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Tackling the four-loop QCD pressure at high temperature

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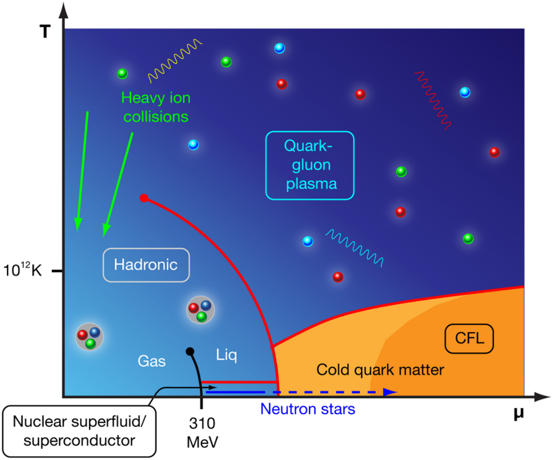
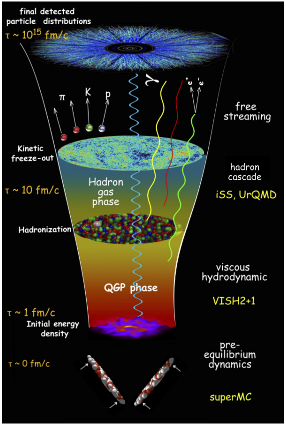
QCD Phase diagram



Motivation



Ongoing experiments:

- LHC: ALICE, ATLAS, CMS
- RHIC: Phenix, STAR

HIC:

- Fireball time $\sim$ 10 fm/c
- $T_c \sim 170$ MeV
- Jet quenching
- Plasma hydrodynamics

Pressure $\longrightarrow$ Hydrodynamics

(+ other things...)

Equilibrium thermodynamics

Thermodynamics derived from the (grand) partition function

$$\mathcal{Z}(\mu, T) = \mathrm{Tr}\exp[-\beta(\hat{H} - \mu\hat{Q})].$$

Imaginary time formalism ($\tau = it$):

$$\mathcal{Z}(T) = \int_{\mathrm{b.c}} \mathcal{D}[\mathrm{fields}] \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x\, L_E\right\}.$$

- τ: compact dimension of length $\beta\hbar$
- Bosons: periodic $\Longrightarrow$ prop. $[(2\pi n T)^2 + \vec{p}^2 + m^2]^{-1}$
- Fermions: anti-periodic $\Longrightarrow$ $\{i\gamma_0[\pi T(2n+1) + i\mu] + i\gamma_k p_k\}^{-1}$

Equilibrium thermodynamics

- Rest frame of heat bath $\Longrightarrow$ breaking of Lorentz invariance

$$i\int d^4p \longrightarrow T\sum_{n=-\infty}^{\infty}\int d^3\vec{p}, \quad i\int d^4x \longrightarrow \int_0^{\beta} d\tau \int d^3\vec{x}.$$

- Matsubara formalism:

$$\phi^{b/\{f\}}(X) = \sum\!\!\!\!\!\!\!\int_{P/\{P\}} \tilde{\phi}^{b/\{f\}}(P)e^{iP\cdot X}, \quad \sum\!\!\!\!\!\!\!\int_{P/\{P\}} \equiv T\sum_{\text{freq.}}\int \frac{d^d\vec{p}}{(2\pi)^d}.$$

Loop integrals $\longrightarrow$ Sum-integrals!

