Exotic tetraquarks with some heavy quarks in lattice QCD : T_{bc} case study

LHCb meets theory Tbc workshop, CERN

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Abstract

We follow our recent review to be published in Physics Reports arXiv:2212.07793 [16], on tetraquarks and pentaquarks, on the different direct and indirect approaches that lattice QCD has been employing. We now focus in the tetraquarks with heavy quarks, and in particular in the T_{bc}.

Introduction: many exotics with some heavy quarks

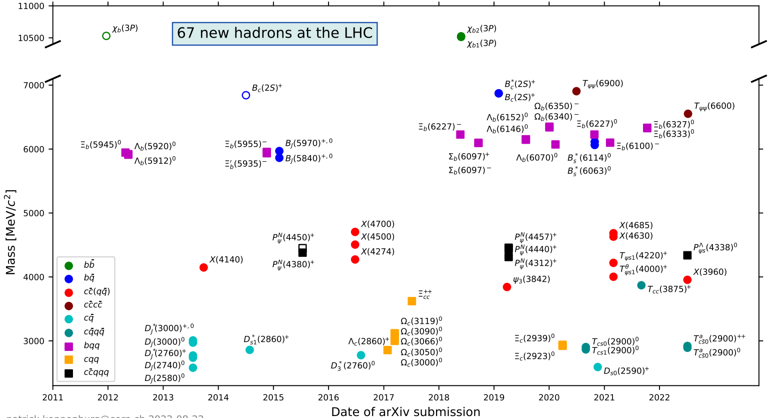

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Figure: New hadrons discovered at the LHC only, most of them at LHCb, including baryons, tetraquarks and pentaquarks, plotted as mass versus preprint submission date [56, 75].

- Using the bag model, Jaffe proposed tetraquarks in 1977 [69, 70].
- Richard and colleagues proposed heavy-light tetraquarks in 1981 [3, 12, 116, 57].
- Since 2008, forty different experiments observed multiquarks.

state	qnumber	δmass	width	decay mode	significance	experiment / lattice
$Tcc(3874)$	$ud\bar{c}\bar{c}$?	-360 ± 44 KeV	48 ± 16 KeV	virtual $D^0 D^{*+}$	$15.5 \pm 6.5\sigma$	experimental LHCb [1, 2]
	$ud\bar{c}\bar{c}$ 1^+	-23 ± 11 MeV	0	-	-	dynamical lattice QCD [71]
		vBS ~ -9 MeV	0	-	-	scattering lattice QCD [95]
$Tccs$	$us\bar{c}\bar{c}$ 1^+	-8 ± 8 MeV	0	-	-	dynamical lattice QCD [71]
$X_0,\ X_1,\ Tcs$	$ud\bar{c}\bar{s}$ $1^+, 0^+$	~ 0	-	-	-	heavy quark lattice QCD [63]
Tbs	$ud\bar{b}\bar{s}$ $1^+, 0^+$	~ 0	-	-	-	heavy quark lattice QCD [63]
Tbc	$ud\bar{b}\bar{c}$ $1^+, 0^+$	~ 0	-	-	-	heavy quark lattice QCD [71]
		$-38 \pm 23 MeV$	0	-	-	heavy quark lattice QCD [55]
		~ 0	-	-	-	heavy quark lattice QCD [63]
		$\sim -40 \pm 50$ MeV	0	-	-	heavy quark lattice QCD [111, 98, 87]
$Tbcs$	$us\bar{b}\bar{c}$ $1^+, 0^+$	~ 0	-	-	-	heavy quark lattice QCD [63]
Tbb	$ud\bar{b}\bar{b}$ 1^+	-90 ± 43 MeV	0	-	-	static lattice QCD [26, 22, 23, 18, 24]
		-59 ± 38 MeV	0	-	-	2×2 static lattice QCD [25]
		-189 ± 13 MeV	0	-	-	heavy quark lattice QCD [54]
		~ -113 MeV	0	-	-	heavy quark lattice QCD [55, 63]
		-143 ± 34 MeV	0	-	-	heavy quark lattice QCD [71]
		-128 ± 34 MeV	0	-	-	heavy quark lattice QCD [79]
		~ -120 MeV	0	-	-	heavy quark lattice QCD [46]
		-112.0 ± 13.2 MeV	0	-	-	heavy quark lattice QCD [65, 64]
		-154.8 ± 37.2 MeV	0	-	-	scattering lattice QCD [7]
		-83.0 ± 30.2 MeV	0	-	-	scattering lattice QCD [7]
		-103 ± 8 MeV	0	-	-	scattering lattice QCD [114, 98]
	$ud\bar{b}\bar{b}$ 0^+	-50.0 ± 5.1 MeV	0	-	-	static lattice QCD [33]
		-5 ± 18 MeV	0	-	-	heavy quark lattice QCD [71]
$Tbbs$	$us\bar{b}\bar{b}, ds\bar{b}\bar{b}$ 1^+	-98 ± 10 MeV	0	-	-	heavy quark lattice QCD [54]
		~ -36 MeV	0	-	-	heavy quark lattice QCD [55, 63]
		-87 ± 32 MeV	0	-	-	heavy quark lattice QCD [71]
		~ -80 MeV	0	-	-	heavy quark lattice QCD [99]
		-46.4 ± 12.3 MeV	0	-	-	heavy quark lattice QCD [65, 64]
		-86 ± 32 MeV	0	-	-	scattering lattice QCD [111, 98, 87]
$Tbbc$	$uc\bar{b}\bar{b}$ 1^+	-6 ± 11 MeV	0	-	-	heavy quark lattice QCD [71]
		~ 0	-	-	-	heavy quark lattice QCD [63]
$Tbbcs$	$sc\bar{b}\bar{b}$ 1^+	-8 ± 3 MeV	0	-	-	heavy quark lattice QCD [71]
		~ 0	-	-	-	heavy quark lattice QCD [63]
$Tbbbb = X$	$bb\bar{b}\bar{b}$?	~ 0	-	-	-	heavy quark lattice QCD [66]

