Hamiltonian Dynamics

Lecture 2

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Accelerator case

Consider a circulating accelerator with particles moving around the ring at relativistic velocities.

- Start with the Hamiltonian for a relativistic particle in an electromagnetic field.
- Transform into convenient coordinates (Frenet-Serret).
- Change the independent variable from time to coordinate s.
- Convert to small dynamic variables (normalised transverse momenta and energy deviation).
- Introduce potential of each magnet element

Hamiltonian - General electromagnetic fields

The Lagrangian in general EM fields $U(x, \dot{x}, t) = e(\phi - \boldsymbol{v} \cdot \boldsymbol{A})$ is given by

$$L(x, \dot{x}, t) = -mc^2\sqrt{1-\beta^2} - e\phi + e\boldsymbol{v} \cdot \boldsymbol{A}. \tag{1}$$

the conjugate momentum is

$$P_i = \frac{\partial L}{\partial \dot{x}_i} = \frac{m\dot{x}_i}{\sqrt{1-\beta^2}} + eA_i \tag{2}$$

i.e. the field contributes to the momentum.
The Hamiltonian

$$H(q, P, t) = \sum_i P_i \dot{q}_i - L = \frac{mc^2}{\sqrt{1-\beta^2}} + e\phi. \tag{3}$$

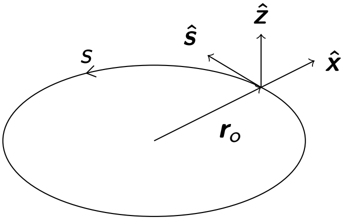
As before use

$$\frac{mc^2}{\sqrt{1-\beta^2}} = \gamma mc^2 = c\sqrt{m^2c^2 + \boldsymbol{p}^2}$$

to obtain

$$H(q, P, t) = c\sqrt{m^2c^2 + (\boldsymbol{P} - e\boldsymbol{A})^2} + e\phi. \tag{4}$$

Frenet-Serret coordinates



For the transverse plane we can specify motion with respect to a reference orbit we label $\boldsymbol{r}_0(s)$. s is the arc length along the closed orbit from some reference point.

Then the tangential unit vector

$$\hat{\boldsymbol{s}}(s) = \frac{\mathrm{d}\boldsymbol{r}_0(s)}{\mathrm{d}s}, \tag{5}$$

The principle unit normal vector, perpendicular to the tangent vector

$$\hat{\boldsymbol{x}}(s) = -\rho(s)\frac{\mathrm{d}\hat{\boldsymbol{s}}(s)}{\mathrm{d}s} \tag{6}$$

where $\rho(s)$ defines the local radius of curvature.

The unit binormal vector, orthogonal to the transverse plane

$$\hat{\boldsymbol{z}}(s) = \hat{\boldsymbol{x}}(s) \times \hat{\boldsymbol{s}}(s). \tag{7}$$

These vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{s}}$ form the orthonormal basis for the right handed Frenet-Serret curvilinear coordinate system. In the planar case, the particle orbits are

$$\boldsymbol{r}(s) = \boldsymbol{r}_0(s) + x\hat{\boldsymbol{x}}(s) + z\hat{\boldsymbol{z}}(s). \tag{8}$$

It can be shown Hamiltonian becomes

$$H(s, x, z, p_s, p_x, p_z, t) =$$

$$c\sqrt{m_o^2 c^2 + \frac{(p_s - eA_s)^2}{\left(1+\frac{x}{\rho}\right)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2} + e\phi, \quad (9)$$

Note: the Hamiltonian for a straight beamline is obtained in the limit $x/\rho \to 0$. The equations of motion follow

$$\dot{s} = \frac{\partial H}{\partial p_s}, \quad \dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{z} = \frac{\partial H}{\partial p_z}$$
$$\dot{p}_s = -\frac{\partial H}{\partial s}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}. \quad (10)$$