Energy scales in hot QCD

- Asympotic freedom at high temperature
- Loop expansion parameters:

$$\epsilon_f \sim g^2, \quad \epsilon_b \sim \frac{g^2 T}{m}.$$

- Fermions are completely perturbative
- Energy of typical particle in the plasma $\Longrightarrow$ hard scale T
- Screening of chromoelectric fields $\Longrightarrow$ soft scale $m_E \sim gT$
- Non-perturbative magnetic scale $\Longrightarrow$ ultrasoft scale $m_G \sim g^2 T$
- Hierarchy of scales $T \gg gT \gg g^2 T \Longrightarrow$ effective field theory

QCD pressure

Non-trivial weak-coupling expansion:

$$p_{\text{QCD}}(T) = \lim_{V\to\infty} \frac{T}{V} \log \int_{\text{b.c}} \mathcal{D}[A^a_\mu, \bar{\psi}, \psi, \bar{c}^a, c^a] \exp\left(- \int_0^\beta d\tau \int d^{3-2\varepsilon}x \, L_{\text{QCD}} \right),$$
$$= p_0 + p_2 g^2 + p_3 g^3 + (p_4 + p_4' \log g) g^4 + p_5 g^5 + (p_6 + p_6' \log g) g^6,$$
$$+ \mathcal{O}(g^7).$$

Non-analyticity in g^2 due to dynamically generated scales.

- Non-perturbative effects enter at $\mathcal{O}(g^6)$
- State of the art: p_6 [PN/Schröder]
- Generalizations: $\mu \neq 0$ [Vuorinen]; SM [Gynther/Vepsäläinen]

Dimensionally-reduced EFT

Integrate out massive modes and define Electrostatic QCD:

$$S_{\text{EQCD}} = \frac{1}{T}\int d^3x \left\{ \frac{1}{4}\bar{F}^a_{ij}\bar{F}^a_{ij} + \frac{1}{2}(\mathcal{D}^{ab}_i \bar{A}^b_0)^2 + m_E^2 \text{Tr}[\bar{A}_0^2] + \lambda_E^{(1)}(\text{Tr}[\bar{A}_0^2])^2 \right.$$
$$\left. + \lambda_E^{(2)} Tr[\bar{A}_0^4] + \text{higher order operators} \right\}.$$

Integrate out $\bar{A}_0$ mode and define Magnetostatic QCD (MQCD):

$$S_{\text{MQCD}} = \frac{1}{T}\int d^3x \left\{ \frac{1}{4}\bar{\bar{F}}^a_{ij}\bar{\bar{F}}^a_{ij} + \text{higher order operators} \right\}.$$

- Obtain effective parameters via perturbative matching of correlators, e.g.

m_E^2 : compare pole in static A_0 prop. for QCD and EQCD

The g^6 term of the QCD pressure

- Pressure of hot QCD [Braaten/Nieto '96]

$$p_{\mathrm{QCD}} = p^{(h)ard} + p^{(s)oft} + p^{(u)ltrasoft}.$$

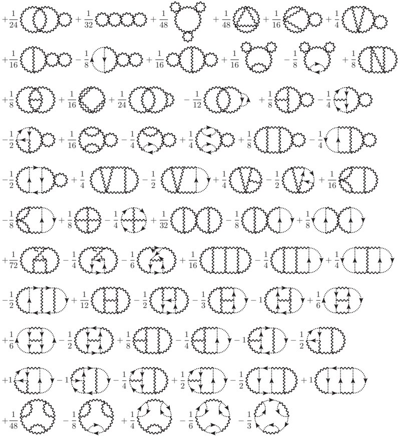
- All scales entering at $\mathcal{O}(g^6) \Longrightarrow$ physical leading order

$$p_{\mathrm{QCD}}\Big|_{\mathcal{O}(g^6)} = p_6^{(u)} + p_6^{(s)} + p_6^{(h)},$$

- $p_6^{(u)}$: lattice 3d YM + 4-loop lattice [Renzo/Schröder '04-'06]
- $p_6^{(s)}$: Weak-coupling expansion in EQCD [Kajantie *et. al* '03]

