Faster, Higher, Stronger *

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* A(lso)-K(nown)-A(s): "Citius, Altius, Fortius"

Talk Surely Inspired by Olympic principles, but mostly by hadron collider challenges

- ✓ Collider Phenomenology: where are we and where do we go from here?
- ✓ Top physics: demanding and challenging
  - ✓ The role of soft gluon resummation
  - ✓ What can we hope to achieve?
  - ✓ Towards NNLO.
- ✓ Dimuons, tt-bar FB, etc.

The nature of the problem

What we do is try to quantify the equation

Experiment – SM = ?

In practice, we phrase it like

Experiment – LO = ?

Or

Experiment – NLO = ?

Does it make a difference; does it matter how we phrase our questions?

We do not pay too much attention because we all believe we know what we mean;

But do we, really?

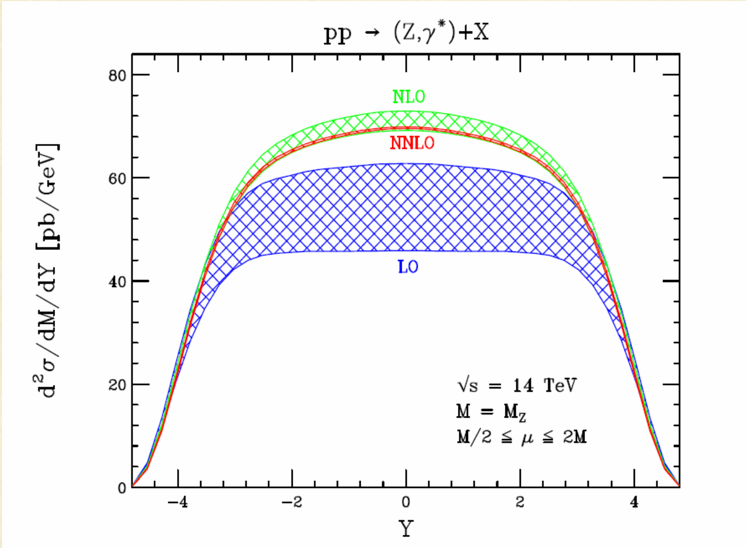
It is not a gamble to predict that these "nuances" will be central to the LHC program

3 examples →

Collider Phenomenology: where are we and where do we go from here?

Drell-Yan: vector boson rapidity distribution

Anastasiou, Dixon, Melnikov, Petriello '03

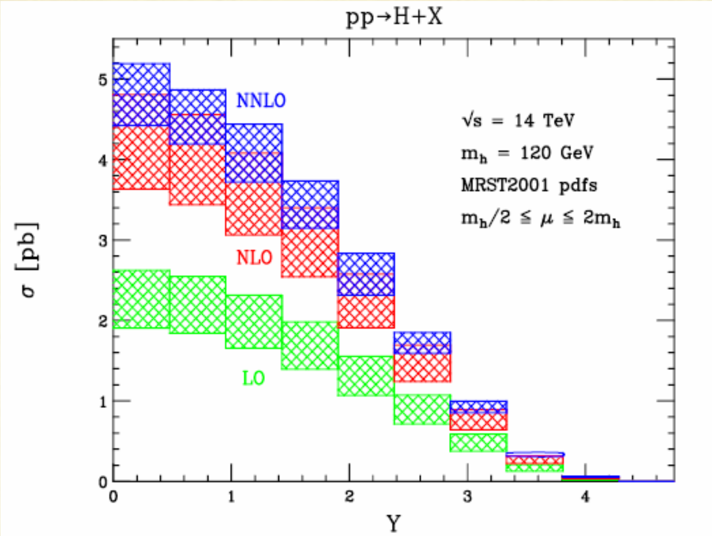


Notice how meaningless a representation of the total uncertainty the scale variations is !

Collider Phenomenology: where are we and where do we go from here?

Differential Higgs production

Anastasiou, Melnikov, Petriello `04

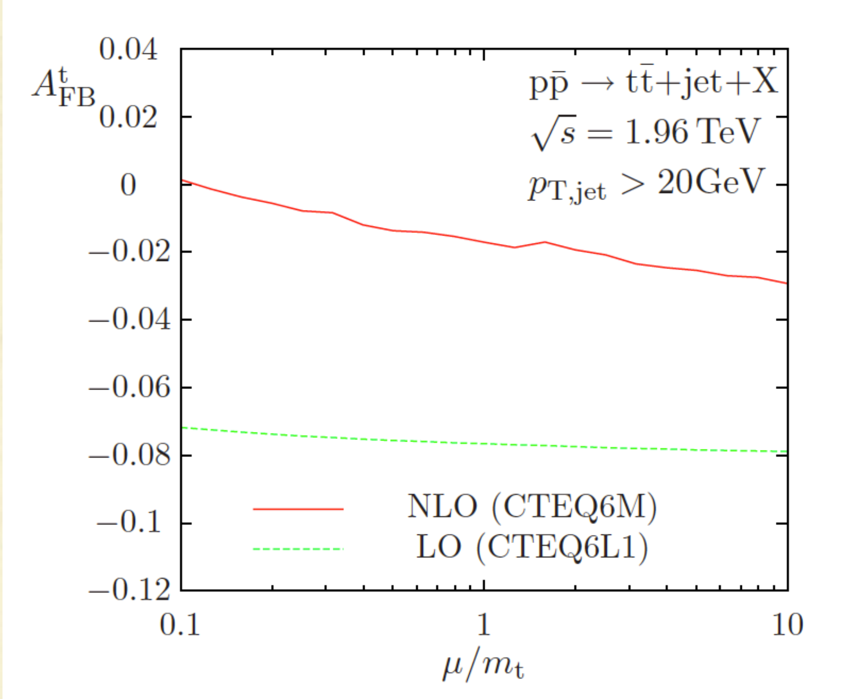


Notice how meaningless a representation of the total uncertainty the scale variations is !

Collider Phenomenology: where are we and where do we go from here?

F_B asymmetry in tT+jet

Dittmaier, Uwer, Weinzierl '07



- ✓ Scale variation at LO suggests uncertainty well below 10%
- ✓ Scale variation at NLO grows substantially
- ✓ NLO corrections around – 100%

Scale variation completely misrepresents the theoretical uncertainty!

Collider Phenomenology: where are we and where do we go from here?

This discussion might not appear very relevant if it was not for the fact that it is at the center-stage, today. Again.

- ✓ F-B asymmetry in tT production
- ✓ W+2jets anomaly: “hey, what is happening in 2jets”
- ✓ Same-sign dimuon excess at the Tevatron (D0)

The NLO revolution

Finally, we have reached the point of "everything at NLO".

(very impressive recent talk by Frixione @ CERN)

It is a truly great development.

But why did it take so long? All essential ingredients are known for at least 15 years.

A Phys. Rep. type of overview appeared 3 days ago (a great reading!):

Ellis, Kunszt, Melnikov, Zanderighi

Here is an excerpt from the Abstract:

> *The success of the experimental program at the Tevatron re-inforced the idea that <u>precision physics at hadron colliders is desirable and, indeed, possible</u>. The Tevatron data strongly suggests that one-loop computations in QCD describe hard scattering well. Extrapolating this observation to the LHC, <u>we conclude that knowledge of many short-distance processes at next-to-leading order may be required</u> to describe the physics of hard scattering …*

Do we still need to justify using NLO ??

On the wings of the NLO revolution and beyond

My point is: LO, or NLO or NNLO, etc. is irrelevant.
What matters is the level of control we have over theory.

So, I argue, going to NNLO in many case is a necessity, that can also be a source of deep satisfaction.

And a disclaimer:

What I do not do, and am not advocating, is to go for partial and unquantifiable higher order corrections.

What I do is to calculate the next largest source of uncertainty.

The line between the two is fine, and I will be sure to point it out in examples.

Top-pair production at hadron colliders

Top – some history

Top-pair production is known within NLO/NLL QCD

Main feature:

- ✓ Very large NLO corrections ~50%
- ✓ Appears in many bSM models
- ✓ Great prospects for experimental improvements down to 5%

How to do NNLL resummation is now completely understood

Czakon, Mitov, Sterman '09
Beneke, Falgari, Schwinn '09

Threshold approximation to NNLO also available

Beneke, Czakon, Falgari, Mitov, Schwinn '09

Is NNLL resummation/threshold approximation a substitute for the NNLO?

Is the threshold region dominant?

