QCD MASTER CLASS
SAINT-JACUT-de-la-MER, FRANCE

Helicity Evolution at Small x: Analytic Solution and Updates

2304.06161, 2406.11647 (preprint)

Jeremy Borden
The Ohio State University

Motivation

Proton Spin Sum Rule

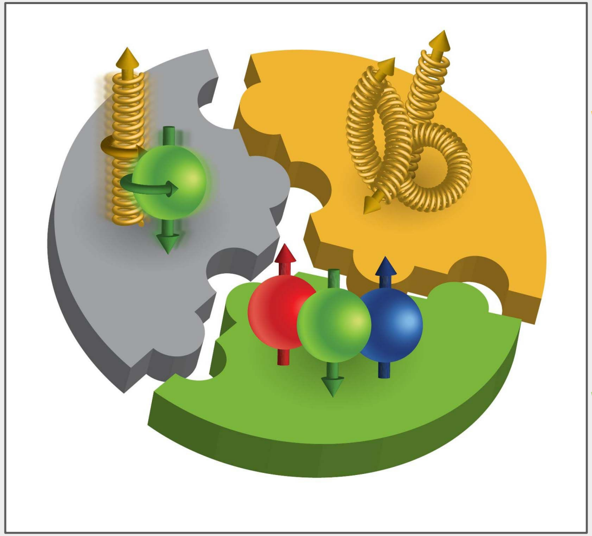
(Jaffe, Manohar) 10.1016/0550

$$S_q + L_q + S_G + L_G = \frac{1}{2}$$

$$S_G(Q^2) = \int_0^1 \mathrm{d}x \Delta G(x, Q^2)$$

$$S_q(Q^2) = \frac{1}{2} \int_0^1 \mathrm{d}x \Delta\Sigma(x, Q^2)$$

S_G , S_q are integrals over all values of Bjorken x



Motivation

$$S_q(Q^2) = \frac{1}{2}\int_0^1 \mathrm{d}x \Delta\Sigma(x, Q^2)$$

$$S_q(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.15 \div 0.20$$

for $x \in [0.001, 0.7]$

$$S_G(Q^2) = \int_0^1 \mathrm{d}x \Delta G(x, Q^2)$$

$$S_G(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.13 \div 0.26$$

for $x \in [0.05, 0.7]$

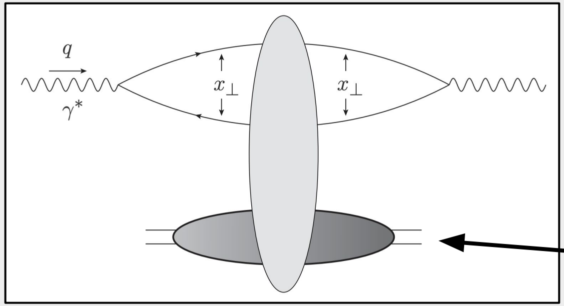
(see e.g. 1212.1701v3)

$$S_q + L_q + S_G + L_G = \frac{1}{2}$$

But current measured values still short of ½

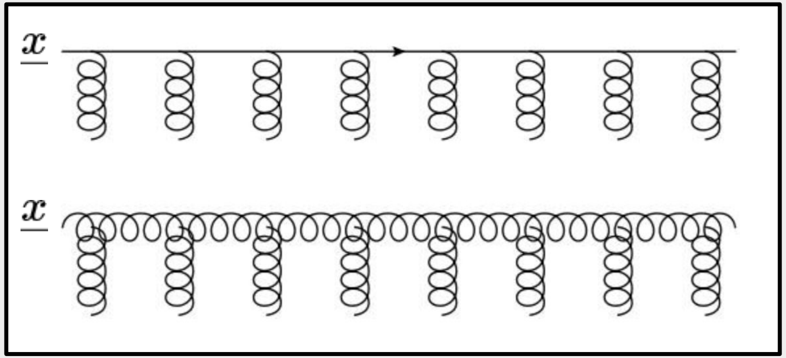
How much spin at small-*x*?

