Quantum transport theory for neutrinos with flavor and particle-antiparticle mixing

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Theory of neutrino oscillations

- Proposed over 60 years ago
- Multiple approaches: wave packets, S-matrix + Feynman diagrammatic technique, density matrices, mean field equations ...
- How to include
  - neutrino-antineutrino coherence?
  - collision term for coherent neutrino states?
  - heavy neutrinos?
- Conceptual problems: decoherence effects (quantum entanglement), separation of oscillations scales, helicity/spin coherence...

  $\Rightarrow$ Need more general approach and quantum kinetic equations (QKE) to answer
- Practically useful quantum kinetics equations have been a long-term goal

Quantum kinetic equations

- We derive transport equations for mixing neutrinos from fundamental field theory formalism
  - Include all flavor and particle-antiparticle coherences
  - Only adiabatic background fields are assumed
  - Equations valid for heavy and light neutrinos (UR-limit is not assumed)
  - Generalized Feynman rules to compute collision terms which include coherently mixing species
- Set of scalar equations, “easy” to use in numerical applications

Real time Kadanoff-Baym equations in Wigner space

$$\hat{\not{K}} S^p(k,x) - e^{-\frac{i}{2}\partial_x^\Sigma \cdot \partial_k}\left[\Sigma^p_{\rm out}(\hat{K},x) S^p(k,x)\right] = 1, \qquad \text{(pole)}$$

$$\hat{\not{K}} S^s(k,x) - e^{-\frac{i}{2}\partial_x^\Sigma \cdot \partial_k}\left[\Sigma^r_{\rm out}(\hat{K},x) S^s(k,x)\right] = e^{-\frac{i}{2}\partial_x^\Sigma \cdot \partial_k}\left[\Sigma^s_{\rm out}(\hat{K},x) S^a(k,x)\right] \qquad \text{(statistical)}$$

- Exact to a given approximation for the self-energy function
  $\Rightarrow$ information about flavor and all particle-antiparticle coherences
- Need for an approximation scheme which does not lose information about coherence

$$iS^<(u,v) \equiv \langle \bar{\psi}(v)\psi(u)\rangle$$
$$iS^>(u,v) \equiv \langle \psi(u)\bar{\psi}(v)\rangle$$
$$S^r(u,v) \equiv \quad \theta(u_0 - v_0)(S^> + S^<)$$
$$S^a(u,v) \equiv -\theta(v_0 - u_0)(S^> + S^<)$$
$$\hat{\not{K}} = \not{k} + \tfrac{i}{2}\not{\partial}_x \quad p = <,>, \quad s = r,a$$
$$g(k,x) \equiv \int \mathrm{d}^4 r\, e^{ik\cdot r} g(x + \tfrac{1}{2}r, x - \tfrac{1}{2}r)$$
$$\Sigma_{\rm out}(k,x) \equiv e^{\frac{i}{2}\partial_x^\Sigma \cdot \partial_k^\Sigma}\Sigma(k,x)$$

Reduction of the KB equations

- KB equations are non-local, and they couple pole equations and dynamical equations
  $\Rightarrow$ Need to *decouple* and *localize* to get practically useful density matrix equations
- Decoupling: Split the statistical function into a background part (closely related to pole functions) and into a perturbation whose equation then formally decouples
- Localization: reduction of the infinite order gradient expansion
  $\Rightarrow$ Assumption of adiabaticity or enforced by integrating over the momentum variables

General transport equations

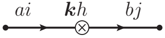
$$\overbrace{\partial_t f^{<ee'}_{\boldsymbol{k}hij} + (\mathcal{V}^{e'e}_{\boldsymbol{k}hij})_{aa'}\hat{\boldsymbol{k}}\cdot\nabla f^{<aa'}_{\boldsymbol{k}hij}}^{\text{Liouville term}} = \overbrace{-2i\Delta\omega^{ee'}_{\boldsymbol{k}ij} f^{<ee'}_{\boldsymbol{k}hij}}^{\text{Leading oscillation scale}} + \overbrace{i[(\mathcal{W}^{\mathrm{H}e'e}_{\boldsymbol{k}hji})^{l}_{a}]^{*} f^{<ea}_{\boldsymbol{k}hil} - i(\mathcal{W}^{\mathrm{H}ee'}_{\boldsymbol{k}hij})^{l}_{a} f^{<ae'}_{\boldsymbol{k}hlj}}^{\text{Generalized forward scattering terms}} + \overbrace{\bar{\mathcal{C}}^{<hee'}_{\mathrm{H},\boldsymbol{k}hij}}^{\text{Collision term}}$$

- Complexity not due to cumbersome notation, but due to generality
- Describes flavor and all particle-antiparticle coherence effects
- Holds for light and heavy neutrinos, UR-limit not assumed
- Scalar equation, “easy to solve”!
- Reduces to the usual density matrix equation in suitable limits

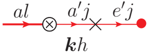
How about the collision terms?