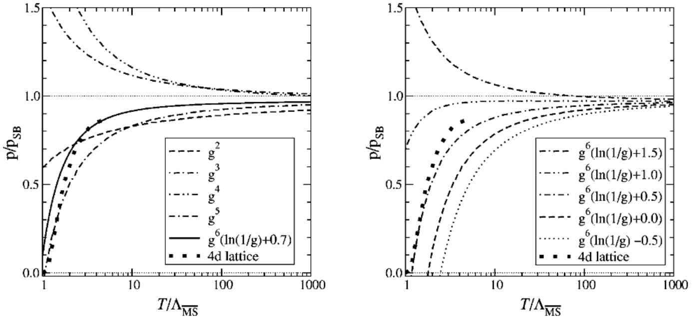
$$p_6^{(h)} = \sum (\text{connected 4-loop vacuum diagrams in 4d thermal QCD})\,.$$

Yang-Mills sector



- 65 Feynman diagrams
- Hardest: $2^9\,6^6 \approx 24M$ terms
- Sum-ints to solve: 176119
- Poly. in gauge parameter ξ^6
- Reduction algorithm $\to$ 21
- Explicit gauge invariance in d dimensions [PN/Schröder]
- Finally: solve 21 sum-ints
- Only 9 unknown sum-ints
- Improvements?
- Also works for the fermion sector ($\mu \neq 0$)! [PN/Paatelainen/Vuorinen]

Pressure up to order $g^6 \log g$



Outlook

- Quark masses? [Laine/Schröder '06]
- A lot of room for improvement on sum-integral technology
- Improvements: Integration-by-parts identities
- Direct comparison with lattice results [Weber *et. al* '21]
- $\mathcal{N} = 4$ Super Yang-Mills [Strickland *et. al* '22]

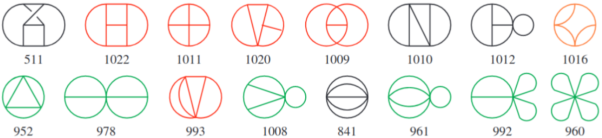
Generalization to finite $\mu \Longrightarrow$ NNNLO cold and dense Quark Matter [see K. Seppänen's talk]:

- 52 Feynman diagrams $\Longrightarrow$ 12 are zero
- Possibility to generalize reduction algorithm
- Systematic procedure for computing integrals $\Longrightarrow$ cutting rules

The End

Status of the four-loop QCD pressure

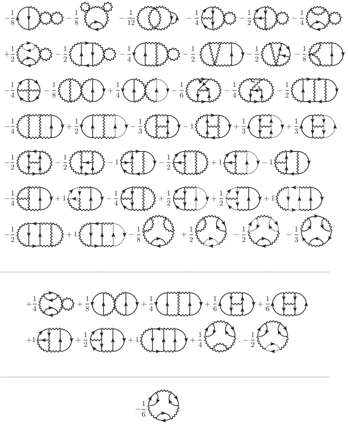
- 2 factorized as (1-loop)^4: all known
- 3 factorized as (1-loop)$^2\times$(2-loop): all known
- 6 factorized as (1-loop)$\times$(3-loop): all known [PN/Schröder '22]
- 10 genuine 4-loops: 9 unknown



$$\sum\!\!\!\!\!\!\!\int_P [\Pi(P)]^3 = \frac{T^4}{(4\pi)^4}\frac{1}{16\epsilon^2}\left[1+\epsilon t_{11}+\epsilon^2 t_{12}+\mathcal{O}(\epsilon^3)\right]$$ [Gynther *et. al* '07],

$$\Pi(P) = \sum\!\!\!\!\!\!\!\int_Q \frac{1}{Q^2(P-Q)^2}.$$ only known 4-loop sum-int !

Backup: Fermionic sector

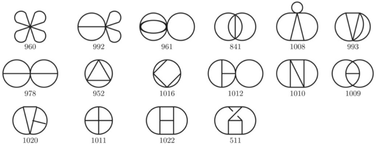


- 52 fermionic Feynman diags:
  - N_f^1: 41; N_f^2: 10; N_f^3: 1
- Poly. in gauge parameter up to ξ^5
- Nr. of sum-ints to solve: 106212
- Apply shifts $\rightarrow$ 22479
- Apply symmetries $\rightarrow$ 1009
- Sum all diags. $\rightarrow$ 134
- Explicit gauge invariance in d dimensions is obtained
- Use IBP $\rightarrow$ 117 $\rightarrow$ Y (in progress)
- Finally: solve Y sum-ints (IBP improvement?)