The observed cross-section is an integral over the product of:

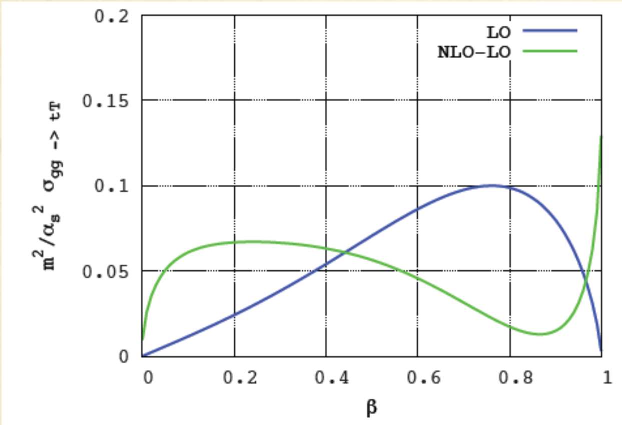
- Partonic cross-section,
- Partonic flux.

$$\sigma(s_{\rm had}) = \sum_{ij} \int_0^{\beta_{\rm max}} d\beta \, \Phi(\beta) \, \hat{\sigma}_{\rm part}(\beta)$$

$$\rho = \frac{4m_t^2}{s}$$

$$\beta = \sqrt{1-\rho}$$

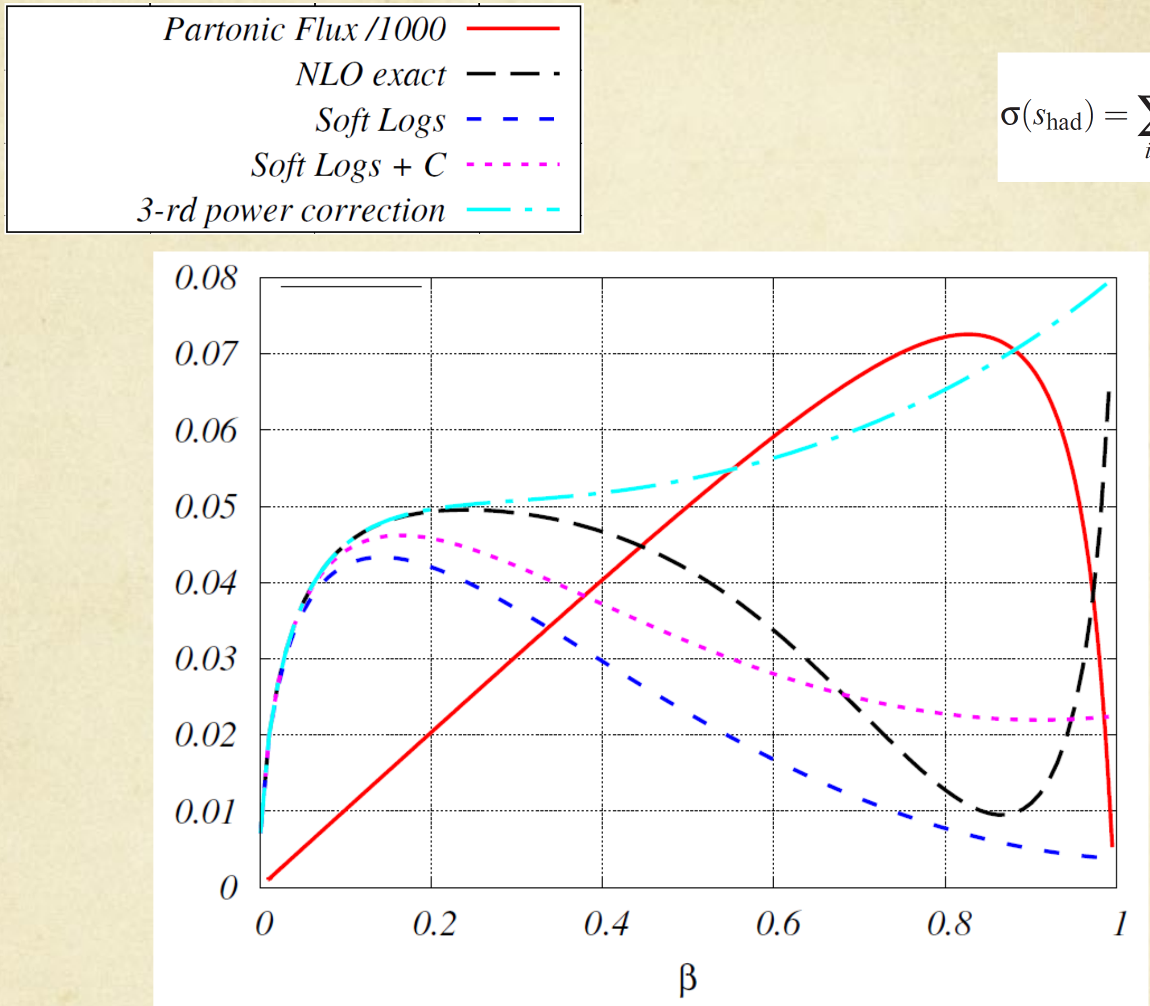
$$\eta = \frac{s}{4m_t^2} - 1$$



Pure LO vs. pure NLO

Very large NLO corrections!

Is the threshold region dominant?



$$\sigma(s_{\rm had}) = \sum_{ij} \int_0^{\beta_{\max}} d\beta \, \Phi(\beta) \, \hat{\sigma}_{\rm part}(\beta)$$

Czakon, Mitov '09

Important for the partonic cross-section (as expected)
Not for the flux.

Top-pair cross-section: 2-loop threshold expansion

Derive NNLO threshold approximation for the cross-section

- ✓ Use soft-gluon expansion (from resummation) — Czakon, Mitov, Sterman '09; Beneke, Falgari, Schwinn '09
- ✓ Extract 2-loop Coulombic terms (from, say, e+e- → tT) — Beneke, Czakon, Falgari, Mitov, Schwinn '09

$$\sigma_{ij,\mathbf{I}}(\beta,\mu,m) = \sigma^{(0)}_{ij,\mathbf{I}}\Bigg\{1+\frac{\alpha_s(\mu^2)}{4\pi}\left[\sigma^{(1,0)}_{ij,\mathbf{I}}+\sigma^{(1,1)}_{ij,\mathbf{I}}\ln\left(\frac{\mu^2}{m^2}\right)\right] + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2\left[\sigma^{(2,0)}_{ij,\mathbf{I}}+\sigma^{(2,1)}_{ij,\mathbf{I}}\ln\left(\frac{\mu^2}{m^2}\right)+\sigma^{(2,2)}_{ij,\mathbf{I}}\ln^2\left(\frac{\mu^2}{m^2}\right)\right]+\mathcal{O}(\alpha_s^3)\Bigg\}$$

$$\sigma^{(2)}_{q\bar{q}} = \frac{3.60774}{\beta^2}+\frac{1}{\beta}\Big(-140.368\ln^2\beta+32.106\ln\beta+3.95105\Big) +910.222\ln^4\beta-1315.53\ln^3\beta+592.292\ln^2\beta+528.557\ln\beta+C^{(2)}_{qq}\,,$$

$$\sigma^{(2)}_{gg} = \frac{68.5471}{\beta^2}+\frac{1}{\beta}\Big(496.3\ln^2\beta+321.137\ln\beta-8.62261\Big) +4608\ln^4\beta-1894.91\ln^3\beta-912.349\ln^2\beta+2456.74\ln\beta+C^{(2)}_{gg}\,,$$

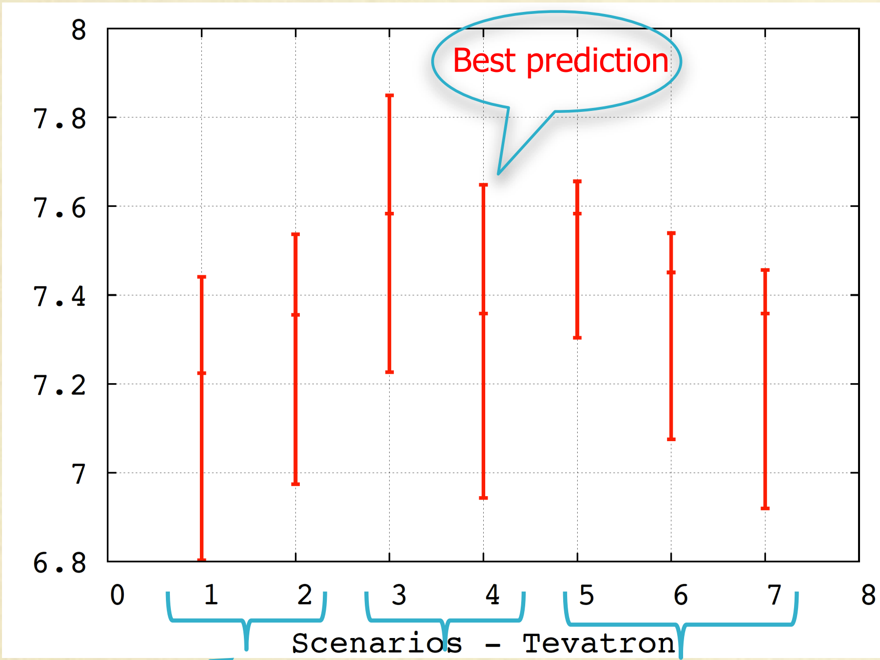
Top-pair total X-section: Tevatron numbers

Being an approximation, how robust are these numbers?