Table: The tetraquark boundstates, or very narrow resonances [16], showing the experimental observations (only the T_{cc}) and the lattice QCD predictions as well and the heavy quark approach.

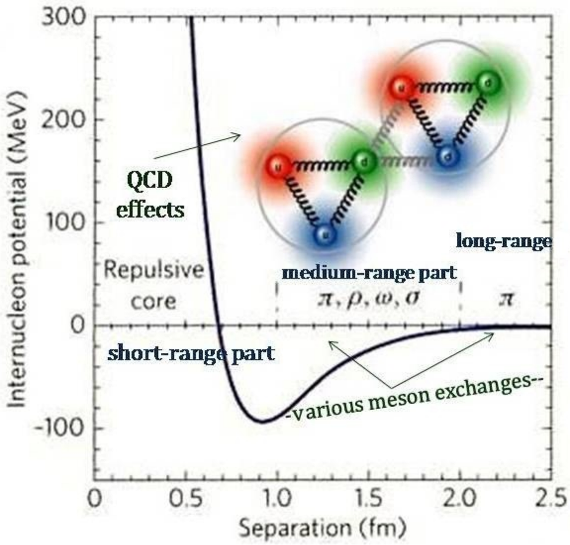


Figure: Three mechanisms for tetraquarks resonances to exist-I : molecular systems, nucleon-like

- The *molecular* tetraquarks, with an interaction similar to the nuclear physics N-N potentials are expected to be close to a threshold. They may be exotic.

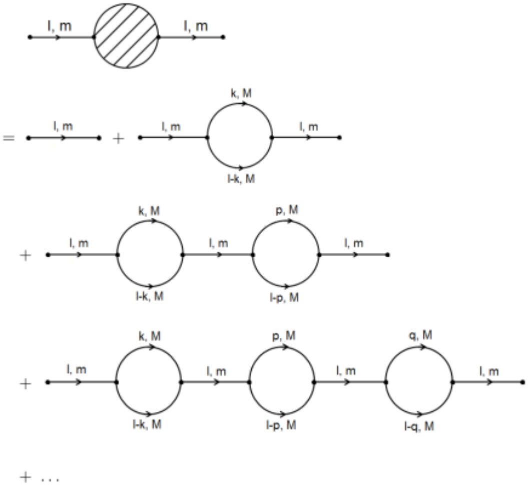


Figure: Three mechanisms for tetraquarks resonances to exist-II: s-pole, sigma meson-like.