Backup: Strategy and reduction algorithm

1. Generate all 4-loop connected vacuum diagrams via `qgraf`
2. Assign Feynman rules in covariant R_ξ gauge
3. Solve Dirac, Lorentz and $SU(N)$ algebra via `FORM` algorithm
4. Map sum-integrals down to master sector, using momentum shifts, symmetries and integration-by-parts (IBP)

- Shifts: linear mapping to master sector
- Symmetries: linear mapping within the same master sector



Final step $\longrightarrow$ solve master sum-integrals

Backup: List notation

For L loops we have $L(L+1)/2$ independent scalar products between loop momenta $\{P_i\}_{i=1}^{L}$. We define the 4-loop momenta family

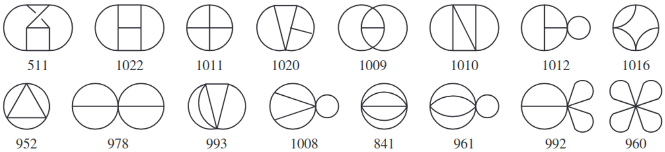
$$\{P_1, P_2, P_3, P_4, P_1 - P_4, P_2 - P_4, P_3 - P_4, P_1 - P_2, P_1 - P_3, P_1 - P_2 - P_3\}.$$

Scalar Feynman integrals can then be expressed as an ordered list of integer numbers, being the exponents of each corresponding propagator, e.g.,

$$\sum\!\!\!\!\!\!\!\int_{P_1} \cdots \sum\!\!\!\!\!\!\!\int_{P_4} \frac{(P_1 - P_2)^2}{[P_1^2]^2 P_2^2 P_3^2 P_4^2 (P_1 - P_2 - P_3)^2} \longleftrightarrow I(2,1,1,1,0,0,0,-1,0,1).$$

- Free to do momentum shifts

Backup: Master sectors



- Sectors are defined assigning a 0 for null and negative exponents, and 1 for positive exponents.
- For each list of 0s and 1s (binary), we can assign a decimal number
- Master sectors are defined by the sectors with the biggest decimal number
- Unique representatives with respect to linear momentum shifts

Backup: Thermal IBP

The idea is exploiting the identity

$$\sum\!\!\!\!\!\!\int_P \partial_{\vec{p}}(\cdots) = 0.$$

- Linear system of eqs. for sum-integrals of interest
- In general, coeffs. are rational functions of the dimension d
- Allows mapping to master sum-integrals
- Computationally expensive method for high orders

A trivial example for 1 loop gives the recursion relation

$$\sum\!\!\!\!\!\!\int_P \frac{(P_0)^{a+2}}{(P^2)^{b+1}} = \left(1 - \frac{d}{2b}\right) \sum\!\!\!\!\!\!\int_P \frac{(P_0)^a}{(P^2)^b}.$$

Backup: Thermal IBP

Non-trivial 2-loop IBP result

The most general massless bosonic 2-loop vacuum sum-integral is

$$L_{s_1 s_2 s_3}^{s_4 s_5 s_6} = \sum\!\!\!\!\!\!\int_{PQ} \frac{(P_0)^{s_4}(Q_0)^{s_5}(P_0 - Q_0)^{s_5}}{(P^2)^{s_1}(Q^2)^{s_2}[(P-Q)^2]^{s_3}}.$$

- Thermal IBP indicates factorization into a product of 1 loop ones [Nishimura/Schröder '12]
- Recently, proof of factorization [Davydychev/PN/Schröder in preparation]
- Trivial to solve in d dimensions in terms of 1-loop sum-integrals, $L = \sum (\text{1-loop}) \times (\text{1-loop})$

Backup: 3-loop Sum-integrals

Main strategy:

1. Disentangling UV/IR div. from 1-loop selfE [Arnold/Zhai '94]
2. Divergences analytically + finite terms numerically
3. Many examples known in the literature

$$
\begin{aligned}
V_1 &= \SumInt_P \SumInt_Q \SumInt_R \frac{1}{P^2(Q^2)^2(Q-P)^2R^2(R-P)^2}\,, \\
&= \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi T^2}\right)^{3\varepsilon}\frac{1}{6\varepsilon^3}\Big\{1+3\varepsilon+\Big(13-3\zeta_3+\frac{9}{2}\zeta_2-6\left(\gamma_E^2+2\gamma_1\right)\Big)\varepsilon^2 \\
&+\Big\{51-42(\gamma_E^2+2\gamma_1)+24\zeta_2\left(\frac{19}{16}+\log 2\pi-12\log G\right)+2\log 2\left(12-12\gamma_E^2-24\gamma_1-\zeta_3\right) \\
&+6\gamma_E\left(3\zeta_3-4-4\gamma_1\right)-36\gamma_2+\frac{25}{2}\zeta_3-16\zeta_3'+6c_1+6c_2+6c_3\Big\}\varepsilon^3+\mathcal{O}(\varepsilon^4)\Big\}, \\
c_2 &= \sum_{n=1}^{\infty}\int_0^{\infty}dx\,\frac{2e^{-x}}{n}\left[e^x\mathrm{Ei}(-x)+\gamma_E+\log\frac{x}{4n^2}\right]\times \\
&\times\left[\psi(n+1)+e^xB(e^{-x/n},n+1,0)+e^x\mathrm{Ei}(-x)-\log(1-e^{-x/n})+\log\frac{x}{n^2}-\frac{x}{12n^2}\right] \\
&\approx -3.2020672566(1).
\end{aligned}
$$

Backup: Status

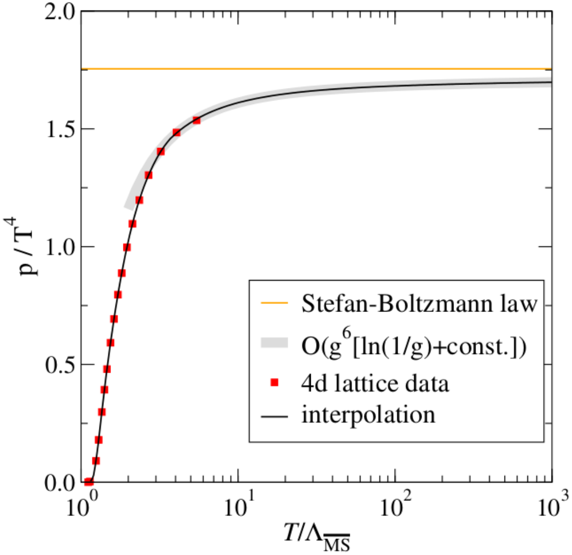
For the contribution coming from topology 1016, need to evaluate the beasts

$$\sumint_P \sumint_Q \sumint_R \sumint_S \frac{[(P-Q-R)^2]^2}{P^2 Q^2 R^2 (S^2)^2 (P-S)^2 (Q-S)^2 (R-S)^2},$$
$$\sumint_P \sumint_Q \sumint_R \sumint_S \frac{[(P-Q-R)^2]^3}{P^2 Q^2 R^2 (S^2)^3 (P-S)^2 (Q-S)^2 (R-S)^2}.$$

After tensor decomposition, need to evaluate [PN/Schröder '22]

$$\sumint_P \frac{1}{(P^2)^2} \Pi \Pi^{\mu\nu} \Pi^{\mu\nu}, \quad \sumint_P \frac{1}{(P^2)^3} \Pi^{\mu\nu} \Pi^{\nu\sigma} \Pi^{\sigma\mu}; \quad \Pi^{\mu\nu} \equiv \sumint_Q \frac{Q^\mu Q^\nu}{Q^2 (P-Q)^2}.$$

Backup: QCD pressure



Backup: Entropy density