- ✓ Try to understand the physics;
- ✓ Stress-test in all possible ways;
- ✓ Quantify the sensitivities.

Defined on the next slide

Construct a number of NLO+NNLL and NNLO_aprox “scenarios” to analyze:



NNLL Resummation

Approximate NNLO

For m_top=171GeV

Plotted for each scenario are:
- ✓ central values
- ✓ scale uncertainty

Used independent variation of:
- ✓ renormalization scale
- ✓ factorization scale

See Cacciari et al ‘08

Top-pair total X-section: Tevatron numbers

Two approaches to NNLO_approx (depends on how the unknown const. are treated)

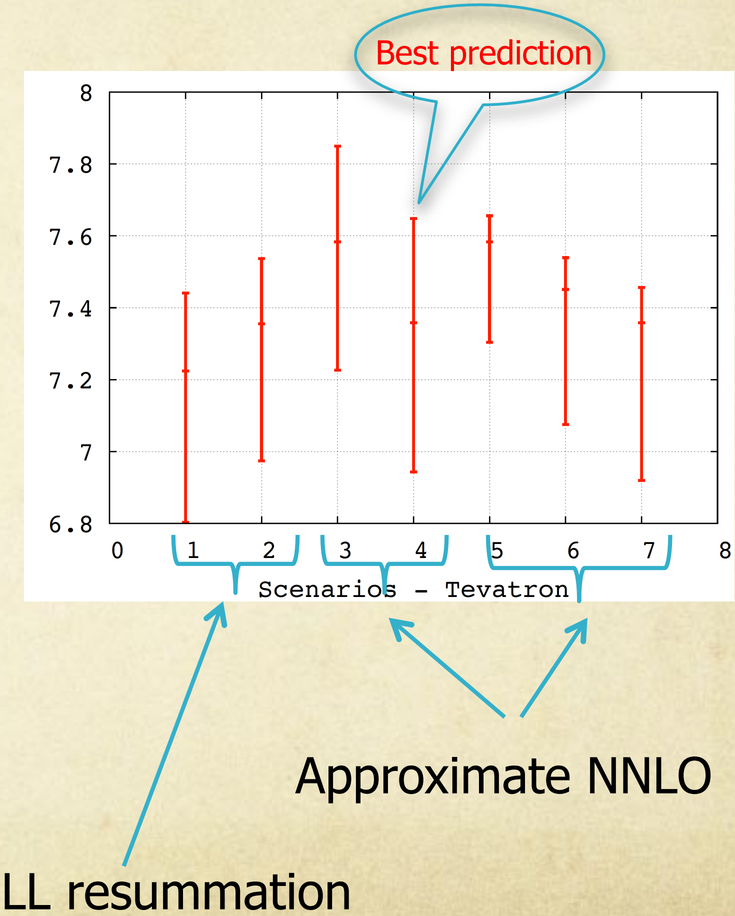
① Unknown constants AND log(mu) terms are omitted:

- ✓ Larger scale variation
- ✓ Consistent approximation
- ✓ Uncertainty = scale variation

② Unknown constants' log(mu) terms INCLUDED:

- ✓ Much smaller scale variation (~ the true NNLO)
- ✓ Uncertainty =
  scale variation AND constant variation
- ✓ Constant varied in a "reasonable range"

Both approaches are mutually consistent



Top-pair total X-section: Tevatron numbers

NNLO approximations vs. exact NLO vs NLO approximations

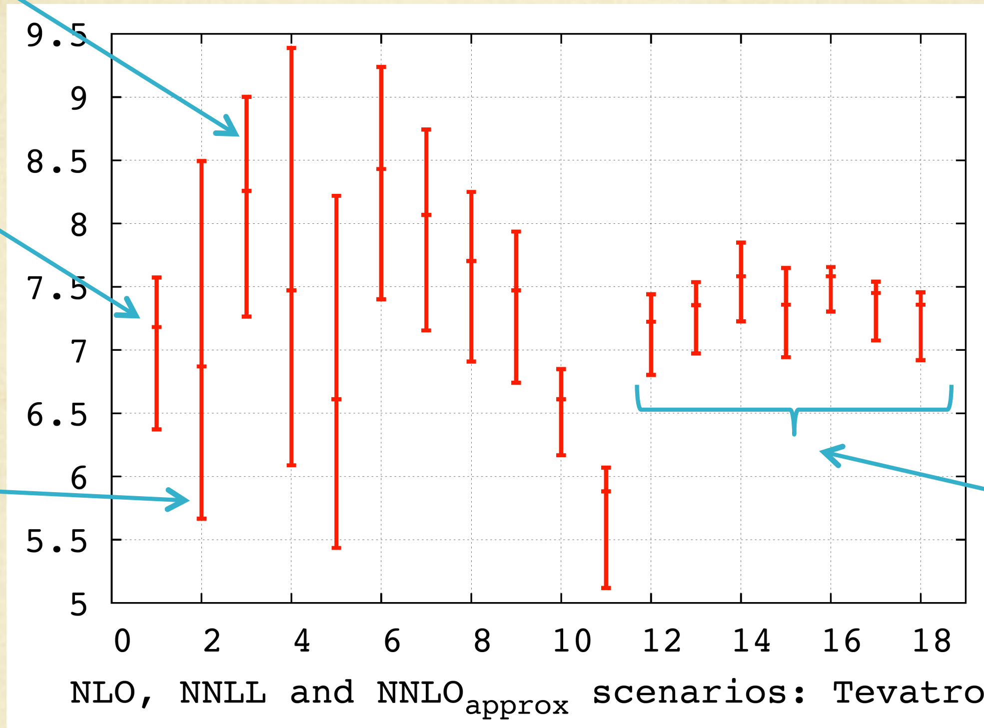
NLO: threshold approximation

For m_top=171GeV

Exact NLO

LO+NLL

NNLL/NNLO_approx approximations



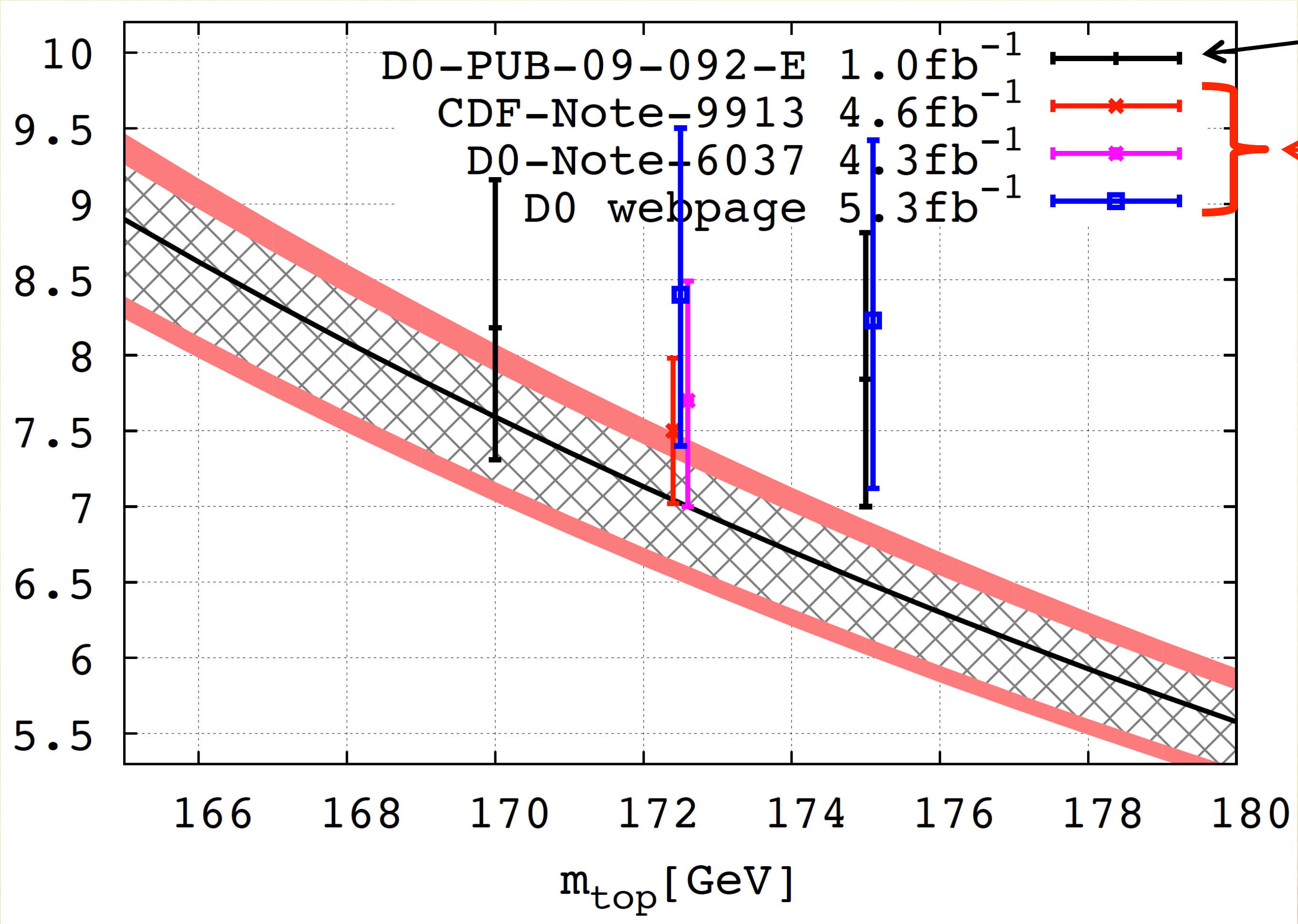
- ✓ NLO/NNLO Pdf sets consistently used
- ✓ Reduction in sensitivities going from NLO to NNLO