- The non-perturbative sigma-like Mandelstam *s pole* tetraquarks, occur as non-perturbative scattering poles of the s-matrix, similar to the σ meson.

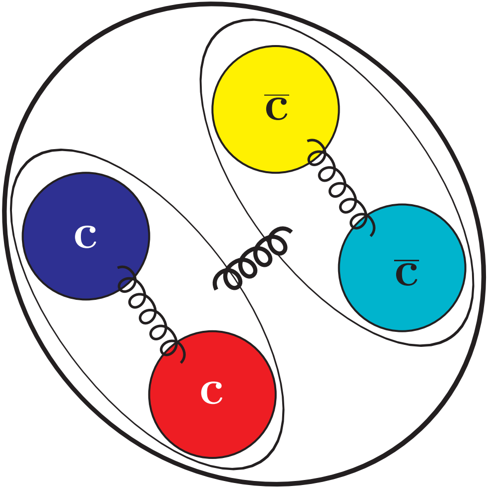
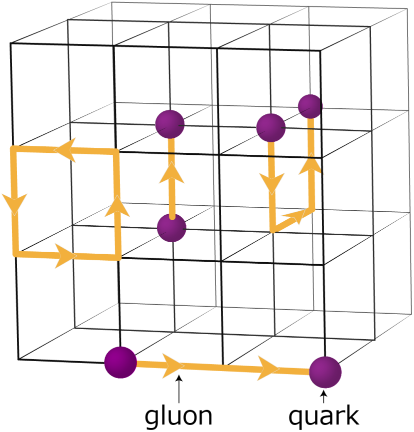


Figure: Three mechanisms for tetraquarks resonances to exist-III : new diquark-antidiquark systems.

- The *diquark* tetraquarks may form exotic multiplets, they are not linked to thresholds, this is a novel mechanism never found previously in hadrons.



Lattice QCD is a discretization of QCD, requiring,

- ensembles of many configurations,
- a large volume V,
- a small lattice spacing *a*
- and a physical pion mass m_π.

It uses the Euclidean time evolution of correlations to compute the energies of observables.

For instance, denoting $|O_1(t)\rangle$ the physical quantum state corresponding to the operator O_1 at time (t), it can be decomposed in eigenvectors $|v_i\rangle$ of the hamiltonian, and the matrix element between two operators at different times is

$$\langle O_2(t)|O_1(0)\rangle = \sum_i c_{2\,i}^{*}\, c_{1\,i}\, e^{-\omega_i t} , \tag{1}$$

clearly for a time long enough the groundstate dominates this matrix element.

Static potentials and colour field densities

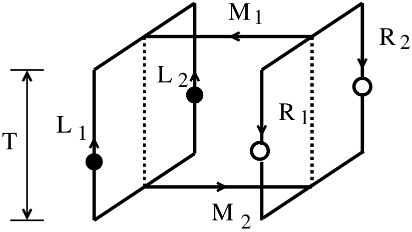


Figure: The tetraquark (4Q) Wilson loop for the calculation of the 4Q potential $V_{4\mathrm{Q}}$ [91].

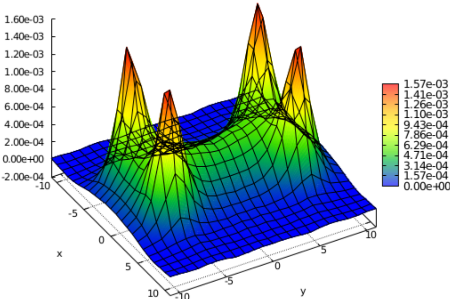


Figure: Lagrangian density 3D plot for a tetraquark [39], showing a diquark-antidiquark flux tube.

- The first lattice QCD computations for tetraquarks used static quarks [5, 92, 91].
- The potential $V_{4\mathrm{Q}}$ is fitted by a Coulomb plus a four-body double-Y potentials,

$$V_{4\mathrm{Q}} = -\alpha_{4\mathrm{Q}}\{(\frac{1}{r_{12}} + \frac{1}{r_{34}}) + \frac{1}{2}(\frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} + \frac{1}{r_{24}})\} + \sigma_{4\mathrm{Q}} L_{\mathrm{min}} + C_{4\mathrm{Q}} \quad (2)$$

- The colour Lagrangian square field densities [39, 40] show two Steiner junctions.