Change of independent variable

We would like to change independent variable from t to s. Our new canonical variables become

$$(x, p_x), \quad (y, p_y), \quad (-t, H) \tag{11}$$

Our new Hamiltonian is $H_1(t, x, z, -H, p_x, p_z, s) = -p_s$.
Then our new canonical equations in terms of s are

$$\begin{aligned} t' &= \frac{\partial p_s}{\partial H}, \quad x' = -\frac{\partial p_s}{\partial p_x}, \quad z' = -\frac{\partial p_s}{\partial p_z} \\ H' &= -\frac{\partial p_s}{\partial t}, \quad p_x' = \frac{\partial p_s}{\partial s}, \quad p_z' = \frac{\partial p_s}{\partial z}. \end{aligned} \tag{12}$$

$$H_1 = -p_s = \\ -eA_s - \left(1 + \frac{x}{\rho}\right)\sqrt{\frac{1}{c^2}(H - e\phi)^2 - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2} \tag{13}$$

Reference momentum

It makes sense to construct a Hamiltonian with reference to a reference momentum P_0. This allows simplification in the case of small momentum spread.
We end up with

$$\tilde{H} = -ea_s - \left(1 + \frac{x}{\rho}\right) \sqrt{\frac{(E - e\phi)^2}{P_0^2 c^2} - \frac{m^2 c^2}{P_0} - (\tilde{p}_x - ea_x)^2 - (\tilde{p}_z - ea_z)^2} \tag{14}$$

where

$$p_i \to \tilde{p}_i = \frac{p_i}{P_0}, \quad H_1 \to \tilde{H} = \frac{H_1}{P_0}, \quad A_i \to \mathbf{a} = e\frac{A_i}{P_0} \tag{15}$$

Change of longitudinal coordinates

Define new longitudinal coordinates with respect to the reference particle.

$$\delta_E = \frac{E}{P_0 c} - \frac{1}{\beta_0}, \quad S = c\Delta t = \frac{s}{\beta_0} - ct \tag{16}$$

where δ_E is known as the energy deviation and β_0 is relativistic beta.
Invoking the generating function

$$F_2(x, P_x, z, P_z, -t, \delta_E, s) = xP_x + xP_z + \left(\frac{s}{\beta_0} - ct\right)\left(\frac{1}{\beta} + \delta_E\right) \tag{17}$$

The 'Accelerator Hamilton'

We find that the transverse variables are unchanged and the new Hamiltonian $H = \tilde{H} + \frac{\partial F_2}{\partial s}$ can be, after some manipulation, shown to be

$$H = -(1+hx)\sqrt{(\frac{1}{\beta_0} + \delta_E - \frac{e\phi}{P_0 c})^2 - (\tilde{p}_x - ea_x)^2 - (\tilde{p}_z - ea_z)^2 - \frac{1}{\beta_0^2\gamma_0^2}} - (1+hx)a_s + \frac{\delta_E}{\beta_0} \quad (18)$$

where $h = \frac{1}{\rho}$ is the curvature.
The Hamiltonian for each element in an accelerator can be found by substituting the corresponding potential a_s or ϕ.

Multipole magnets

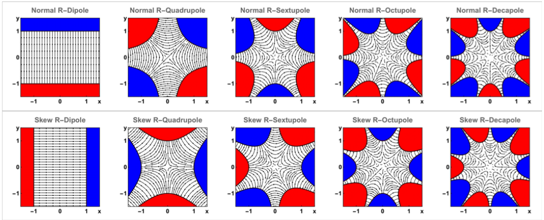


FIG. 4. Normal and skew $2n$-pole magnets in Cartesian coordinates. Each figure shows magnetic (electric) field streamlines and poles' shape in transverse cross section. North (positive electrostatic potential) and south (negative electrostatic potential) poles are shown in red and blue and are given by $(\mathcal{B}, \mathcal{A})_n = \mp R_p^n$ respectively, where R_p is the distance to the pole's tip.

The vector potential for a straight multipole magnet with axial symmetry is

$$A_x = 0, \quad A_z = 0, \quad A_l = -\mathcal{R} \sum_{n=1}^{\infty} (b_n + i a_n) \frac{(x + iz)^n}{n r_0^{n-1}} \tag{19}$$

giving magnetic field ($\mathbf{B} = \nabla \times \mathbf{A}$)

$$B_z + iB_x = -\frac{\partial A_l}{\partial x} + i \frac{\partial A_l}{\partial y} = \mathcal{R} \sum_{n=1}^{\infty} (b_n + i a_n) \frac{(x + iz)^{n-1}}{r_0} \tag{20}$$

