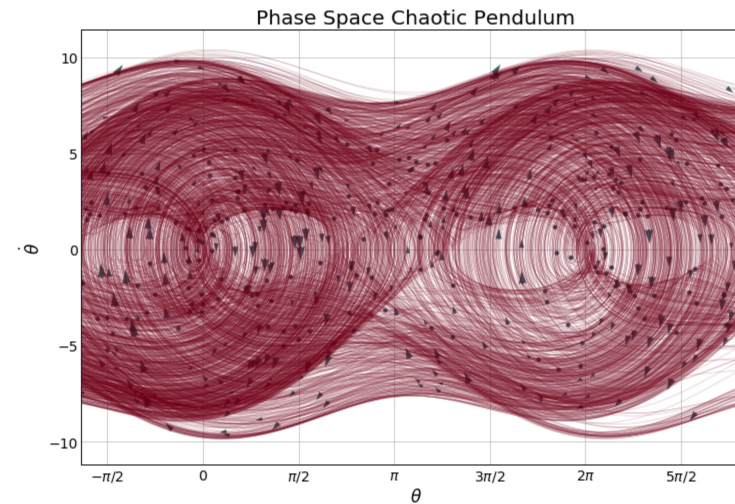
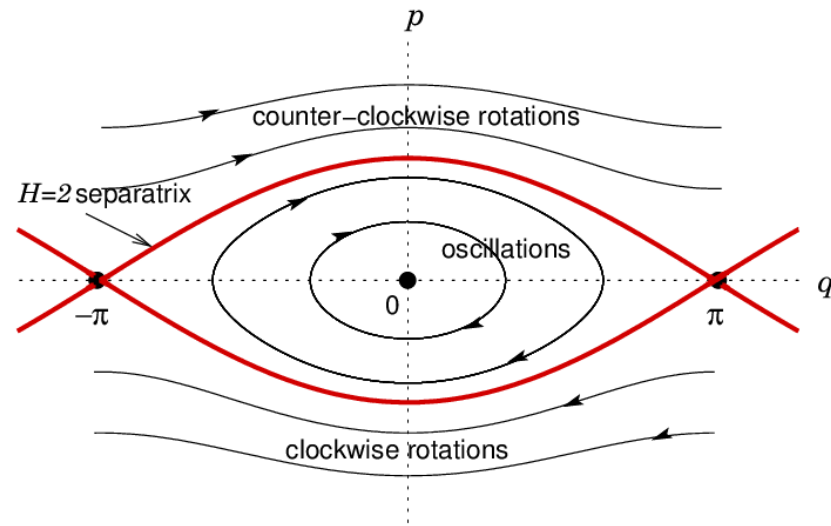
Time series



phase space

# Phase space (2)

- The phase space may exhibit features such as bounded/unbounded motion, regular or chaotic motion, stable and unstable fixed points, resonances etc.

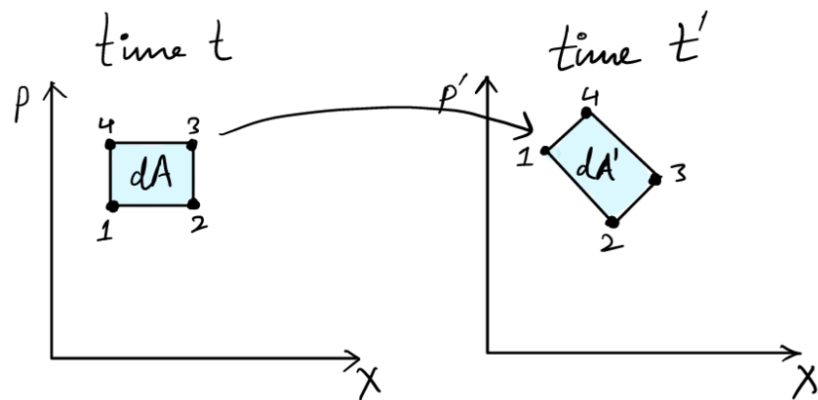


# Liouville's theorem

- Consider the particle distribution function,  $f(p_i, q_i; t)$ .
- Liouville's theorem states that, for a system subject only to non-dissipative forces ( $dH/dt = 0$ ) the phase space distribution function is constant along the trajectory of the motion, i.e.

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^n \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = 0$$

- The phase space acts like an incompressible fluid. The phase space density cannot be increased unless a non-conservative (dissipative) force is added (e.g. charge exchange injection).