String flip-flop potentials with static quarks

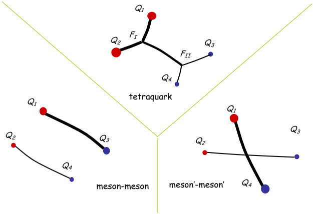


Figure: Triple string flip-flop potential [14].

- We can have three different, non-orthogonal, groundstate colour singlets
  - two different 1 1 and one $\bar{3}\,3$,
  - the orthogonal colour singlet to $\bar{3}\,3$ is $6\,\bar{6}$
  - the orthogonal colour singlet to 1 1 is 8 8.
- The string flip-flop was observed in the tetraquark potentials [91] and in the colour field densities [37, 15].

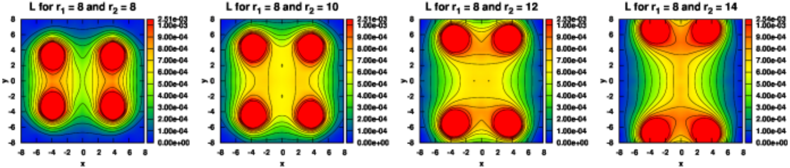


Figure: Flip-flop of the flux tube in the colour field densities for the $QQ\bar{Q}\bar{Q}$ system [37].

- However, the spin-dependence of the flip-flop potential remains to be computed.

Potentials with static heavy quarks for the T_{QQ} family

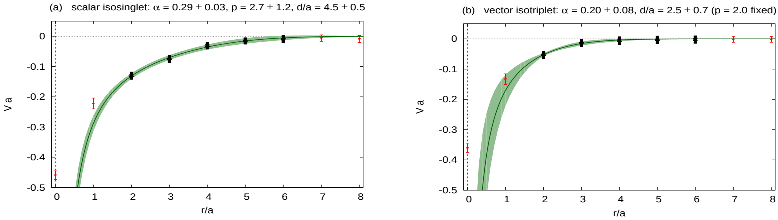


Figure: The lattice QCD potentials fitted in Ref. [112, 113, 26], left scalar-isoscalar and right vector-isovector.

- The family of T_{bb} tetraquarks has two light quarks and two heavy antiquarks.
- Lattice QCD used static heavy quarks and dynamical light quarks, quenched [89, 88, 51, 50] or fully dynamical [112, 113, 9, 33, 26].
- The operators create two static-light mesons [25]

$$(\mathcal{C}\Gamma)_{AB}\Big(\bar{Q}_C(\mathbf{r}_1)\psi_A^{(1)}(\mathbf{r}_1)\Big)\Big(\bar{Q}_C(\mathbf{r}_2)\psi_B^{(2)}(\mathbf{r}_2)\Big), \tag{3}$$

used the chiral limit [26, 22, 23, 25, 18, 24] and light quarks *u*, *d*, *s* and *c* [22].

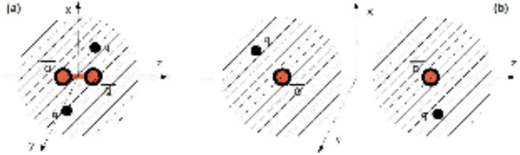


Figure: The picture of perturbative one-gluon exchange at short distances and of meson wavefunction screening at large distances, shown in Ref. [26].

channel	α	d/a	p	χ^2/dof
scalar isosinglet	0.293(33)	4.51(54)	2.74(1.20)	0.35
vector isotriplet	0.201(77)	2.48(69)	2.0 (fixed)	0.06

Table: χ^2 minimizing fit of the ansatz (4) to the lattice static antiquark-antiquark potential; lattice spacing $a \approx 0.079\,\mathrm{fm}$

Our best ansatz for a fit is,

$$V(r) = -\frac{\alpha}{r} \exp\left(-\left(\frac{r}{d}\right)^p\right), \tag{4}$$

We expect:

- at short distances a Coulomb potential between the two heavy antiquarks in a triplet colour state 3, s-wave, spin 1;
- at large distances we expect a screening typical of the static-light wavefunction.