Curl in curvilinear coordinates

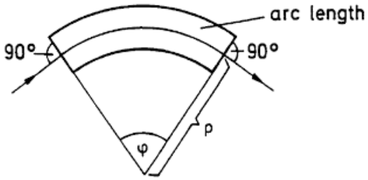
The curl in curvilinear coordinates is

$$B_x = [\nabla \times A]_x = \frac{\partial A_s}{\partial z} - \frac{1}{1+hx}\frac{\partial A_z}{\partial s} \tag{21}$$

$$B_z = [\nabla \times A]_z = \frac{1}{1+hx}\frac{\partial A_x}{\partial s} - \frac{h}{1+hx}A_s - \frac{\partial A_s}{\partial x} \tag{22}$$

$$B_s = [\nabla \times A]_s = \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \tag{23}$$

Dipole magnet (n=1)



Starting with the following vector potential components

$$A_x = 0, \quad A_z = 0, \quad A_s = -B_0 \left(x - \frac{hx^2}{2(1+hx)} \right) \tag{24}$$

using the curl equations one finds the field components

$$B_x = 0, \quad B_z = B_0, \quad B_s = 0 \tag{25}$$

Dipole magnet: Hamiltonian

Using the vector potential for a dipole, the following Hamiltonian results

$$
\begin{aligned}
H &= -(1+hx)\sqrt{\left(\frac{1}{\beta_0}+\delta_E\right)^2 - \tilde{p}_x^2 - \tilde{p}_z^2 - \frac{1}{\beta_0^2\gamma_0^2}} \\
&\quad + (1+hx)\, k_0 \left(x - \frac{hx^2}{2(1+hx)}\right) + \frac{\delta_E}{\beta_0} \qquad (26)
\end{aligned}
$$

where the normalised dipole field strength is $k_0 = \frac{e}{P_0} B_0$.

As long as the dynamical variables are small the Hamiltonian can be expanded to second order as

$$H_2 = \frac{p_x^2}{2} + \frac{p_z^2}{2} + (k_0 - h)x + \frac{hk_0x^2}{2} - \frac{hx\delta_E}{\beta_0} + \frac{\delta_E^2}{2\beta_0^2\gamma_0^2} \qquad (27)$$

The following observations can be made:

- The $(k_0 - h)x$ term results in a change in p_x. It is zero if $k_0 = h$, i.e. when the dipole field bends with the design curvature.
- The $\frac{1}{2}hk_0x^2$ term is the weak focusing term.
- The $\frac{hx\delta_E}{\beta_0}$ term represents first order dispersion.
- Note - the high order terms that are ignored when expanding the square root are known as *kinematic terms*.

Apply Hamilton's equation (assume $k_0 = h$)

$$x' = \frac{dx}{ds} = \frac{\partial H}{\partial p_x} = p_x \tag{28}$$

$$p_x' = \frac{dp_x}{ds} = -\frac{\partial H}{\partial x} = -hk_0 x + \frac{h\delta_E}{\beta_0} \tag{29}$$

In the case of $\delta_E = 0$, it follows $x'' = p_x' = -hk_0 k$, the solution is that of a harmonic oscillator

$$x(s) = x(0)\cos\left(\sqrt{hk_0}s\right) + \frac{p_x(0)}{\sqrt{hk_0}}\sin\left(\sqrt{hk_0}s\right) \tag{30}$$

The complete solutions for the transverse coordinates (assuming $k_0 = h$):

$$
\begin{aligned}
x(s) &= x(0)\mathrm{cos}\omega s + p_x(0)\frac{sin\omega s}{\omega} + \delta_E(0)\frac{h}{\beta_0}\left(\frac{1-\mathrm{cos}\omega s}{\omega^2}\right) && (31)\\
p_x(s) &= -x(0)\omega\mathrm{sin}\omega s + p_x(0)cos\omega s + \delta_E(0)\frac{h}{\beta_0}\frac{\mathrm{sin}\omega s}{\omega} && (32)\\
z(s) &= z(0) + p_z(0)s && (33)\\
p_z(s) &= p_z(0) && (34)
\end{aligned}
$$

where $\omega = \sqrt{hk}$ and $(x(0), p_x(0), z(0), p_z(0))$ are the initial transverse coordinates. Note the oscillatory terms in the horizontal plane - the effect of weak focusing.