- How to compute collision rates for arbitrary neutrino masses and kinematics while including local coherence effects?
- Necessary to understand neutrino flavor evolution in dense environments
  - Role of particle-antiparticle coherence?
- Has been an open problem in the literature since 90's

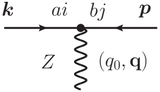
General Feynman rules



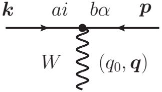
$\sim D^{ab}_{\boldsymbol{k}hij}$



$\sim \frac{1}{2\bar{\omega}^{e'e}_{\boldsymbol{k}ij}} D^{ab}_{\boldsymbol{k}hlj}\gamma^0 D^{e'e}_{\boldsymbol{k}hji}$

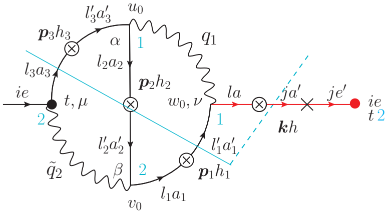


$\sim \frac{ig}{2c_w}\gamma^\mu P_{\mathrm{L}}\bar{U}_{ij}$



$\sim \frac{ig}{\sqrt{2}}\gamma^\mu P_{\mathrm{L}} U_{i\alpha}$

- $D^{ab}_{\boldsymbol{k}hij} \equiv ab\hat{N}^{ab}_{\boldsymbol{k}ij} P_{\boldsymbol{k}h}(\not{k}^a_i + m_i)(\not{k}^b_j + m_j)$
- Simple and straightforward rules to compute the general collision term!
- Correct energy and monemtum conservation rules



Collision terms

- Can always be written as

$$\bar{\mathcal{C}}^{<hee'}_{\mathrm{H},\boldsymbol{k}hij} = \sum\nolimits_Y \frac{1}{2\bar{\omega}^{aa'}_{\boldsymbol{k}lj}} \int \mathrm{dPS}_3 \Big[\frac{1}{2}(\mathcal{M}^2)^{ee'}_{\boldsymbol{k}hij\{\boldsymbol{p}_i,Y\}} \Lambda_{\boldsymbol{k}hj\{\boldsymbol{p}_i,Y\},x} + (h.c.)^{e'e}_{ji}\Big].$$

- Here we defined

$$\Lambda_{\boldsymbol{k}hj\{\boldsymbol{p}_i,\mathrm{Y}\}}(x) = f^{<aa'}_{\boldsymbol{k}hlj}(x)\, f^{>}_{\mathrm{X}_1\boldsymbol{p}_1}(x)\, f^{<}_{\mathrm{X}_2\boldsymbol{p}_2}(x)\, f^{>}_{\mathrm{X}_3\boldsymbol{p}_3}(x) \;-(>\leftrightarrow<),$$

$$\int \mathrm{dPS}_3 \equiv \int \Big[\prod_{i=1,3} \frac{\mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3 2\bar{\omega}_{\boldsymbol{p}_i l_i l'_i}}\Big] (2\pi)^4 \delta^4(k^a_l + p^{a_2}_{2\,l_2} - p^{a'_1}_{1\,l'_1} - p^{a'_3}_{3\,l'_3}).$$

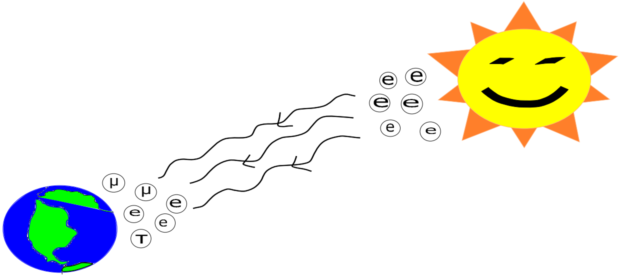
- All summed indices are in curly brackets $Y \equiv \{\mathrm{X}_i, h', a, a', l\}$
- Shorthand notation $f^{s\,a_i a'_i}_{\boldsymbol{p}_i h_i l_i l'_i}(x) \equiv f^{s}_{\boldsymbol{p}_i \mathrm{X}_i}(x)$, where h_i is the helicity, a_i energy sign index, and l_i flavor index

Applications

- Modelling neutrino distributions in hot and dense astrophysical environments:
  - nascent neutron stars
  - supernova explosions
  - compact object mergers
- Big bang nucleosynthesis
- Heavy neutrino oscillations
- Quantitative way to study how observer-system interference affects the evolution of the system
- Answers to "philosophical" issues: same energy/momentum, decoherence effects, quantum entanglement ...