Building blocks of helicity evolution



Dipole picture of DIS

Target proton



In the unpolarized case, we work with Wilson lines

$$V_{\underline{x}}[x_f^-, x_i^-] = \mathcal{P}\exp\left[ig\int_{x_i^-}^{x_f^-} \mathrm{d}x^- \, A^+(0^+, x^-, \underline{x})\right]$$

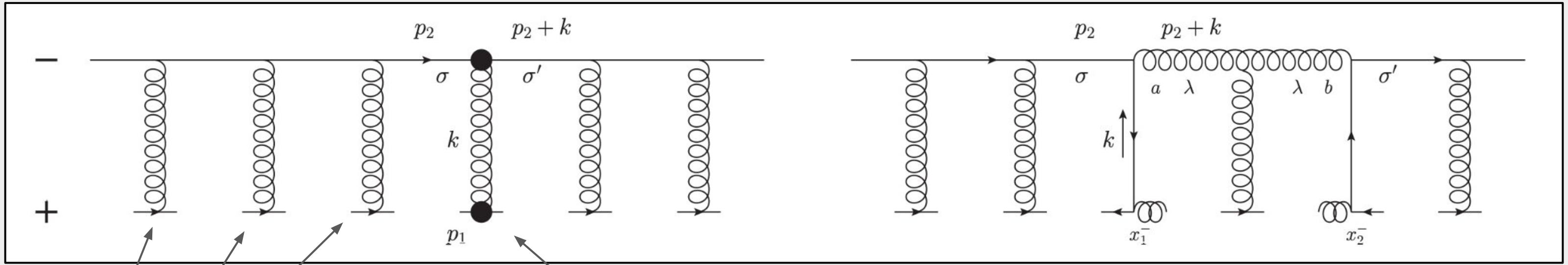
$$U_{\underline{x}}[x_f^-, x_i^-] = \mathcal{P}\exp\left[ig\int_{x_i^-}^{x_f^-} \mathrm{d}x^- \, \mathcal{A}^+(0^+, x^-, \underline{x})\right]$$

Light-cone coordinates: $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right)$

<u>But Wilson lines do not couple to the target proton's helicity</u>

Building blocks of helicity evolution

<u>We need ‘Polarized Wilson Lines’</u>



Regular soft gluon exchanges

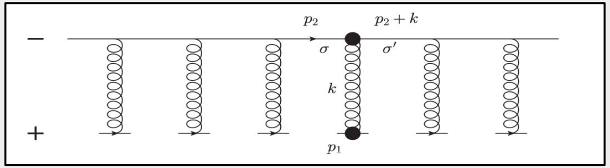
Energy suppressed, polarization-dependent interaction

These are Wilson lines with one extra polarization-dependent, sub-eikonal interaction inserted

(sub-eikonal = 1 extra power of energy suppression)

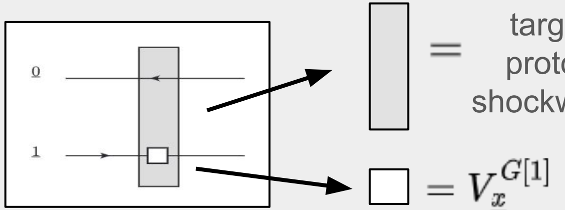
Building blocks of helicity evolution

Polarized Wilson line: $V_{\underline{x}}^{G[1]} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-] F^{12}(x^-, \underline{x}) V_{\underline{x}}[x^-, -\infty] \subset$



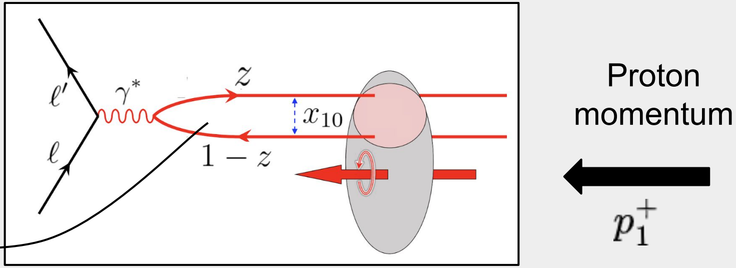
$$\mathrm{tr}\left[(\text{polarized Wilson line}) \times (\text{regular Wilson line})^\dagger\right] = \text{polarized dipole amplitude}$$

$$G_{10}(zs) = \frac{1}{2N_c} \mathrm{Re} \left\langle\left\langle \mathrm{T\,tr}\left[V_{\underline{0}} V_{\underline{1}}^{G[1]\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{1}}^{G[1]} V_{\underline{0}}^\dagger\right] \right\rangle\right\rangle(zs) \supset$$