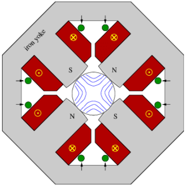
Dipole magnet: Transfer Matrix

It is convenient to express the map of a dipole magnet in the form of a transfer matrix

$$M = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos\omega L}{\omega\beta_0} \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin\omega L}{\beta_0} & -\frac{1-\cos\omega L}{\omega\beta_0} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{\omega L-\sin\omega L}{\omega\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

where L is the dipole length. This will multiply the following phase space vector $(x, p_x, y, p_y, S, \delta)$.

Quadrupole magnet (n=2)



Starting with the following vector potential components

$$A_x = 0, \quad A_z = 0, \quad A_s = -\frac{b_2}{2r_0}\left(x^2 - z^2\right) \tag{36}$$

using the curl equations one finds the field components

$$B_x = \frac{b_2}{r_0} z, \quad B_z = \frac{b_2}{r_0} x, \quad B_s = 0 \tag{37}$$

leading to Hamiltonian (the normalised quadrupole gradient $k_1 = \frac{qb_2}{P_0 r_0}$).

$$H = \frac{\delta_E}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta_E\right)^2 - p_x^2 - p_z^2 - \frac{1}{\beta_0^2 \gamma_0^2}} + \frac{1}{2} k_1 \left(x^2 - z^2\right) \tag{38}$$

To second order the Hamiltonian becomes

$$H_2 = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \frac{k_1 x^2}{2} - \frac{k_1 z^2}{2} + \frac{1}{2\beta_0^2\gamma_0^2}\delta_E^2 \tag{39}$$

If $k_1 > 0$ this leads to focusing in x and defocusing in z. The transfer matrix for a "focusing" quadrupole follows

$$M = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & \cosh\omega L & \frac{\sinh\omega L}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh\omega L & \cosh\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

where $\omega = \sqrt{k_1}$.

Symplectic integration of a Harmonic oscillator

The Hamiltonian for a harmonic oscillator in one dimension is

$$H(p, q; \tau) = \frac{1}{2}\left(p^2 + q^2\right) \tag{41}$$

where the potential energy is $U(q) = \frac{q^2}{2}$. The equations of motion are

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p \\ \dot{p} &= -\frac{\partial H}{\partial q} = -q \end{aligned}$$

The exact evolution is given by

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = M \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \tag{42}$$

Note the symplectic condition ($M\Omega M = \Omega$) is met

$$\begin{pmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (43)$$

This condition must be satisfied to preserve the phase space volume under evolution (Liouville). Next, expand the cosine and sine to first order

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \quad (44)$$

The symplectic condition is not satisfied in this case and furthermore

$$\left| det \begin{pmatrix} 1 & \tau \\ -\tau & 1 \end{pmatrix} \right| = 1 + \tau^2 \quad (45)$$

The energy after one timestep

$$H_{integrated} = \frac{1}{2}\left(p(\tau)^2 + q(\tau)^2\right) = \frac{1}{2}(1+\tau^2)\left(p^2+q^2\right) \tag{46}$$

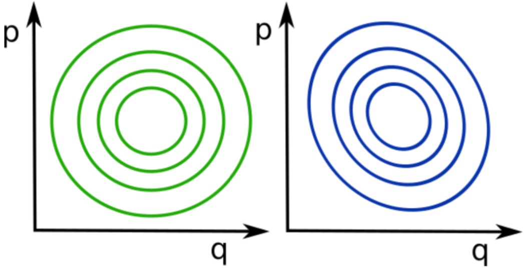
The increase in energy will cause the trajectory to spiral outwards. A *symplectic* integration scheme (one the preserves phase space volume) can be created as follows

$$\begin{pmatrix} q(\tau) \\ p(\tau) \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ -\tau & 1-\tau^2 \end{pmatrix} \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} \tag{47}$$

Although the symplectic condition is met we find after one time step

$$H_{integrated} = \frac{1}{2}\left(p^2+q^2\right) + \frac{\tau}{2}pq \tag{48}$$

the integrated Hamiltonian differs from the true one.