# Symplecticity

- A map  $M$  is used to track particles from one part of a ring to another or turn-by-turn. Quantities such as betatron tune and other optics parameters can be obtained from the map itself.

$$\begin{bmatrix} x_f \\ p_f \end{bmatrix} = M \begin{bmatrix} x_i \\ p_i \end{bmatrix}$$

- How do we ensure the map is consistent with the Hamiltonian? Let's write Hamilton's equations in matrix form

$$\begin{bmatrix} \dot{q}_i \\ \dot{p}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q_i} \\ \frac{\partial H}{\partial p_i} \end{bmatrix}$$

- Define a vector  $\zeta = (q_i, p_i)$  and write Hamilton's equations in vector form

$$\dot{\zeta} = \Omega \nabla H(\zeta) \quad \text{where } \Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

- It can be shown that the corresponding map  $M$  given by  $\zeta(t) = M\zeta(t_0)$  ( $\Omega$  is a skew-symmetric matrix)

has the symplectic property

$$M^T \Omega M = \Omega$$

# Canonical transformations

- It often proves useful to transform from one set of phase space coordinates  $(q,p)$  to another  $(Q,P)$ . The transformation is said to be canonical if it preserves the form of Hamilton's equations.
- Consider the transformation from  $H(q, p, t)$  to  $K(Q, P, t)$ . From the gauge invariance of the Lagrangian we can write

$$\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{dF}{dt} \quad (\text{Assume the case } \lambda=1)$$

- The function  $F$  is a generating function that can depend on various combinations of old and new phase space coordinates.
- Consider the case  $F = F_1(q,Q,t)$ , known as a type 1 generating function. Then by the partial derivative chain rule

$$p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F_i}{\partial t}$$

Rearranging terms

$$\left(p_i - \frac{\partial F}{\partial q_i}\right)\dot{q}_i - \left(P_i + \frac{\partial F}{\partial Q_i}\right)\dot{Q}_i + K - \left(H + \frac{\partial F_i}{\partial t}\right) = 0$$

To allow separately independent coordinates the coefficients must be zero

$$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}, K = H + \frac{\partial F_1}{\partial t}$$

# Canonical transformation – generating functions

Generating function	Transformation equations	
$F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$
$F_2(q, P, t)$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$
$F_3(p, Q, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$P_i = -\frac{\partial F_3}{\partial Q_i}$
$F_4(p, P, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$

# Action-angle coordinates (1)

- The canonical transformation to action-angle coordinates helps simplify the dynamics. Define canonical variables  $(\theta, I)$  such as the Hamiltonian depends only on action,  $H = H(I)$ . Then

$$\dot{I} = -\frac{\partial H}{\partial \omega} = 0, \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I)$$

- Let's apply this transformation for the case of a simple harmonic oscillator with Hamiltonian

$$H = \frac{\omega}{2}(q^2 + p^2)$$

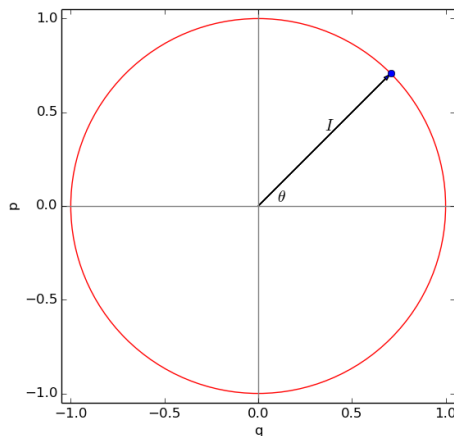
- Try a transformation to action-angle coordinates

$$q = \sqrt{\frac{2}{\omega}} f(P) \sin Q, \quad p = \sqrt{\frac{2}{\omega}} f(P) \cos Q$$

$$\Rightarrow p = q \cot Q, \quad K = H = f^2(P) (\sin^2 Q + \cos^2 Q) = f^2(P)$$

This is independent of  $f(P)$ , and has the form of the  $F_1(q, Q, t)$  type of generating function

$$p = \frac{\partial F_1}{\partial q}$$





# Action-angle coordinates (2)

- The corresponding generating function is given by

$$F_1(q, Q) = \frac{1}{2}q^2 \cot Q$$

$$\Rightarrow P = -\frac{\partial F_1(q, Q)}{\partial Q} = \frac{1}{2} \frac{q^2}{\sin^2 Q}$$

- Rearrange for q

$$q = \sqrt{2P} \sin Q$$

- By comparing with equation for q on previous slide, we obtain f(P) and K.