Summary

- Developed general formalism to model neutrino evolution with all local coherences and the full collision term for coherent neutrino states



Thank you for your attention!

Projective representation

- Due to adiabatic background fields the Hamiltonian of the system is essentially helicity diagonal, i.e. the system is locally homogeneous and isotropic
  $\Rightarrow$ $S^{<}_{\boldsymbol{k}ij}$ can be constructed using 8 of the 16 basis elements of the Dirac algebra
- Convenient choice is to use energy and (shell independent) helicity projection operators:

$$P^{e}_{\boldsymbol{k}i} \equiv \frac{1}{2}\Big(\mathbb{1} + e\frac{\mathcal{H}_{\boldsymbol{k}i}}{\omega_{\boldsymbol{k}i}}\Big) \qquad \text{and} \qquad P^{h}_{\boldsymbol{k}} \equiv \frac{1}{2}\big(\mathbb{1} + h\hat{h}_{\boldsymbol{k}}\big)$$

with

$$\mathcal{H}_{\boldsymbol{k}ij} = (\boldsymbol{\alpha}\cdot\boldsymbol{k})_i\delta_{ij} - \gamma^0 m_i\delta_{ij}, \quad \hat{h}_{\boldsymbol{k}} \equiv \boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}}\gamma^5$$

$\Rightarrow$

$$\overline{S}^{<}_{ij}(k,x) = \sum_{he}\Big(P^{hee}_{\boldsymbol{k}ij} f^{<hee}_{\boldsymbol{k}ij}\delta(k_0 - e\overline{\omega}_{\boldsymbol{k}ij}) + P^{he-e}_{\boldsymbol{k}ij} f^{<he-e}_{\boldsymbol{k}ij}\delta(k_0 - e\Delta\omega_{\boldsymbol{k}ij})\Big)$$

Projective representation

- Shorthand notation:

$$P^{hee'}_{\boldsymbol{k}ij} = N^{ee'}_{\boldsymbol{k}} P^{h}_{\boldsymbol{k}} P^{e}_{\boldsymbol{k}i} \gamma^0 P^{e'}_{\boldsymbol{k}j}$$

  where the normalization factors are chosen as

$$N^{ee'}_{\boldsymbol{k}ij} \equiv \mathrm{Tr}\Big[P^{h}_{\boldsymbol{k}} P^{e}_{\boldsymbol{k}i} \gamma^0 P^{e'}_{\boldsymbol{k}j} \gamma^0\Big]^{-1/2} = \left[\frac{2\omega_{\boldsymbol{k}i}\omega_{\boldsymbol{k}j}}{\omega_{\boldsymbol{k}i}\omega_{\boldsymbol{k}j} + ee'(m_i m_j - |\boldsymbol{k}|^2)}\right]^{1/2}$$

- Normalization factors can be chosen freely, but the above ones simplify the dynamical equations the most
- Apparent singularity at $\omega_{\boldsymbol{k}_i}\omega_{\boldsymbol{k}_j} = ee'(|\boldsymbol{k}|^2 - m_i m_j)$ do not cause problems, since they cancel at the end

Transport equations

$$\partial_t f^{lee'}_{\boldsymbol{k}hij} + (\mathcal{V}^{e'e}_{\boldsymbol{k}hij})_{aa'}\boldsymbol{k}\cdot\boldsymbol{\nabla} f^{laa'}_{\boldsymbol{k}hij} = -2i\Delta\omega^{ee'}_{\boldsymbol{k}ij} f^{lee'}_{\boldsymbol{k}hij} - i(\mathcal{W}^{ee'}_{\boldsymbol{k}hij})^l_a f^{lae'}_{\boldsymbol{k}hlj} + i[(\mathcal{W}^{e'e}_{\boldsymbol{k}hji})^l_a]^* f^{lea}_{\boldsymbol{k}hli} + \bar{\mathcal{C}}^{<hee'}_{\mathrm{H},\boldsymbol{k}hij}$$