Top-pair total X-section: Tevatron numbers

With: Czakon; Beneke et al; Cacciari, Mangano, Nason; Moch, Uwer

Reconcile past differences and come up with common theory systematics

Prediction based on NLO+NNLL / NNLO_approx



Published data

Preliminary data

✓ Scale and PDF variation

✓ MSTW2008 (NNLO)

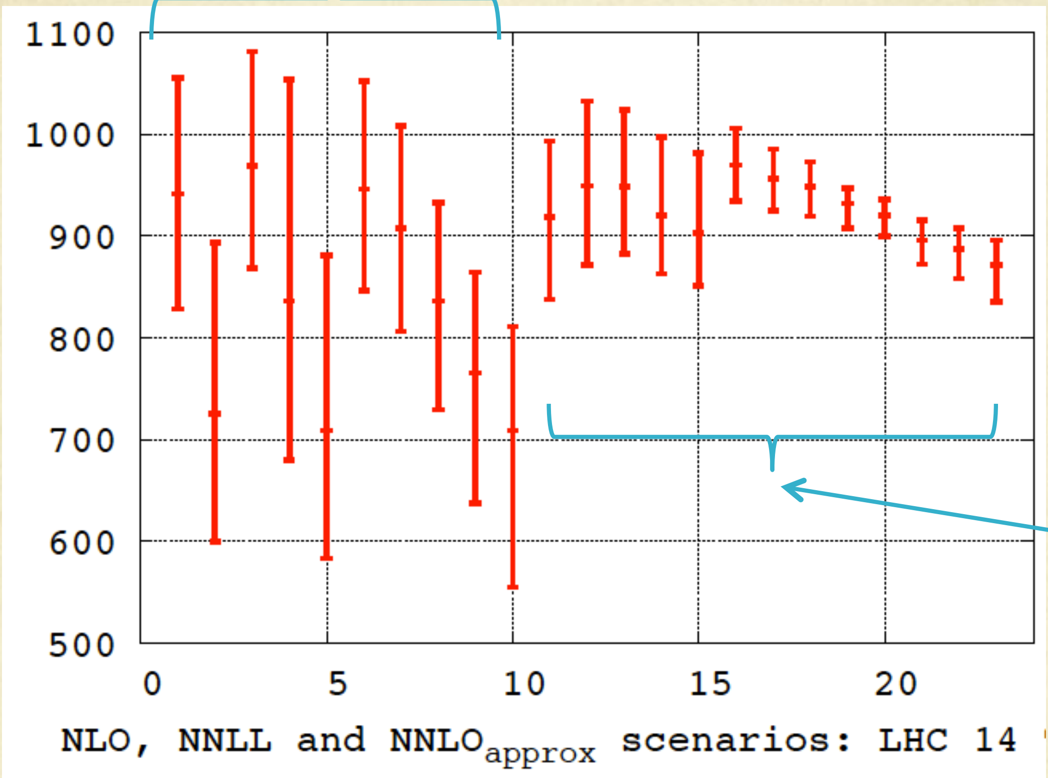
Note O(10%) decrease compared to MRST2006 (NNLO)

Top-pair total X-section: LHC @ 14 TeV numbers

LHC 7 – very similar!

Various NLO/NLL approximations

For m_top=171GeV



NNLL/NNLO_approx approximations

- ✓ NLO/NNLO Pdf sets consistently used
- ✓ Great reduction in sensitivities expected at full NNLO

Threshold approach: the conclusions

- ✓ Once we get to NNLO, resummation is not as important (tower of soft logs is ~0 beyond NNNLO)

First pointed out by Bonciani, Catani, Mangano, Nason`98

- ✓ A consistent use shows reduction in theoretical uncertainty
- ✓ Recall: theory uncertainty = scale variation + constant variation
- ✓ This has been neglected in the past – and bold conclusions were made
- ✓ It also shows the potential for improving theory once NNLO is known:
  - ✓ up to factor of 2 (Tevatron) – down to ~ 3%
  - ✓ up to a factor of 3 (LHC) – down to ~ 2-3%

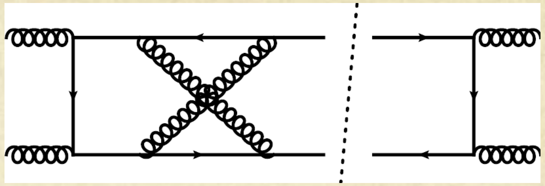
Towards NNLO for tT production and more

Work in progress with M. Czakon

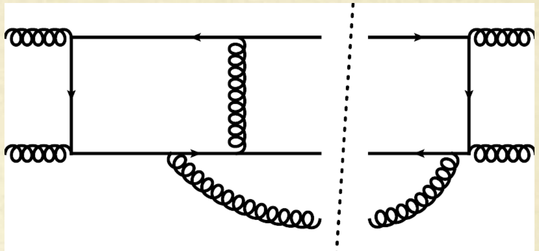
Towards NNLO

There are 3 principle contributions:

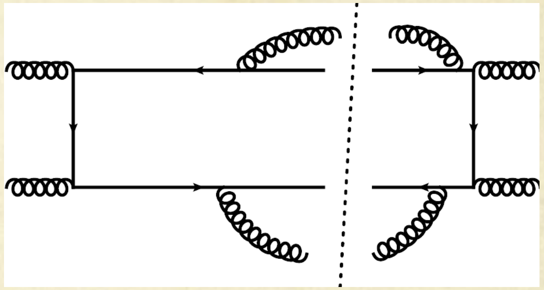
- ✓ 2-loop virtual corrections (V-V)



- ✓ 1-loop virtual with one extra parton (R-V)



- ✓ 2 extra emitted partons at tree level (R-R)



And 2 secondary contributions:

- ✓ Collinear subtraction for the initial state

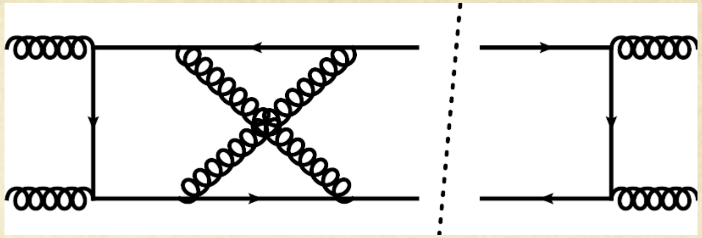
Known, in principle. To be done numerically.

- ✓ One-loop squared amplitudes

Korner, Merebashvili, Rogal `07

Towards NNLO: V-V

Required are the two loop amplitudes:
qq → QQ and gg → QQ.



✓ Their high energy limits and their poles are known analytically

Czakon, Mitov, Moch '07
Czakon, Mitov, Sterman '09
Ferroglia, Neubert, Pecjak, Yang '09

✓ The qq→QQ amplitude is known numerically

Czakon `07

✓ Numerical work underway for the gg→QQ

Czakon, Bärnreuther

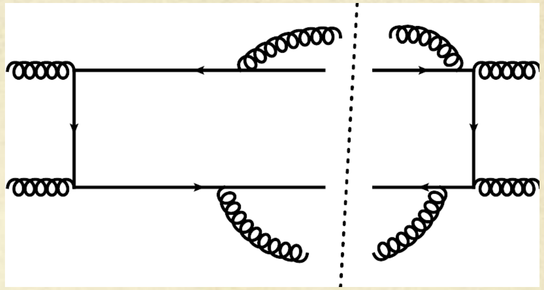
Comments about the 2-loop amplitudes

- ✓ Czakon's calculation is in principle straightforward (highly tedious):
  - ✓ Derive a large set of masters,
  - ✓ Derive differential equations for them (2-dim pde)
  - ✓ Derive numerically boundary conditions in the high energy limit (Mellin-Barnes)
  - ✓ Solve numerically the pde's
- ✓ That works for qq → QQ and for gg → QQ
- ✓ This will work for any 2-to-2 amplitude, obviously. But hardly beyond.