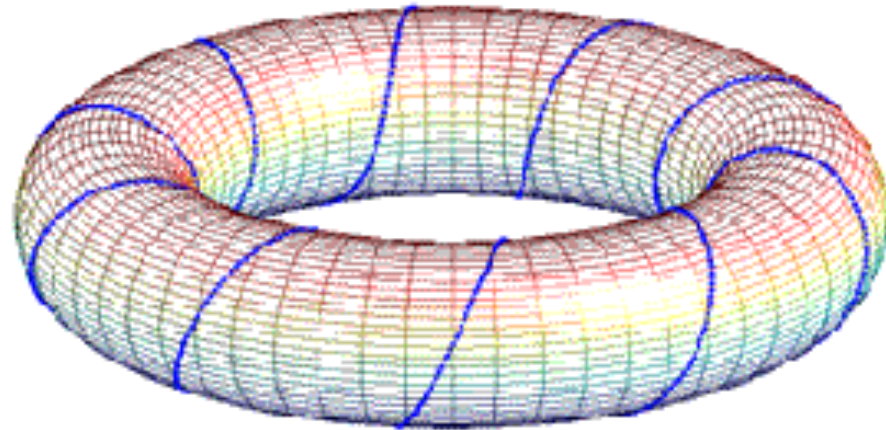
$$f = \sqrt{\omega P}, \quad K = \omega P$$

- From the equations of motion for P, Q we see action P is constant and depends on energy, while angle Q increases monotonically in time.

$$P = \frac{K}{\omega} \qquad \dot{Q} = \frac{\partial K}{\partial P} = \omega, \quad Q = \omega t + C$$

# Integrability

- The Liouville-Arnold theorem states that existence of  $n$  invariants of motion is enough to fully characterize a for an  $n$  degree-of-freedom system. In that case a canonical transformation exists to action angle coordinates in which the Hamiltonian depends only on the action.
- For an ideal linear lattice, the motion in both horizontal and vertical planes can be separately transformed into action-angle coordinates. The motion remains bounded and regular indefinitely in this case.



# Poisson Brackets

- Introduce functions of the canonical variables  $u(q,p)$  and  $v(q,p)$ . The Poisson bracket of  $u$  and  $v$  is defined as

$$[u, v]_{p,q} = \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial q}$$

- For the phase space coordinates we have

$$[q_i, q_j] = [p_i, p_j] = 0$$

$$[q_i, p_j] = -[p_i, q_j] = \delta_{i,j}$$

- Poisson bracket is invariant under canonical transformation.

$$[u, v]_{p,q} = [u, v]_{P,Q}$$

# Poisson Brackets – Hamilton's equations

- Start with the total differential of a function  $u = (q_i, p_i, t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{p}_i + \frac{\partial u}{\partial t}$$

- Making use of Hamilton's equations

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

- Rewriting in terms of a Poisson bracket

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

- Setting  $u = q$  or  $u = p$ , and assuming no explicit time dependence, Hamilton's equations follow

$$\dot{q} = [q, H], \quad \dot{p} = [p, H]$$

# Lie operator and transformation

- The Lie operator for function  $f(q_i, p_i)$  is defined

$$: f := \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i}$$

- The Lie operator  $f$  operating on the function  $g$  is equivalent to the Poisson bracket of the two functions.

$$: f : g = [f, g]$$

- Powers of Lie operators

$$: f :^0 g = g \quad : f :^1 g = : f : g = [f, g]$$

$$: f :^2 g = : f : g = [f, [f, g]]$$

- The exponential operator is known as a Lie Transformation (allows us to build symplectic transfer maps!)

$$e^{:f:} = \sum_{k=0}^{\infty} \frac{:f:^k}{k!} \quad \exp(:f:)g = \sum_{k=1}^{\infty} \frac{:f:^k g}{k!} = g + [f, g] + [f, [f, g]]/2! + \dots$$

# Lie operators of phase space variables

$$: q_i := \frac{\partial q_i}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial q_i}{\partial p_i} \frac{\partial}{\partial q_i} = \frac{\partial}{\partial p_i}$$

$$: p_i := \frac{\partial p_i}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial p_i}{\partial p_i} \frac{\partial}{\partial q_i} = -\frac{\partial}{\partial q_i}$$

$$: q_i p_i := \frac{\partial(q_i p_i)}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial(q_i p_i)}{\partial p_i} \frac{\partial}{\partial q_i} = p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i}$$

$$: q_i^2 := \frac{\partial q_i^2}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial q_i^2}{\partial p_i} \frac{\partial}{\partial q_i} = 2q_i \frac{\partial}{\partial p_i}$$

$$: p_i^2 := 2p_i \frac{\partial}{\partial q_i}$$

# Symplectic map

- Define a map  $M$  (e.g. transfer matrix) that updates the coordinates over some increment

$$(q_{i+1}, p_{i+1}) = M(q_i, p_i)$$

- The map is symplectic if

$$M^T \Omega M = \Omega$$

$$\text{where } \Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

# Taylor series map

- The phase space coordinates can be expressed as a Taylor power series of the initial coordinates

$$z(i, 1) = \sum_{j=1}^6 R_{ij} z(j, 0) + \sum_{j,k=1, j \leq k}^6 T_{ijk} z(j, 0) z(k, 0) + \dots$$

where R, T are the 1st and 2nd order transfer map matrices,  $(z_i, 0)$  and  $(z_i, 1)$  are the phase space coordinates at the entrance and exit of a lattice element, respectively. In general, the map is not symplectic when truncated at some order.