Polarized dipole amplitude: Depends on transverse positions $\underline{x}_1$, $\underline{x}_0$ and COM energy for next step of evolution zs

z = longitudinal momentum fraction



Building blocks of helicity evolution

Multiple types of polarized Wilson lines ⟶ Multiple types of polarized dipole amplitudes

At large-N_c we have

$$G_{10}(zs) = \frac{1}{2N_c} \mathrm{Re} \left\langle\left\langle \mathrm{T}\, \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{G[1]\dagger}\right] + \mathrm{T}\, \mathrm{tr}\left[V_{\underline{1}}^{G[1]} V_{\underline{0}}^{\dagger}\right] \right\rangle\right\rangle (zs)$$

$$G_{10}^{i}(zs) = \frac{1}{2N_c} \left\langle\left\langle \mathrm{tr}\left[V_{\underline{0}}^{\dagger} V_{\underline{1}}^{iG[2]} + \left(V_{\underline{1}}^{iG[2]}\right)^{\dagger} V_{\underline{0}}\right] \right\rangle\right\rangle (zs)$$

$$\int \mathrm{d}^2 \left(\frac{x_0 + x_1}{2}\right) G_{10}(zs) = G(x_{10}^2, zs)$$

$$\int \mathrm{d}^2 \left(\frac{x_0 + x_1}{2}\right) G_{10}^{i}(zs) = (x_{10})_{\perp}^{i}\, G_1(x_{10}^2, zs) + \epsilon^{ij} (x_{10})_{\perp}^{j}\, G_2(x_{10}^2, zs)$$

Integrating over impact parameter

Building blocks of helicity evolution

Gluon helicity TMD

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{N_c}{\alpha_s 2\pi^4} \int \mathrm{d}^2 x_{10}\, e^{-i\underline{k}\cdot\underline{x}_{10}} \left[1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2}\right] G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)$$

Flavor Singlet quark helicity TMD

$$g_{1L}^{S}(x,k_T^2) = \frac{8iN_cN_f}{(2\pi)^5} \int\limits_{\Lambda^2/s}^{1} \frac{\mathrm{d}z}{z} \int \mathrm{d}^2 x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}} \frac{\underline{x}_{10}}{x_{10}^2} \cdot \frac{\underline{k}}{k^2} \left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

Gluon helicity PDF

$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2}\right) G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]_{x_{10}^2 = 1/Q^2}$$

Flavor Singlet quark helicity PDF

$$\Delta\Sigma(x,Q^2) = -\frac{N_cN_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{\mathrm{d}z}{z} \int\limits_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\}} \frac{\mathrm{d}x_{10}^2}{x_{10}^2} \left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

g_1 structure function

$$g_1(x,Q^2) = -\sum_f \frac{N_c Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{\mathrm{d}z}{z} \int\limits_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\}} \frac{\mathrm{d}x_{10}^2}{x_{10}^2} \left[G(x_{10}^2, zs) + 2G_2(x_{10}^2, zs)\right]$$

Helicity evolution at large-N_c

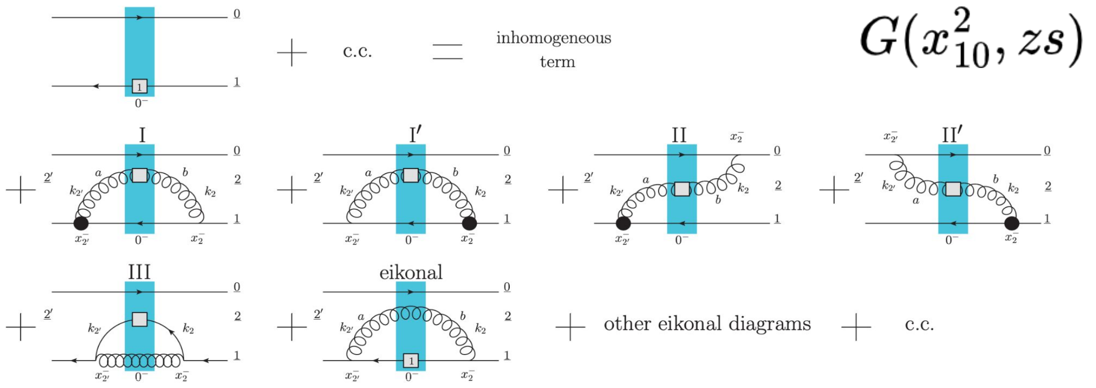
Cougoulic, Kovchegov, Tarasov, Tawabutr 2204.11898v3
{Kovchegov, Pitonyak, Sievert} 1511.06737v3, 1808.09010v1, 1610.06197v1, 1706.04236v3