Since $H_{integrated}$ is conserved, the difference between it and the true Hamiltonian H_{true} is constant and the trajectory is *bounded*. The figure on the left shows *level curves* for H_{true} and on the right for $H_{integrated}$.

Splitting the Hamiltonian (1)

In general a symplectic integrator is constructed by splitting the Hamiltonian into terms R and K that depend on momentum and coordinates, respectively.

$$H = \frac{p_x^2}{2} + V(x) = R(p_x) + K(x) \tag{49}$$

The Lie operator for R becomes

$$:R:x = \frac{\partial R}{\partial x}\frac{\partial x}{\partial p_x} - \frac{\partial R}{\partial p_x}\frac{\partial x}{\partial x} = -\frac{\partial R}{\partial p_x}$$
$$:R:p_x = \frac{\partial R}{\partial x}\frac{\partial p_x}{\partial p_x} - \frac{\partial R}{\partial p_x}\frac{\partial p_x}{\partial x} = 0$$

Similarly for K

$$:K:x = \frac{\partial K}{\partial x}\frac{\partial x}{\partial p_x} - \frac{\partial K}{\partial p_x}\frac{\partial x}{\partial x} = 0$$
$$:K:p_x = \frac{\partial K}{\partial x}\frac{\partial p_x}{\partial p_x} - \frac{\partial K}{\partial p_x}\frac{\partial p_x}{\partial x} = \frac{\partial K}{\partial x}$$

Splitting the Hamiltonian(2)

It follows that the Hamiltonian K (the "kick") updates the momentum only

$$e^{:K:}x = x$$
$$e^{:K:}p_x = p_x + \frac{\partial K}{\partial x}$$

while R (the "drift") updates the position alone

$$e^{:R:}x = x - \frac{\partial R}{\partial p_x}$$
$$e^{:R:}p_x = p_x$$

First order integrator

To first order we can write

$$e^{-t:R(p_x)+K(x)} = e^{-t:R(p_x)}e^{-t:K(x)}$$

This is the symplectic Euler method. Dividing the interval into steps of length h,

$$p_{n+1} = p_n - h\frac{\partial V}{\partial q}(q_n)$$
$$q_{n+1} = q_n + h\frac{\partial T}{\partial q}(p_{n+1})$$

Note, the standard Euler method is non-symplectic as in the lower equation $\frac{\partial T}{\partial q}$ is evaluated at p_n.

Second order integrator

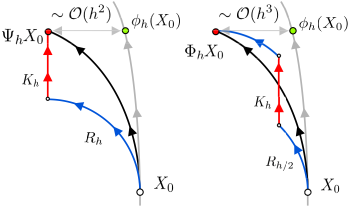


Figure: One step of a symplectic Euler integrator (left) and second order leapfrog (right). [S. Baturin]

It follows from BCH formula that if we split the Hamiltonian as follows

$$H = \frac{1}{2}R(p_x) + K(x) + \frac{1}{2}R(p_x)$$

then to second order

$$e^{-t:\frac{1}{2}R(p_x)+K(x)+\frac{1}{2}R(p_x)} = e^{-t:\frac{1}{2}R(p_x)}e^{-t:K(x)}e^{t:\frac{1}{2}R(p_x)}$$

This is known as the *drift-kick-drift* or leapfrog integrator..

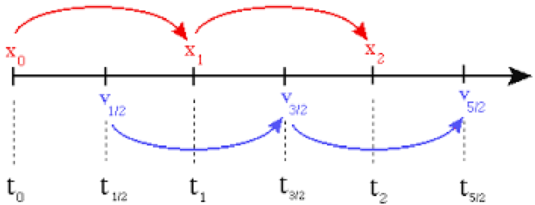
Leapfrog integration

The leapfrog (or *velocity Verlet*) scheme is a second order symplectic integrator. In simplified terms (for step size h) for the *kick-drift-kick* form,

$$p_{n+1/2} = p_n + \frac{h}{2}\frac{\partial V}{\partial q}(q_n)$$

$$x_{n+1} = x_n + hp_{n+1/2},$$

$$p_n = p_{n+1/2} + \frac{h}{2}\frac{\partial V}{\partial q}(q_{n+1})$$



Fourth order integrator

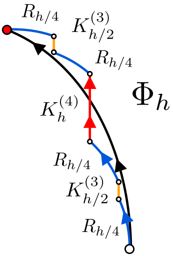


Figure: One step of the fourth order Yoshida integrator. [S. Baturin]

Yoshida found that a set of integrators at order $2n$ can be found building on the second order integrator S_2. This exploits the time reversal symmetry of the system.

$$S_4 = S_2(\gamma t) \circ S_2(\kappa t) \circ S_2(\gamma t) \tag{50}$$

where $\gamma = 1/(2 - 2^{1/3})$, $\kappa = 1/[(2^{1/3})(2 - 2^{1/3})]$.