- We provide dynamics (kinetic energy) to the heavy quarks in the Schrödinger equation with the Born-Oppenheimer approximation [29].
- A boundstate with $E_r = -90 \pm 43$ MeV (-60 ± 45 including heavy spins) , and a resonance are predicted . The quantum numbers for the T_{bb} are $I(J^P) = 0(1^+)$.
- Other tetraquarks such as $ls\bar{b}\bar{b}$, $lc\bar{b}\bar{b}$, $sc\bar{b}\bar{b}$, $ll\bar{c}\bar{b}$, $ll\bar{c}\bar{c}$ are not predicted to bind.

Diquark-antidiquark and meson-meson in a bounstate

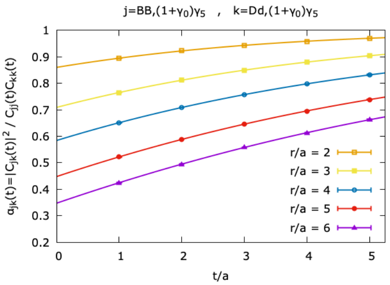


Figure: The squared overlap α_{jk} for for several fixed r distances as a function of t for ensemble B40.24 of Ref. [24]. The normalized trial states are $j = BB, (1+\gamma_0)\gamma_5$, $k = Dd, (1+\gamma_0)\gamma_5$.

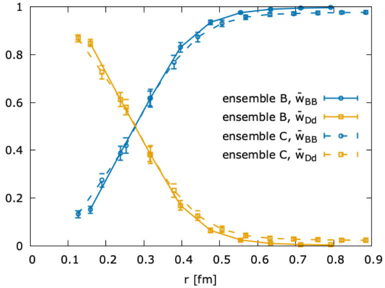


Figure: Comparing the *real* tetraquark *Dd* versus the *molecular BB* in Ref. [24]. Fitted normalized absolute squares of the coefficients $\bar{w}_{BB}$ and $\bar{w}_{Dd}$ as functions of the distance r.

- Ref. [24] used two $\bar{b}$ quarks frozen distance r for the T_{bb}, a tetraquark $\bar{b}\bar{b}ud$,
- and compared two frequently discussed competing structures
  - the *real* tetraquark *Dd*,
  - versus the *molecular BB*.
- Solving a generalized eigenvalue problem (GEVP) [27], they estimated the meson to diquark-antidiquark ratio of this tetraquark is around 60% to 40%.

String breaking potentials for quarkonium, crypto-exotics

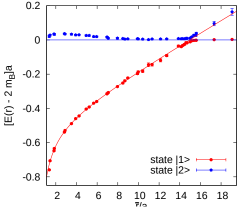


Figure: String breaking with the potential eigenvalues [11].

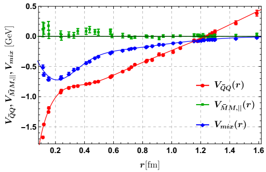


Figure: The matrix elements of the $\bar{Q}Q$ and $\bar{M}M$ potential [17, 21].

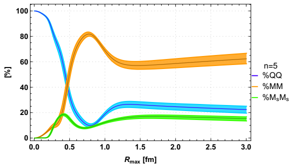


Figure: The composition of the bottomonium state $\Upsilon(10753)$ [21].

- String breaking needs both two quark operators and four quark operators.
- The coupling potentials, Figs. 13, 15, between two static quark-antiquark and two static-light mesons have been computed with light u, d [11] or u, d, s [34] quarks.

$$V_0(r) = \begin{pmatrix} V_{\bar{Q}Q}(r) & V_{\text{mix}}(r) \\ V_{\text{mix}}(r) & V_{\bar{M}M,||}(r) \end{pmatrix} \tag{5}$$

- The poles of the bottomonium resonances [17, 21, 20] are found with a Born-Oppenheimer approach and scattering theory [38], see Fig. 14.
- Two non-perturbative dynamical extra states are found, in the s-wave and d-wave spectra [20, 19], both compatible with the recent $\Upsilon(10753)$ observed at BELLE.

Potentials with static heavy quarks $Q\bar{Q}$ for Z_b family

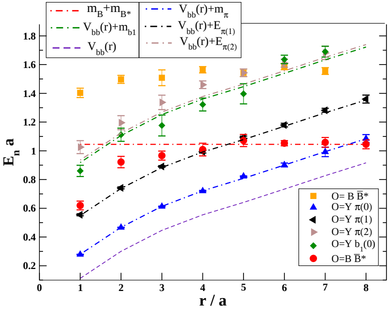


Figure: Eigen-energies of $\bar{b}b\bar{d}u$ system, computed in Ref. [102], for various static quarks b and $\bar{b}$ separations. The eigenstate dominated by $B\bar{B}^*$ (red circles) has energy well below $m_B + m_{B*}$.