What's the future here?

- ✓ I believe that right now this is the biggest (and perhaps only) obstacle for NNLO phenomenology on a mass scale.
- ✓ Two-loop unitary – is it a mirage?

Towards NNLO: R-R



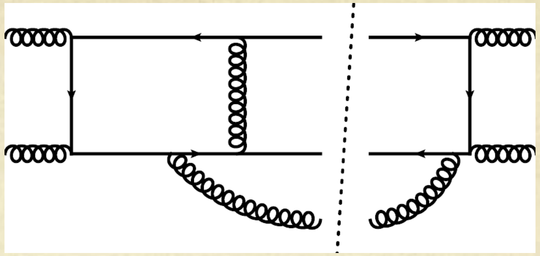
✓ All is done ☺

Czakon `10-11

✓ The method is general.

✓ Explicit contribution to the total cross-section given

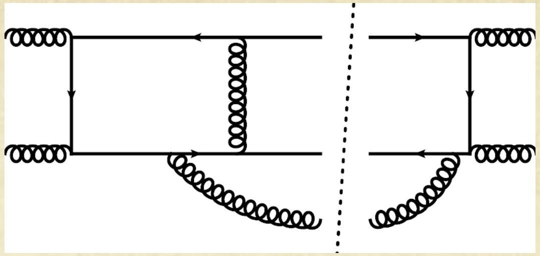
Towards NNLO: R-V



- The idea here is to get the NLO revolution to work
- All that is needed are the finite terms of the amplitude.
  No subleading terms in Epsilon needed (use counterterms - more later).
- Existing packages can handle any problem (in principle).
- Speed seems to be an issue (even with OPP)
- An analytical evaluation in terms of masters might be needed.
- The poles of any 1-loop amplitude can be written analytically.
- So, in principle, all is known.

The only missing piece are the subtraction terms in the soft and collinear limits

Towards NNLO: R-V



✓ Why subtractions?

We need to integrate over the real gluon. That generates additional divergences when the gluon is soft and/or collinear to external legs.

✓ Idea: devise a CT which approximates |amplitude|^2 in the soft/collinear limits:

$$|M|^2 = \underbrace{|M|^2 - CT}{} + CT$$

✓ Singular phase-space integration

✓ Simple function

✓ Finite in all limits. Can be integrated numerically in 4d.

✓ Note: the poles of the amplitude are known. Assumed subtracted beforehand.

Counter-terms for 1-loop amplitudes

How to devise CT for any 1-loop amplitude (masses and all)?

✓ Collinear limits are easy: emissions off massive lines are finite; massless - known

Bern, Del Duca, Kilgore, Schmidt '98-99

✓ Soft limit is an open problem. We have solved it; paper to appear.

Bierenbaum, Czakon, Mitov - to appear.

Soft limit of 1-loop massive amplitudes

Consider an (n+1) - point amplitude, with one external gluon (momentum q)

When the gluon becomes soft (q → 0) the amplitude becomes singular:

$$M_a(n+1;q) = J_a(q)M(n) + \mathcal{O}(\lambda^p) \qquad q \to \lambda q,\ \ \lambda \to 0$$

J : the soft-gluon (eikonal) current. It is process-independent!

The one-loop soft-gluon current

✓ So, the singular (soft) limit of 1-loop amplitudes is controlled by J:

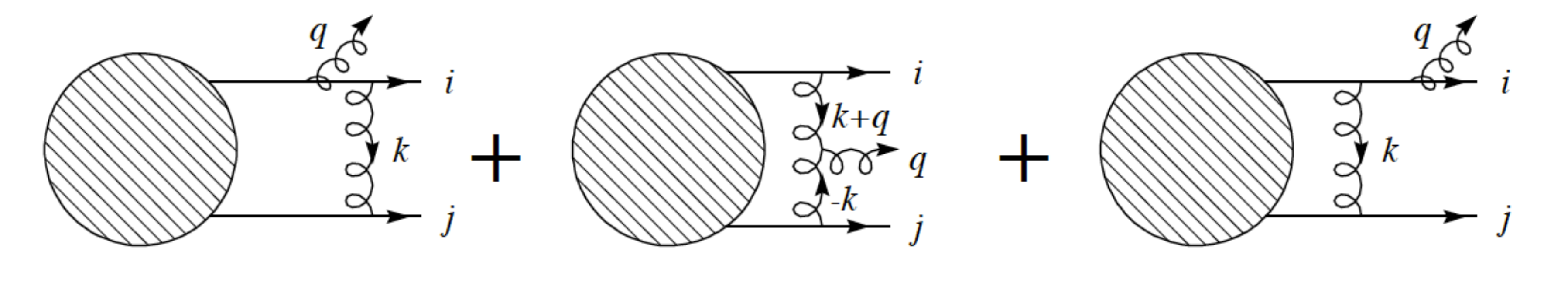
$$J_a(q) \quad = \quad g_S \mu^\epsilon \left(J_a^{(0)}(q) + J_a^{(1)}(q) + \mathcal{O}(\alpha_S^2) \right)$$

$$J_a^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q} \equiv \sum_{i=1}^{n} T_i^a e_i^\mu$$

✓ The one-loop correction J^(1) is known in the massless case:

Bern, Del Duca, Kilgore, Schmidt '98-99
Catani, Grazzini `00

Process independent calculation of eikonal diagrams:



The one-loop soft-gluon current

✓ Result expressed in terms of 3 scalar integrals (simple, complicated, harsh)

$$M_1 \equiv \Phi \int \frac{d^dk}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][-p_j \cdot k]}$$
$$M_2 \equiv \Phi \int \frac{d^dk}{i(2\pi)^d} \frac{1}{[k^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]}$$
$$M_3 \equiv \Phi \int \frac{d^dk}{i(2\pi)^d} \frac{1}{[k^2][(k+q)^2][p_i \cdot k + p_i \cdot q][-p_j \cdot k]}$$

Note:
these are not scaleless integrals!

✓ M_1 and M_2 through Gauss hypergeometric functions.

✓ M_3 is very hard. Involves multiple polylogs (also appear in one-loop squared)

$$F_c(x_1, x_2) = \int_0^1 dt \frac{\ln(1-t)\ln\left(1 - t\frac{x_2}{x_1}\right)}{\frac{1}{x_2} - t}$$

✓ Calculate 3 kinematical configurations:
- ✓ (i,j) massive/massless, or
- ✓ (i,j) incoming/outgoing

✓ Understood the analytical continuation spacelike → timelike.

✓ Number of checks (small mass limit; poles; numerical checks)

The one-loop soft-gluon current

✓ Here is the result for the IM parts of the one-mass case (simplest thing to show):

$$g_{ij}^{(1)}(Case\ 1) \quad = \quad R_{ij}^{[C1]} + i\pi I_{ij}^{[C1]} \equiv \left(\frac{2(p_i\cdot p_j)\mu^2}{2(p_i\cdot q)2(p_j\cdot q)}\right)^{\epsilon} \sum_{n=-2}^{2} G_{ij}^{(n)[C1]}\epsilon^n$$

The leading q-dependence factors out in d-dimensions; the rest is homogeneous!