# Map from Lie Transformations

- Symplectic maps can be created using Lie transformations

$$z(t) = \exp(t : H :) z_0$$

with  $M = \exp(t : H :)$ . A map to a given order in phase space coordinates can be created by composition

$$M = e^{:f^1:} e^{:f^2:} e^{:f^3:} \dots e^{:f^k:} + \mathcal{O}(k)$$

The map can be truncated at order  $k$  and it remains symplectic (Dragt-Finn factorisation theorem). Make use of the Baker-Campbell-Hausdorff (BCH) formula

$$e^{:A:} e^{:B:} = e^{:C:}$$

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] \dots$$

# Lie Operators for a drift

The map for a drift is simply  $M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$  Since there is no potential term,  $H_{drift} = \frac{p^2}{2}$

The equivalent Lie operator is  $exp(: Lp^2/2 :)$

To show this expand the transformation as follows

$$exp(: Lp^2/2 :)x = x + [Lp^2/2, x] + [Lp^2/2, [[Lp^2/2, x]]/2 + \dots$$

$$exp(: Lp^2/2 :)p = p + [Lp^2/2, p] + [Lp^2/2, [[Lp^2/2, p]]/2 + \dots$$

Noting  $[Lp_x^2/2, p_x] = Lp_x \frac{\partial p_x}{\partial x} = Lp_x$ ,  $[Lp_x^2/2, p_x] = Lp_x \frac{\partial p_x}{\partial x} = 0$  and the higher order terms are zero,

# Lie Operators for Accelerator elements

Table 1: Lie Operators for Common Accelerator Elements

Element	Map	Lie Operator
Drift space	$x = x_0 + Lp_0$ $p = p_0$	$\exp(: -\frac{1}{2}Lp^2 :)$
Thin-lens quadrupole	$x = x_0$ $p = p_0 - \frac{1}{f}x_0$	$\exp(: -\frac{1}{2f}x^2 :)$
Thin-lens kick	$x = x_0$ $p = p_0 + \lambda nx_0^{n-1}$	$\exp(: \lambda x^n :)$
Thick focusing quad	$x = x_0 \cos \sqrt{k}L + \frac{p_0}{\sqrt{k}} \sin \sqrt{k}L$ $p = -kx_0 \sin \sqrt{k}L + p_0 \cos \sqrt{k}L$	$\exp(: -\frac{1}{2}L(kx^2 + p^2) :)$
Thick defocusing quad	$x = x_0 \cosh \sqrt{k}L + \frac{p_0}{\sqrt{k}} \sinh \sqrt{k}L$ $p = -kx_0 \sinh \sqrt{k}L + p_0 \cosh \sqrt{k}L$	$\exp(: -\frac{1}{2}L(kx^2 - p^2) :)$
Coordinate shift	$x = x_0 - b$ $p = p_0 + a$	$\exp(: ax + bp :)$
Coordinate rotation (Phase advance $\mu$ )	$x = x_0 \cos \mu + p_0 \sin \mu$ $p = -x_0 \sin \mu + p_0 \cos \mu$	$\exp(: -\frac{\mu}{2}(x^2 + p^2) :)$
Full-turn Hamiltonian	(lots of things)	$\exp(C : H_{\text{eff}} :)$ or $\exp(: -\frac{\mu}{2}(\gamma x^2 + 2\alpha xp + \beta p^2) :)$

# Cheat Sheet

- Hamiltonian formal definition, the Legendre transform  $H = \sum_{i=1}^n p_i \dot{q}_i - L$
- For a conservative system  $H = T + V$
- Hamilton's equations...  $\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$
- ...may be expressed in terms of Poisson brackets

$$\dot{q} = [q, H], \dot{p} = [p, H]$$

- The following fundamental concepts follow
  - **Liouville's theorem:**  $\frac{\partial f}{\partial t} + [f, H] = 0$
  - **Symplecticity**  $M^T \Omega M = \Omega$  where  $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$
  - **Integrability** (#invariants = # degrees of freedom)

Symplectic maps can be constructed using Lie Transformations

$$z(t) = \exp(t : H :) z_0$$