- With the Z_b flavour $u\bar{d}b\bar{b}$, we have two meson-meson channels, $u\bar{b} = B^{+(*)}$ and $\bar{d}b = \bar{B}^{0(*)}$ or $u\bar{d}$ and $b\bar{b}$, say π^+ and a quarkonium meson such as an Υ.
- A boundstate is suggested in the $B^{+(*)} - \bar{B}^{0(*)}$ channel [96, 97, 103, 102, 106, 4].
- A difficulty resides in identifying these non-orthogonal, open coupled channels.

HAL QCD non-static potentials

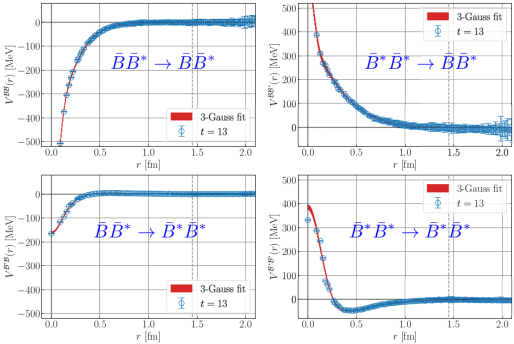


Figure: HAL QCD Coupled channel potentials [6] using dynamical light quarks and non-relativistic heavy quarks.

The HAL QCD method computes potentials with dynamical quarks [67, 8, 68], extracting them from the Schrödinger wavefunction or the Nambu-Bethe-Salpeter amplitude,

$$V(\boldsymbol{r}) = \frac{\Delta\phi(\boldsymbol{r})}{2m\phi(\boldsymbol{r})} + E\,. \qquad (6)$$

using operators of two hadrons, composed of quarks.

- Ref. [6] studied the T_{bb} ,see Fig. 17, with a 2×2 time dependent coupled channel of a BB^* pair and a B^*B^* pair .
- There is evidence for OGEP attraction in the BB^*, as with static quarks [25].
- Moreover, the HAL QCD shows a new evidence for OPEP attraction in the B^*B^*.

Search for tetraquark resonances high in the spectrum

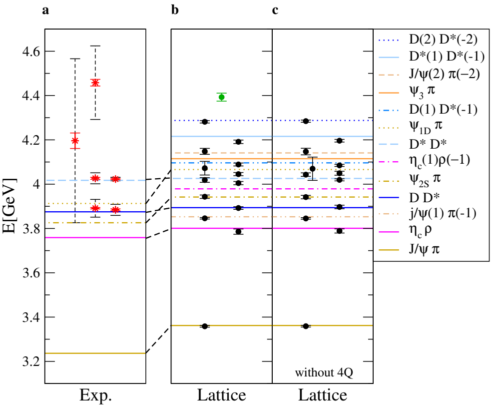


Figure: The Z_c spectrum for quantum numbers $I^G(J^{PC}) = 1^+(1^{+-})$ from Ref. [104].

(a) Position of the experimental Z_c^+ candidates [30].

(b) Lattice simulation energies based on complete 22 × 22 matrix of interpolators.

(c) Lattice simulation energies based on the 18 × 18 correlator matrix without diquark-antidiquark interpolating fields $\mathcal{O}^{4q}_{1-4}$.

- Z_c was the first exotic tetraquark discovered, confirmed in several experiments.
- The first lattice QCD calculations [105, 104, 59, 44] used the technique of comparing the spectrum just with several meson-meson operators,

$$\mathcal{O}_1^{\psi(0)\pi(0)} = \bar{c}\gamma_i c(0)\, \bar{d}\gamma_5 u(0) \tag{7}$$

and the spectrum after adding diquark-antidiquark operators.

$$\mathcal{O}_1^{4q} \propto \epsilon_{abc}\epsilon_{ab'c'}(\bar{c}_b C\gamma_5 \bar{d}_c\; c_{b'}\gamma_i C u_{c'} - \bar{c}_b C\gamma_i \bar{d}_c\; c_{b'}\gamma_5 C u_{c'})\ .$$

- This technique succeeded in identifying the $X(3872)$ state in the spectrum [94], but no evidence for the Z_c was found in the spectrum, see for instance Fig. 18.