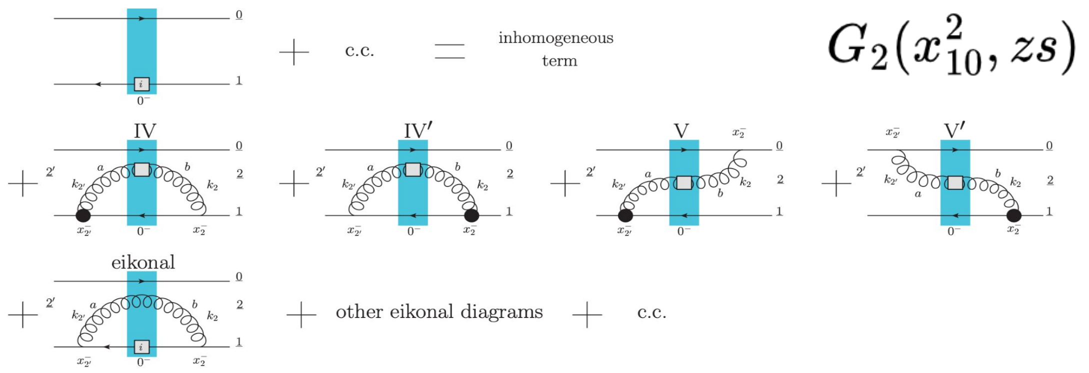
Small-*x* evolution of polarized dipole amplitudes

Double-logarithmic - resumming powers of

$$\alpha_s \ln^2(1/x)$$

Full evolution equations (beyond large-N_c) don't close, like Balitsky hierarchy
See Balitsky 9509348v1, 9812311v1





Helicity evolution at large-N_c

Cougoulic, Kovchegov, Tarasov, Tawabutr
2204.11898v3

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z's) + 3G(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) + 2\Gamma_2(x_{10}^2, x_{21}^2, z's)\right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3G(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) + 2\Gamma_2(x_{10}^2, x_{32}^2, z''s)\right]$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{\max\{x_{10}^2, \frac{1}{z's}\}}^{\min\{x_{10}^2 \frac{z}{z'}, \frac{1}{\Lambda^2}\}} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2G_2(x_{21}^2, z's)\right]$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\Lambda^2/s}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\{x_{10}^2, \frac{1}{z''s}\}}^{\min\{x_{21}^2 \frac{z'}{z''}, \frac{1}{\Lambda^2}\}} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s)\right]$$

Γ and Γ_2 are auxiliary functions ('neighbor dipole amplitudes')

Analytic solution of large-$\mathbf{N_c}$ equations

Write both polarized dipole amplitudes as double-inverse-Laplace transforms

$$G_2(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$G(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{\omega\gamma}$$

Neighbor dipole amplitudes
$\Gamma(x_{10}^2, x_{21}^2, z's)$ and $\Gamma_2(x_{10}^2, x_{21}^2, z's)$
depend on an additional transverse separation - complicates things, but still solvable

Analytic solution of large-N_c equations

Borden, Kovchegov 2304.06161

$$G_2(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zs x_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$G(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zs x_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{\omega\gamma}{2\overline{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) - 2 G_{2\omega\gamma} \right]$$

$$\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

$$G_{2\omega\gamma} = G_{2\omega\gamma}^{(0)} + \frac{\overline{\alpha}_s}{\omega\left(\gamma - \gamma_\omega^-\right)\left(\gamma - \gamma_\omega^+\right)} \left[2\left(\gamma - \delta_\omega^+\right)\left(G_{\delta_\omega^+ \gamma}^{(0)} + 2 G_{2\delta_\omega^+ \gamma}^{(0)} \right) - 2\left(\gamma_\omega^+ - \delta_\omega^+\right)\left(G_{\delta_\omega^+ \gamma_\omega^+}^{(0)} + 2 G_{2\delta_\omega^+ \gamma_\omega^+}^{(0)} \right) + 8\delta_\omega^- \left(G_{2\omega\gamma}^{(0)} - G_{2\omega\gamma_\omega^+}^{(0)} \right) \right]$$

$$\delta_\omega^\pm = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}} \right]$$

$$\gamma_\omega^\pm = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}} \right]$$