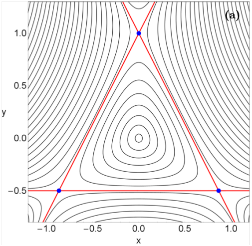
Linear Integrable systems

- The ideal linear Hamiltonian

$$H = Q_x J_x + Q_y J_y \tag{51}$$

  has two invariants of motion, the transverse actions J_x, J_y. This ensures the system is integrable.
- However, the addition of nonlinearities may compromise this integrability and lead to a reduction in the dynamic aperture.
- Nonlinear magnets may be added intentionally, for example sextupole magnets to correct chromaticity, or arise from magnet imperfections or other sources.

A non-integrable Hamiltonian - the Henon-Heiles system



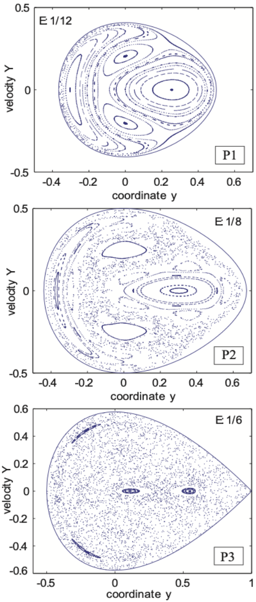
The Hénon-Heiles potential can be written

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3 \tag{52}$$

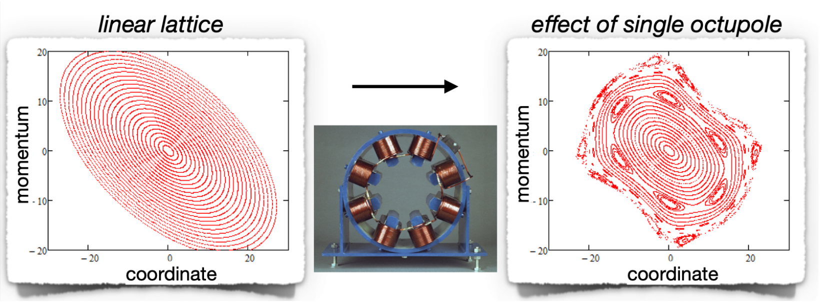
with Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3 = E \tag{53}$$

The Hamiltonian is integrable only for limited number of initial conditions.



Poincare section in the Henon-Heiles cases for increasing values of E. The motion is increasingly chaotic as E approaches the escape value $E = 1/6$.


linear lattice
momentum
coordinate
20
10
0
- 10
- 20
- 20
0
20
effect of single octupole
momentum
coordinate
20
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- 10
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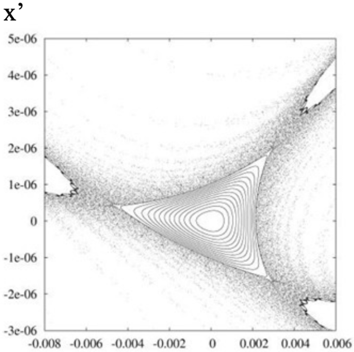
20

Dynamic Aperture



- The dynamic aperture is largest amplitude in phase space inside of which the motion is regular and bounded in the time range of interest.
- Outside the dynamic aperture there is chaotic motion (but there may also be regular motion - islands of stability).