Boundstate search with non-relativistic bottom quarks

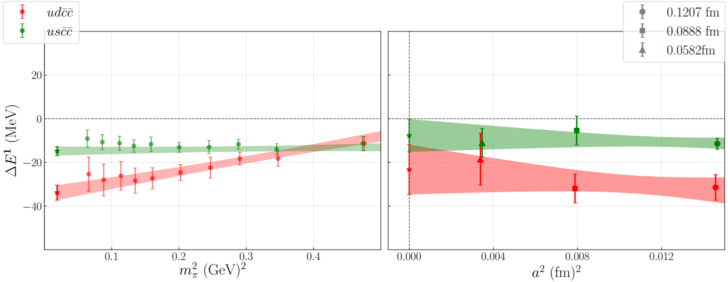


Figure: Results of Ref. [71] for the T_{cc} and T_{ccs}, $ud\bar{c}\bar{c}$ and $us\bar{c}\bar{c}$ doubly charm tetraquark states. Left panel: chiral extrapolation for several pion masses at $a = 0.1207$ fm for each of the states. Right panel: Continuum extrapolation at the chiral extrapolation to the physical pion mass.

- The T_{bb} fully exotic tetraquark has been studied [55, 79, 71], using the NRQCD lattice action [109, 78, 86, 10, 32, 47, 80, 81] for the bottom quark propagators.
- All the lattice QCD computations agree on bound T_{bb} and T_{bbs} tetraquarks.
- There is still some tension in T_{cc}, T_{ccs} and T_{bbc} only seen by Ref. [71], see Fig. 19 where Ref. [71] checks the physical m_π and the small a limit.
- There is a null evidence for a full bottom $bb\bar{b}\bar{b}$ tetraquark boundstate in Ref. [66].

Scattering study with the Lüscher method for phase shifts

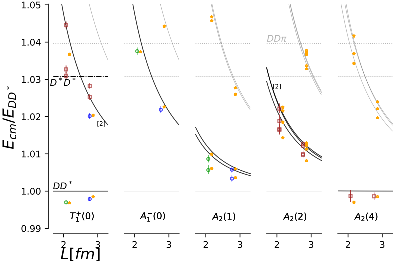


Figure: The center-of-momentum energy $E_{cm} = (E^2 - \vec{P}^2)^{1/2}$ of the $cc\bar{u}\bar{d}$ system [95] in various lattices. The free DD^* energies are shown by lines.

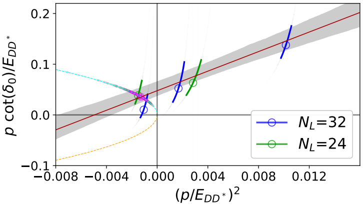


Figure: $p\cot\delta_{l=0}^{(J=1)}$ for DD^* scattering (red line) and $ip = +|p|$ (cyan line) versus p^2. The intersection shows a virtual bound state [95].

- The Lüscher technique extracts phase shifts from the momenta and energy of open channels [84, 85, 84, 83, 82, 31, 93].
- In a single channel with two particles, the effective range approximation [45, 90],

$$k^{2l+1}\cot\delta_l(k) = \frac{-1}{a_l} + \frac{r_l}{2}k^2 + o(k^4) \tag{8}$$

  with scattering length a_l and effective range r_l, a pole implies $\cot\delta_l + i = 0$.
- The T_{bb} boundstate was obtained [6, 111] in the scattering of B - B^* mesons.
- The T_{cc} was studied in the channel DD^* [95]. However only a virtual boundstate pole was found, with $\mathrm{Im}(k) < 0$, for a boundstate we should have $\mathrm{Im}(k) > 0$.