$$\begin{aligned}
I_{ij}^{(-2)[C1]} &= 0\,, \\
I_{ij}^{(-1)[C1]} &= -\frac{1}{2}\,, \\
R_S\, I_{ij}^{(0)[C1]} &= 2m_i^2(p_j\cdot q)\ln\left(\frac{\alpha_i}{2}\right)\,, \\
R_S\, I_{ij}^{(1)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right) + m_i^2(p_j\cdot q)\ln^2\left(\frac{\alpha_i}{2}\right) \\
&\quad + \pi^2\frac{-2(p_i\cdot p_j)(p_i\cdot q) + m_i^2(p_j\cdot q)}{2}\,, \\
R_S\, I_{ij}^{(2)[C1]} &= 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\left[\mathrm{Li}_3\left(1-\frac{\alpha_i}{2}\right) + \mathrm{Li}_3\left(\frac{\alpha_i}{2}\right)\right] - \zeta_3\frac{40(p_i\cdot p_j)(p_i\cdot q) - 26m_i^2(p_j\cdot q)}{3} \\
&\quad + 2\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\ln\left(1-\frac{\alpha_i}{2}\right)\ln^2\left(\frac{\alpha_i}{2}\right) + \frac{m_i^2(p_j\cdot q)}{3}\ln^3\left(\frac{\alpha_i}{2}\right) \\
&\quad + \ln\left(\frac{\alpha_i}{2}\right)\left(\pi^2\frac{-4(p_i\cdot p_j)(p_i\cdot q) + m_i^2(p_j\cdot q)}{6} + 4\left[(p_i\cdot p_j)(p_i\cdot q) - m_i^2(p_j\cdot q)\right]\mathrm{Li}_2\left(1-\frac{\alpha_i}{2}\right)\right)
\end{aligned}$$

$$R_S = 4\left[m_i^2(p_j\cdot q) - 2(p_i\cdot p_j)(p_i\cdot q)\right]\ ,\ \alpha_i = \frac{m_i^2(p_j\cdot q)}{(p_i\cdot q)(p_i\cdot p_j)}$$

Derivation of CT

The current reads:

$$J_a^{\mu(1)}(q) = i f_{abc} \sum_{i \neq j=1}^{n} T_i^b T_j^c \left(e_i^\mu - e_j^\mu\right) g_{ij}^{(1)}(\epsilon, q, p_i, p_j)$$

where:

$$g_{ij}^{(1)} \equiv R_{ij} + i\pi I_{ij}$$

Then, the square of the Born diagram becomes:

$$\langle M_a^{(0)}(n+1;q)|M_a^{(0)}(n+1;q)\rangle =$$
$$-4\pi\alpha_S\mu^{2\epsilon}\left\{\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle \;+\; \sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(0)}(n)\rangle\right\} + \mathcal{O}(\lambda^p)$$

... and the one-loop amplitude reads:

Note!

$$\langle M_a^{(0)}(n+1;q)|M_a^{(1)}(n+1;q)\rangle + c.c. = -4\pi\alpha_S\mu^{2\epsilon}\Bigg\{$$
$$2C_A\sum_{i\neq j=1}^{n}\left(e_{ij}-e_{ii}\right)R_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(0)}(n)\rangle - 4\pi\sum_{i\neq j\neq k=1}^{n} e_{ik}I_{ij}\langle M^{(0)}(n)|f^{abc}T_i^aT_j^bT_k^c|M^{(0)}(n)\rangle$$
$$+\left(\sum_{i\neq j=1}^{n} e_{ij}\langle M^{(0)}(n)|T_i\cdot T_j|M^{(1)}(n)\rangle + c.c.\right) + \left(\sum_{i=1}^{n}\mathcal{C}_i e_{ii}\langle M^{(0)}(n)|M^{(1)}(n)\rangle + c.c.\right)\Bigg\} + \mathcal{O}(\lambda^p)\,, \quad (14)$$

Comment on subtractions (dipole or otherwise)

- ✓ So, CT are derived. What's next?
- ✓ Integrate them over phase space. It is a problem.
- ✓ I first tried Catani-Seymour type of approach. Realized fast that Nature cannot be that cruel ☺.
- ✓ The way out seems to be FKS.
- ✓ No details at that point, just few comments:
  - ✓ Dipoles are not enough beyond NLO (known from many places now)
  - ✓ FKS: no shift in variables; no divergent integration.
    Works similarly to sector decomposition.
  - ✓ FKS does not really care about the color structure (which is very involved);
    It is all about the kinematics.

Summary and Conclusions

- We have developed methods for NNLO calculations in any 2-to-2 process.
- Very soon to produce results for the tT-total cross-section at Tevatron.
- That will be followed by F-B asymmetry and fully exclusive observables.
- Applications for dijets and heavy flavor production in DIS at NNLO.
- Methods are fully exclusive and numeric.
- Down the road – produce partonic Monte Carlo at NNLO.

For the tT total cross-section:

- Discussed relation between fixed order calculations and resummation in tT
- Applicability of threshold approximation (many speculative statements out there)
- Potential for tT at NNLO is at the 2-3% level.

Backup Slides

Singularities of Massive Gauge Theory Amplitudes

Amplitudes: the basics

- Gauge theory amplitudes: UV renormalized, S-matrix elements
- The amplitudes are not observables:
  - UV renormalized gauge amplitudes are not finite due to IR singularities.
  - Assume they are regulated dimensionally $d=4-2\varepsilon$

What was known before (massive case):

✓ Explicit expression for the IR poles of any one-loop amplitude derived

Catani, Dittmaier, Trocsanyi '00

✓ The small mass limit is proportional to the massless amplitude

Mitov, Moch '06
Becher, Melnikov '07

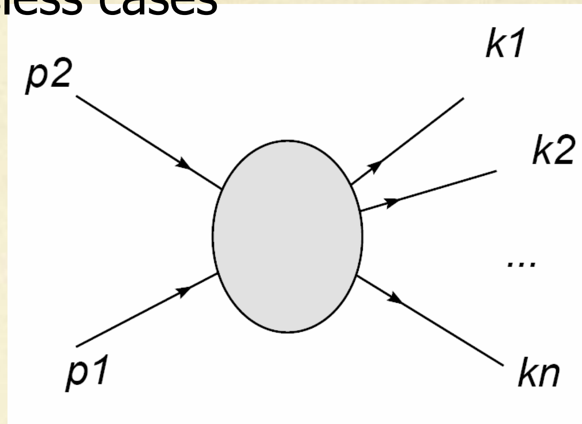
Note: predicts not just the poles but the finite parts too (for $m \to 0$)!

Factorization: “divide and conquer“

Structure of amplitudes becomes transparent thanks to factorization th.

$$M_I(\varepsilon,\mu_R,s_{ij},m_i) = J(\varepsilon,\mu_R,\mu_F,m_i)\cdot S_{IJ}(\varepsilon,\mu_R,\mu_F,s_{ij},m_i)\cdot H_J(\varepsilon,\mu_R,\mu_F,s_{ij},m_i)$$

Note: applicable to both massive and massless cases



I,J – color indexes.

J(…) – “jet” function. Absorbs all the collinear enhancement.

S(…) – “soft” function. All soft non-collinear contributions.

H(…) – “hard” function. Insensitive to IR.

Factorization: the Jet function

$$M_I(Q,m,\epsilon) = J(m,\epsilon) \cdot S_{IJ}(Q,m,\epsilon) \cdot H_J(Q,m)$$

For an amplitude with n-external legs, J(…) is of the form:

$$J(m,\epsilon) = \prod_{i=1}^{n} J_i(m,\epsilon)$$

i.e. we associate a jet factor to each external leg.

Some obvious properties:
- Color singlets,
- Process independent; i.e. do not depend on the hard scale Q.

J_i not unique (only up to sub-leading soft terms).

A natural scheme: J_i = square root of the space-like QCD formfactor.

Sterman and Tejeda-Yeomans '02

Scheme works in both the massless and the massive cases.

The massive form-factor's exponentiation known through 2 loops

Mitov, Moch '06

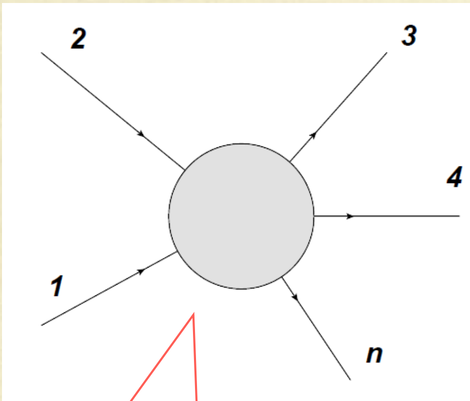
Factorization: the Soft function

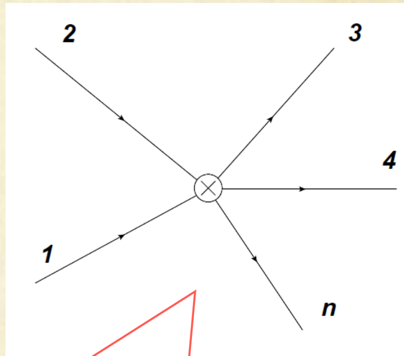
$$M_I(Q,m,\epsilon) = J(m,\epsilon)\cdot S_{IJ}(Q,m,\epsilon)\cdot H_J(Q,m)$$

Soft function is the most non-trivial element
(recall: it contains only soft poles).

But we know that the soft limit is reproduced by the eikonal approximation.