$$\begin{aligned}
(\mathcal{V}^{e'e}_{\boldsymbol{k}hij})_{aa'} &= \delta_{a'e'}\mathcal{V}^{eae'}_{\boldsymbol{k}hij} + \delta_{ae}\mathcal{V}^{a'e'e}_{\boldsymbol{k}hji},\\
\mathcal{V}^{abc}_{\boldsymbol{k}hij} &\equiv \frac{1}{2}N^{ac}_{\boldsymbol{k}ij}N^{bc}_{\boldsymbol{k}ij}\left(\frac{1}{\omega_{\boldsymbol{k}i}}\left[\frac{a}{(N^{bc}_{\boldsymbol{k}ij})^2} + \frac{b}{(N^{ac}_{\boldsymbol{k}ij})^2}\right] - \frac{c}{\omega_{\boldsymbol{k}j}}\delta_{a-b}\right),\\
2\Delta\omega^{ee'}_{\boldsymbol{k}ij} &\equiv e\omega_{\boldsymbol{k}i} - e'\omega_{\boldsymbol{k}j},\\
(\mathcal{W}^{ee'}_{\boldsymbol{k}hij})^l_a &\equiv \mathrm{Tr}\left[P^{e'e}_{\boldsymbol{k}hji}\bar{\Sigma}^{\mathrm{H}}_{\mathrm{eff}\,\boldsymbol{k}il}P^{ae'}_{\boldsymbol{k}hlj}\right].
\end{aligned}$$

Transport equations

- In this article we consider vector-like gauge interactions (a_{ij} and b_{ij} are flavor matrices, and u is the plasma 4-velocity):

$$\begin{aligned}\bar{\Sigma}_{\mathrm{H},ij}(k,x) &= \gamma^0(a_{ij}\not{k} + b_{ij}\not{u})P_L \\ &= \big(k_0 a_{ij} + b_{ij}\big)P_L - a_{ij}\boldsymbol{\alpha}\cdot\boldsymbol{k}P_L, \\ &\to \Big((k_0 + h|\boldsymbol{k}|)a_{ij} + b_{ij}\Big)P_L \equiv V_{\boldsymbol{k}hij}(k_0,x)P_L.\end{aligned}$$

- In this case the forward scattering tensor coefficients read

$$\begin{aligned}(\mathcal{W}^{\mathrm{L}e'e}_{\boldsymbol{k}hij})^l_{a'} &= \sigma^{a'e'e}_{\boldsymbol{k}hjil}V_{\boldsymbol{k}hlj}(a'\omega_{\boldsymbol{k}l},x), \\ (\mathcal{W}^{\mathrm{R}e'e}_{\boldsymbol{k}hij})^l_{a} &= \sigma^{eae'}_{\boldsymbol{k}hlji}V_{\boldsymbol{k}hil}(a\omega_{\boldsymbol{k}l},x), \\ \sigma^{abc}_{\boldsymbol{k}hlji} &\equiv \frac{1}{2}N^{ca}_{\boldsymbol{k}il}N^{bc}_{\boldsymbol{k}ji}\Big(\frac{\hat{P}^a_{\boldsymbol{k}hl}}{(N^{bc}_{\boldsymbol{k}ji})^2} + \frac{\hat{P}^b_{\boldsymbol{k}hj}}{(N^{ca}_{\boldsymbol{k}il})^2} - \hat{P}^c_{\boldsymbol{k}hi}\Big(\frac{1}{(N^{ab}_{\boldsymbol{k}lj})^2} - ab\frac{m_l m_j}{\omega_{\boldsymbol{k}l}\omega_{\boldsymbol{k}j}}\Big)\Big).\end{aligned} \tag{1}$$

Transport equations in the UR-limit

- In the UR-limit tensors $\sigma^{haee'}_{\boldsymbol{k}ijl}$ and $\mathcal{V}^{haee'}_{\boldsymbol{k}ij}$ become very simple:

$$\mathcal{V}^{haee'}_{\boldsymbol{k}ij} = \frac{e'}{|\boldsymbol{k}|}\delta_{a,e'}(\delta_{e,e'} + \delta_{e,-e'})$$

$$\sigma^{haee'}_{\boldsymbol{k}ijl} = \delta_{a,e'}(\delta_{e,e'}\delta_{h,-e} + \delta_{e,-e'}\delta_{h,e})$$

- Particle-antiparticle coherence average out at timescales relevant for neutrino oscillations everywhere except in the collision terms
  - This can be seen by using a coarse-graining factor over which one integrates
- According to above, the transport equations can be written in the UR-limit as

$$\partial_t f^{hee}_{\boldsymbol{k}ij} - e\hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} f^{hee}_{\boldsymbol{k}ij} = -2i\Delta\omega^{ee}_{\boldsymbol{k}ij} f^{hee}_{\boldsymbol{k}ij} - i\delta_{e,-h}\sum_l \left(V^e_{\boldsymbol{k}il} f^{hee}_{\boldsymbol{k}lj} - (V^e_{\boldsymbol{k}})^T_{jl} f^{hee}_{\boldsymbol{k}il}\right) + \overline{\mathcal{C}}^{A,hee'}_{\boldsymbol{k}ij}$$