Chaotic motion

One can test whether the motion is chaotic by calculating the rate of divergence between two initially close points in phase space. For regular motion the distance d between the two tracks grows linearly with the number of turns N

$$d(N) \propto N \tag{54}$$

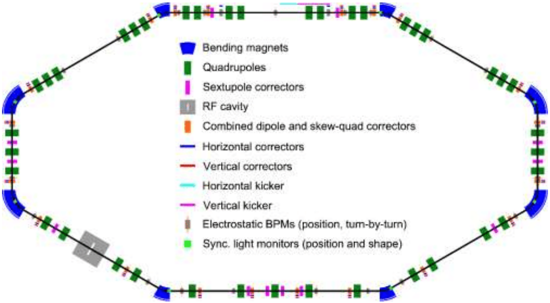
while for chaotic motion the separation increases exponentially

$$d(N) \propto e^{\lambda N} \tag{55}$$

where λ is the Lyapunov exponent formally defined as

$$\lambda = \lim_{N \to \infty} \lim_{d(0) \to 0} \frac{1}{N} d(N) d(0) \tag{56}$$

Nonlinear Integrable systems



- It has been proposed to build an accelerator based on a nonlinear integrable Hamiltonian.
- As well as reducing chaos in single particle motion, the strong tune spread in such a machine may help stem collective instabilities via *Landau damping*.
- As before, the Hamiltonian needs to possess two integrals of motion. A solution was found by Danilov and Nagaitsev (2010).

Start with the Hamiltonian

$$H = \frac{p_x^2}{2} + \frac{p_z^2}{2} + k(s)\left(\frac{x^2}{2} + \frac{z^2}{2}\right) + V(x, z, s) \tag{57}$$

Choose s-dependence of nonlinear potential V so that the Hamiltonian is time-independent in normlised variables $(x_N, p_{xN}, z_N, p_{zN})$.

$$\begin{aligned} H_N &= \frac{p_{xN}^2 + p_{zN}^2}{2} + \frac{x_N^2 + z_N^2}{2} + \beta(\psi) V(x_N\sqrt{\beta(\psi)}, z_N\sqrt{\beta(\psi)}), s(\psi)) \\ &= \frac{p_{xN}^2 + p_{zN}^2}{2} + \frac{x_N^2 + z_N^2}{2} + U(x_N, z_N, \psi) \end{aligned}$$

H_N is an integral of motion for any choice of V(x,z,s) so long as it scales with β appropriately.

Octupole case

If we use an octupole for the nonlinear element then the potential should be scaled by $1/\beta^3$.

$$V(x,z,s) = \frac{\alpha}{\beta(s)^3}\left(\frac{x^4}{4} + \frac{z^4}{4} - \frac{3x^3y^3}{2}\right) \tag{58}$$

where α sets the octupole strength. Then the normalised Hamiltonian becomes

$$H_N = \frac{p_{xN}^2 + p_{zN}^2}{2} + \frac{x_N^2 + z_N^2}{2} + \alpha\left(\frac{x_N^4}{4} + \frac{z_N^4}{4} - \frac{3x_N^3y_N^3}{2}\right) \tag{59}$$

In this case H_N is the only integral of motion. This solution is known as quasi-integrable.

Special potential

A nonlinear potential that results in a second integral of motion arises from the Bertrand-Darboux partial differential equation[1].

$$xz(U_{xx} - U_{zz}) + (z^2 - x^2 + c^2)U_{xz} + 3zU_x - 3xU_z = 0 \tag{60}$$

The equation has general solution

$$U(x,z) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2} \tag{61}$$

where f and g are arbitrary functions of the elliptic coordinates

$$\xi = \frac{\sqrt{(x+c)^2 + z^2} + \sqrt{(x-c)^2 + z^2}}{2c}$$

$$\eta = \frac{\sqrt{(x+c)^2 + z^2} - \sqrt{(x-c)^2 + z^2}}{2c}$$

[1] The coordinates are normalised but the N is omitted

As before, the normalised Hamiltonian is one invariant

$$H = \frac{p_x^2 + p_z^2}{2} + \frac{x^2 + z^2}{2} + \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2} \tag{62}$$

but there is now a second invariant

$$I(x, z, p_x, p_z) = (xp_z - zp_x)^2 + c^2 p_x^2 + 2c^2 \frac{f(\xi)\eta^2 + g(\eta)\xi^2}{\xi^2 - \eta^2} \tag{63}$$

See *V. Danilov and S. Nagaitsev, PRST-AB 13 084002 (2010)* for details.

IOTA

The concept is currently being investigated at the Integrable Optics Test Accelerator (IOTA), Fermilab.