➔ Extract S(…) from the eikonalized amplitude:





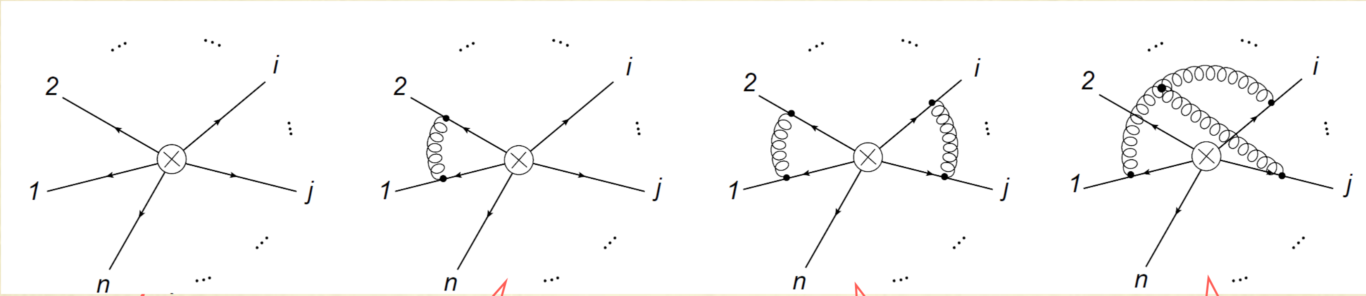
The LO amplitude M(…)

The eikonal version of the amplitude.
(the blob is replaced by an effective n-point vertex)

Factorization: the Soft function

Calculation of the eikonal amplitude:

consider all soft exchanges between the external (hard) partons



The fixed order expansion of the soft function takes the form:

$$
\begin{aligned}
S^{(1)}_{IJ}(\varepsilon, s_{ij}, m_i) &= \frac{1}{\varepsilon}\Gamma^{(1)}_{IJ}(s_{ij}, m_i) + O(\varepsilon^0)\,,\\
S^{(2)}_{IJ}(\varepsilon, s_{ij}, m_i) &= -\frac{\beta_0}{4\varepsilon^2}\Gamma^{(1)}_{IJ}(s_{ij}, m_i) + \frac{1}{2}\left(S^{(1)}_{IJ}(\varepsilon, s_{ij}, m_i)\right)^2 + \frac{1}{\varepsilon}\Gamma^{(2)}_{IJ}(s_{ij}, m_i) + O(\varepsilon^0)\,.
\end{aligned}
$$

… as follows from the usual RG equation:

$$
\left(\mu\frac{\partial}{\partial\mu} + \beta(g,\varepsilon)\frac{\partial}{\partial g}\right) S_{IJ}(\varepsilon, s_{ij}, m_i) = -\Gamma_{IK}(\varepsilon, s_{ij}, m_i)\; S_{KJ}(\varepsilon, s_{ij}, m_i)
$$

➔ All information about S(…) is contained in the anomal's dimension matrix Γ_{IJ}

the Soft function at 1 loop

Here is the result for the anomalous dim. matrix at one loop

$$\mathbf{\Gamma}_S^{(1)} = \underbrace{\frac{1}{2}\sum_{(i\neq j)=1}^{n} T_i\cdot T_j \ln\left(-\frac{\mu^2}{\sigma_{ij}}\right)}_{\text{The massless case}} + \underbrace{\frac{1}{2}\sum_{(i\neq j)\in\mathcal{N}_m} T_i\cdot T_j \left[\ln\left(1+x_{ij}^2\right) + \frac{2x_{ij}^2}{1-x_{ij}^2}\ln(x_{ij})\right]}_{\text{O(m) corrections in the massive case}}$$

where:
- all masses are taken equal,
- written for space-like kinematics (everything is real).

$$\frac{m^2}{s_{ij}} = -\frac{x_{ij}}{(1-x_{ij})^2} \quad , \quad x_{ij} = \frac{\sqrt{1-\frac{4m^2}{s_{ij}}}-1}{\sqrt{1-\frac{4m^2}{s_{ij}}}+1}$$

$$s_{ij} = (p_i+p_j)^2 \text{ and } \sigma_{ij} = 2p_i\cdot p_j = s_{ij} - m_i^2 - m_j^2$$

The Soft function at 2 loops

The simplest approach is the following. Start with the Ansatz:

$$\underbrace{\mathbf{\Gamma}_S^{(2)} = \frac{1}{2}\sum_{(i\neq j)=1}^{n} T_i\cdot T_j\,\frac{K}{2}\,\ln\left(-\frac{\mu^2}{\sigma_{ij}}\right)}_{\text{Reproduces the massless case}} + \underbrace{\frac{1}{2}\sum_{(i\neq j)\in\mathcal{N}_m} T_i\cdot T_j\, P_{ij}^{(2)}}_{\text{Parametrizes the O(m) corrections to the massless case}} + 3E\ \text{terms}$$

Then note: the function $P^{(2)}_{ij}$ depends on (i,j) only through s_{ij}

$\boxed{\rightarrow P^{(2)}_{ij} = P^{(2)}(s_{ij})}$ This single function can be extracted from the known n=2 amplitude: the massive two-loop QCD formfactor.

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi `04

Gluza, Mitov, Moch, Riemann '09

The Soft function at 2 loops

The complete result for the 2E reads: $$P^{(2)} = \frac{K}{2} P^{(1)} + P^{(2),\mathrm{m}}$$

$$\begin{aligned} P^{(2),\mathrm{m}}(x) \;=\; & \frac{C_A}{(1-x^2)^2}\Bigg\{ -\frac{(1+x^2)^2}{2}\mathrm{Li}_3(x^2) + \left(\frac{(1+x^2)^2}{2}\ln(x) - \frac{1-x^4}{2}\right)\mathrm{Li}_2(x^2) \\ & + \frac{x^2(1+x^2)}{3}\ln^3(x) + x^2(1-x^2)\ln^2(x) \\ & + \left(-(1-x^4)\ln\left(1-x^2\right) + x^2(1+x^2)\zeta_2\right)\ln(x) + x^2(1-x^2)\zeta_2 + 2x^2\zeta_3 \Bigg\}, \end{aligned}$$

This term breaks the simple relation $\Gamma^{(2)}_{S_\mathrm{f}} = \frac{K}{2}\Gamma^{(1)}_{S_\mathrm{f}}$ from the massless case!

Aybot, Dixon, Sterman ’06

Above result derived by 3 different groups:

Kidonakis ’09
Becher, Neubert ‘09
Czakon, Mitov, Sterman ‘09

Kidonakis derived the massive eikonal formfactor;
Becher, Neubert used old results of Korchemsky, Radushkin

The Soft function at 2 loops

What about the 3E contributions in the massive case?

Until recently there existed no indication if they were non-zero!

In particular, the following squared two-loop amplitudes are insensitive to it:

Czakon, Mitov, Sterman '09

Known numerically $\langle M^{(2)}|M^{(0)}\rangle(q\bar{q} \to Q\overline{Q})$ Czakon '07

Poles reported $\langle M^{(2)}|M^{(0)}\rangle(gg \to Q\overline{Q})$ Czakon, Bärnreuther '09

3E correlators not vanish if at least two legs are massive – direct position-space calculation for Euclidean momenta (numerical results)

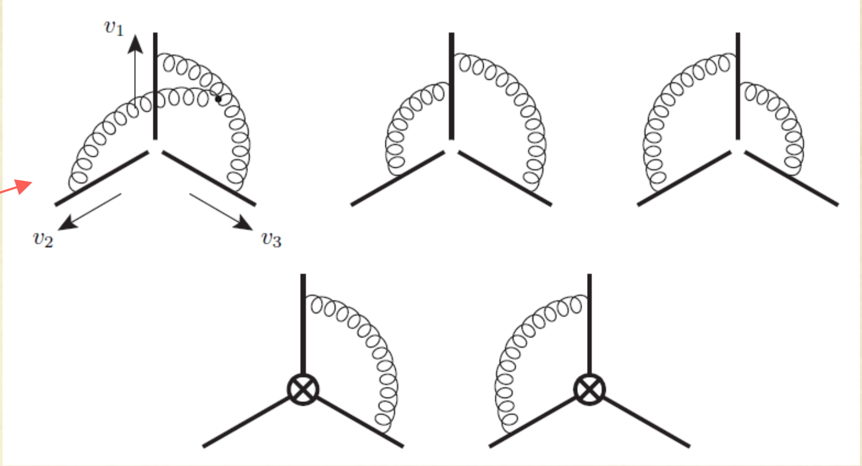
Mitov, Sterman, Sung '09

Exact result computed analytically

Ferroglia, Neubert, Pecjak, Yang '09

The Soft function at 2 loops. Massive case.