T_{bc} case study with potentials

- In our most detailed potential study arXiv:1505.00613 [22] we only studied systems with two equal heavy quarks. Now we are staring to study other possible flavours such as T_{bc}, T_{bbs} and the T_{cc} observed in LHCb.
- The potentials from lattice QCD can be fitted with 2 parameters, Coulomb potential strength α and screening length d. μ is the reduced mass. Approximately,

  $$\text{binding if : } \mu\alpha d > 0.6 \tag{9}$$

  With T_{bb}, considering mb $\sim$ 5 GeV we get $\mu_{bb}\alpha d \sim 1.9$ and binding: the binding energy is $\delta\text{mass} = -90 \pm 43$ MeV.
- In the quark model, we have $m_c \sim m_b/3$. This implies
  $\mu_{bc} \sim \mu_{bb}/2 \sim 0.95$: T_{bc} should be bound,
  $\mu_{cc} \sim \mu_{bb}/3 \sim 0.95$: 0.63, may be bound and actually experimentally is.
- The non-relativistic approximation can be controlled by the binding energy and we get approximately $< p^2/\mu^2 >\sim \delta\text{mass} * \mu/2 \sim 0,036$, which is fine! This is OK as well for the T_{b_c} and T_{cc}if the binding energy is smaller
- The spin is also a next to leading term in a non-relativistic expansion. We can include the spin splittings with the technique of arXiv:1612.02758 [25].

  Notice the two quarks bb in the groundstate s-wave and colour triplet 3 must have spin 1, thus T_{bb}has $J^P = 1^+$, it couples to BB^* and B^*B^* Channels. But the two quarks bc may have either spin 0 or spin 1.
  $T_{bc}1^+ = \frac{1}{2}B^*D - \frac{1}{2}BD^* + \frac{1}{\sqrt{2}}B^* \times D^*$
  $T_{bc}0^+ = \frac{1}{2}BD + \frac{\sqrt{3}}{2}B^* \cdot D^*$
  thus $T_{bc}0^+$ should have a smaller mass, but the BD threshould should even be lower. We will know which binds more when we compute it.

Conclusion and outlook for tetraquarks with lattice QCD

Lattice best predictions are the boundstates of the T family, such as T_{bb} and T_{bbs}. T_{bb} is a benchmark for the different lattice QCD studies and challenges future experiments.

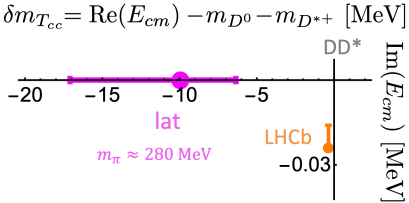


Figure: The pole in the scattering amplitude related to T_{cc} in the complex energy plane: the lattice result [95] (magenta) and the LHCb result (orange).

The T_{cc}, T_{bc} and are excellent case studies for future lattice QCD efforts.

- The extension of the Lüscher technique to several channels and to three particle resonances is under development, [60, 61, 62], this improves the study of the T_{cc}.
- The Master field approach, uses a very large lattice is used. The One Pion Exchange Potential should increase, and this is important for deuteron-like molecules. T_{cc} is expected to have both a Dd-like attraction and a π exchange.

However, for the Z_b, Z_c . . . family, high in the spectrum, with many coupled decay channels, new technical advances are still necessary to study them with lattice QCD.

We expect Tetraquarks, Pentaquarks and Hexaquarks will become a priority for the lattice QCD community, with increased precision, new techniques, and computations to directly address theoretical questions on the properties of tetraquarks, as in Fig. 23.

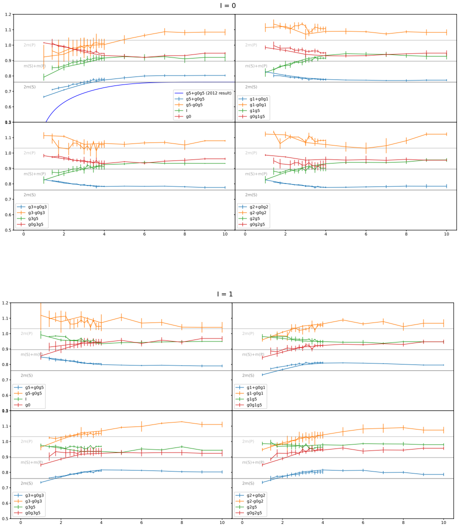


Figure: Very preliminary results for B-B potentials by PB (U. Lisboa), L Müller and M. Wagner (U. Frankfurt) and M Marinkovic (ETH Zürich) obtained with the open source package for lattice QCD **open Q*D** developed by Prof. Marina Krstic Marinkovic (ETH Zürich) et al [36].

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