The types of contributing diagrams:



The analytical result is very simple:

Ferroglia, Neubert, Pecjak, Yang '09

$$F^{(3g)} \sim \sum_{ijk} \epsilon^{ijk} \ln^2(x_{ij})\, r(x_{ik})$$

where:

$$r(x) = -\frac{1+x^2}{1-x^2}\ln(x)$$

The calculation of the double exchange diagrams is very transparent.
Agrees in both momentum and position spaces

A.M., Sterman, Sung '10

Massive gauge amplitudes: Summary

- The results I presented can be used to predict the poles of any massive 2-loop amplitude with:
  - *n* external colored particles (plus arbitrary number of colorless ones),
  - arbitrary values of the masses (usefull for SUSY).

- Results checked in the 2-loop amplitudes:

$$\langle M^{(2)}|M^{(0)}\rangle(q\bar{q}\to Q\overline{Q})$$
$$\langle M^{(2)}|M^{(0)}\rangle(gg\to Q\overline{Q})$$

- Needed in jet subtractions with massive particles at 2-loops

- Input for NNLL resummation (next slides)

The connection to resummation at hadron colliders

How is the threshold resummation done?

The resummation of soft gluons is driven mostly by kinematics:

Sterman '87
Catani, Trentadue '89

- Only soft emissions possible due to phase space suppression (hence kinematics)
- That's all there is for almost all "standard" processes: Higgs, Drell-Yan, DIS, e^+e^-

Key: the number of hard colored partons < 4

In top pair production (hadron colliders) new feature arises:

Color correlations due to soft exchanges (n>=4)

Non-trivial color algebra in this case.

The top cross-section: NNLL resummation

Factorization of the partonic cross-section close to threshold:

Kidonakis, Sterman '97
Czakon, Mitov, Sterman '09

$$\omega_P\left(N,\hat{\eta},\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right) = J_1(N,\alpha_s(\mu^2))\dots J_k(N,M/\mu,m/\mu,\alpha_s(\mu^2)) \times \mathrm{Tr}\left[\mathbf{H}^P\left(\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right)\mathbf{S}^P\left(\frac{N^2\mu^2}{M^2},\frac{M^2}{m^2},\alpha_s(\mu^2)\right)\right] + \mathcal{O}(1/N)$$

N – the usual Mellin dual to the kinematical variable that defines the threshold kinematics:

$$\sigma(N) = \int_0^1 dz z^{N-1}\sigma(z)$$

$z = Q^2/s$ ← Drell-Yan

$z = 4m^2/s$ ← t-tbar total X-section

$z = M_{t\bar{t}}^2/s$ ← t-tbar – pair invariant mass

J's – jet functions (different from the ones in amplitudes)

S,H – Soft/Hard functions. Also different.

The top cross-section: NNLL resummation

Specifically, for top-pair production we have:

$$\sigma^P(N,m^2,\mu^2) = \sigma^P_{\rm Born}(N)\ \left[J^P_{\rm in}(N,m^2,\mu^2)\right]^2\ \left[J_{\rm incl}(N,m^2,\mu^2)\right]^2\ {\rm Tr}\left[\hat{\mathbf{H}}^P(m^2,\mu^2)\mathbf{S}^P(N,m^2,\mu^2)\right] + \mathcal{O}(1/N)$$

where:

- $J^P_{\rm in}$ – is the Drell-Yan/Higgs cross-section
- $J_{\rm incl}$ – observable dependent function (i.e. depends on the final state)

$$J_{\rm incl}(N,m^2,\mu^2) = \exp\left\{\frac{1}{2}\int_0^1 dx \frac{x^{N-1}-1}{1-x}\,\Gamma_{\rm incl}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right\}$$

$$\Gamma_{\rm incl} = \frac{\alpha_s(\mu^2)}{\pi}\,C_F\left[-1-\ln\left(\frac{m^2}{\mu^2}\right)\right] + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left[\frac{K}{2}C_F\left(-1-\ln\left(\frac{m^2}{\mu^2}\right)\right) - \frac{\zeta_3-1}{2}C_F C_A\right]$$

Defines the poles of the massive QCD formfactor in the small-mass limit.

Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi `04
Gluza, Mitov, Moch, Riemann '09
Mitov, Moch '06

The top cross-section: NNLL resummation

Here is the result for the Soft function:

$$
\begin{aligned}
\mathbf{S}\left(\frac{N^2\mu^2}{M^2}, \beta_i\cdot\beta_j, \alpha_s(\mu^2)\right)\Bigg|_{\mu=M} &= \overline{\mathcal{P}}\exp\left\{-\int_{M/\bar{N}}^{M}\frac{d\mu'}{\mu'}\mathbf{\Gamma}_S^\dagger\left(\beta_i\cdot\beta_j, \alpha_s\left(\mu'^2\right)\right)\right\} \\
&\quad\times\mathbf{S}\left(1, \beta_i\cdot\beta_j, \alpha_s\left(M^2/\bar{N}^2\right)\right) \\
&\quad\times\mathcal{P}\exp\left\{-\int_{M/\bar{N}}^{M}\frac{d\mu'}{\mu'}\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j, \alpha_s\left(\mu'^2\right)\right)\right\} \\
&= \overline{\mathcal{P}}\exp\left\{\int_0^1 dx\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S^\dagger\left(\beta_i\cdot\beta_j, \alpha_s\left((1-x)^2M^2\right)\right)\right\} \\
&\quad\times\mathbf{S}\left(1, \beta_i\cdot\beta_j, \alpha_s\left(M^2/N^2\right)\right) \\
&\quad\times\mathcal{P}\exp\left\{\int_0^1 dx\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j, \alpha_s\left((1-x)^2M^2\right)\right)\right\}
\end{aligned}
$$

Note: the Soft function satisfies RGE with the same anomalous dimension matrix as the Soft function of the underlying amplitude!

Therefore: knowing the singularities of an amplitude, allows resummation of soft logs in observables!

The top cross-section: NNLL resummation

We also need to specify a boundary condition for the soft function:

$$\mathbf{S}\left(1, \beta_i \cdot \beta_j, \alpha_s\left(M^2/N^2\right)\right) = \mathbf{S}^{(0)} + \frac{\alpha_s\left(M^2/N^2\right)}{\pi}\, \mathbf{S}^{(1)}\left(1, \beta_i \cdot \beta_j\right) + \ldots$$

For two-loop resummation we need it only at one loop
(since its contribution at two loops is only through the running coupling).

For example, for the total t-tbar cross-section in gg-reaction it reads:

$$\begin{aligned}\mathbf{S}\left(1, \alpha_s\left(Q^2/N^2\right)\right) &= \mathbf{S}^{(0)}\left[1 + \frac{\alpha_s\left(Q^2/N^2\right)}{\pi}\, C_A \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \ldots\right] \\ &= \mathbf{S}^{(0)}\left[1 + C_A \frac{\alpha_s\left(\mu^2\right)}{\pi}\left\{1 + \frac{\alpha_s\left(\mu^2\right)}{\pi}\frac{\beta_0}{4}\ln\left(\frac{N^2\mu^2}{Q^2}\right)\right\}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \ldots\right]\end{aligned}$$

Can be derived by calculating the one-loop eikonal cross-section.

The top cross-section: NNLL resummation

Combining everything we get the following result for the resummed total t-tbar cross-section:

Hard function. Known exactly at 1 loop.

Czakon, Mitov '08
Hagiwara, Sumino, Yokoya '08

$$\frac{\sigma^P(N,m^2,\mu^2)}{\sigma^P_{\rm Born}(N)} = {\rm Tr}\Bigg[\mathbf{H}^P(m^2,\mu^2)\ \exp\Bigg\{\int_0^1 dx \frac{x^{N-1}-1}{1-x} \times \left(\int_{\mu_F^2}^{4m^2(1-x)^2} \frac{dq^2}{q^2}\, 2\, A_P\left(\alpha_s\left[q^2\right]\right)\mathbf{1} + D^P_{Q\overline{Q}}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right)\Bigg\}\Bigg]$$

And the anomalous dimension is:

Jet functions (from Drell-Yan/Higgs)

$$D^P_{Q\overline{Q}} = \frac{\alpha_s(\mu^2)}{\pi}(-C_A)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\left\{D^{(2)}_{\rm P}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \left(-C_A\frac{K}{2} - \frac{\zeta_3-1}{2}C_A^2 - C_A\frac{\beta_0}{2}\right)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}$$

Fixed by the small-mass limit of the massive formfactor!

Czakon, Mitov, Sterman '09
Beneke, Falgari, Schwinn '09

Get the cross-section

How we put all this to work?

- Match fixed order and resummed results:

$$\sigma_{RESUM} = \sigma_{NLO} + \sigma_{SUDAKOV} - \sigma_{OVERLAP}$$

Now known at NNLO

- σ_{NLO} is known exactly,
- $\sigma_{SUDAKOV}$: anomalous dimensions and matching coefficients needed.

Known at NLO

i.e. at present one can derive the NLO+NNLL cross-section