Note $G_{2\omega\gamma}^{(0)}$, $G_{\omega\gamma}^{(0)}$ are the double-Laplace images of the initial conditions $G_2^{(0)}(x_{10}^2, zs)$, $G^{(0)}(x_{10}^2, zs)$

Analytic solution of large-N_c equations

Resummed Anomalous Dimension

Fix some simple initial conditions: $G_2^{(0)}(x_{10}^2, zs) = 1 \qquad G^{(0)}(x_{10}^2, zs) = 0$

Gluon helicity PDF becomes:
$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \int \frac{d\omega}{2\pi i} e^{\omega \ln\left(\frac{1}{x}\right) + \gamma_\omega^- \ln\left(\frac{Q^2}{\Lambda^2}\right)} \frac{1}{\omega}$$

Pure-glue polarized anomalous dimension

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}}\right] = \frac{4\overline{\alpha}_s}{\omega} + \frac{8\overline{\alpha}_s^2}{\omega^3} + \frac{56\overline{\alpha}_s^3}{\omega^5} + \frac{496\overline{\alpha}_s^4}{\omega^7} + \mathcal{O}(\alpha_s^5)$$

$$\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

Agrees with finite-order calculations up to $\mathcal{O}\left(\alpha_s^3\right)$

Altarelli, Parisi 10.1016/0550-3213(77)90384-4
Mertig & van Neerven 9506451v3
Moch, Vermaseren, & Vogt 1409.5131v1
Blümlein, Marquard, Schneider, & Schönwald 2111.12401v2

Analytic solution of large-$\mathbf{N_c}$ equations

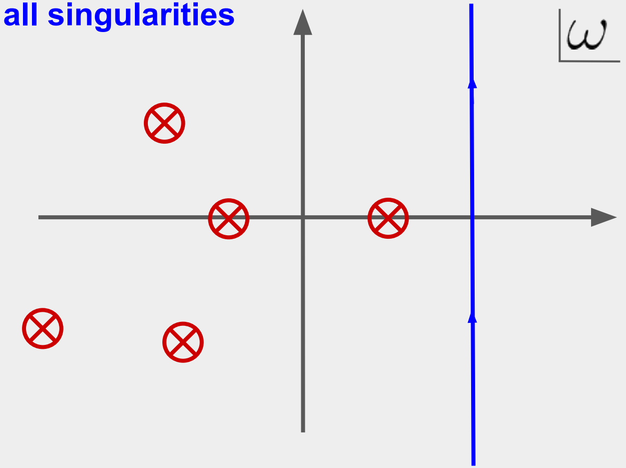
Small-x Asymptotics

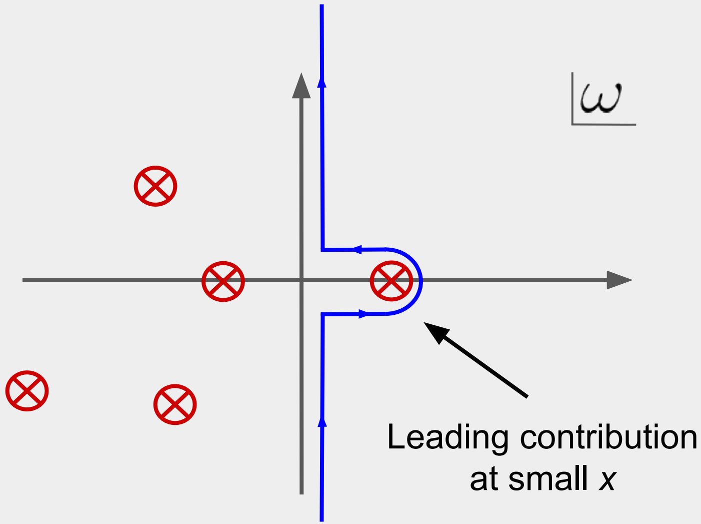
Asymptotics governed by the intercept α_h → $\Delta\Sigma(x,Q^2) \sim \Delta G(x,Q^2) \sim g_1(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h}$

↘ corresponds to the rightmost singularity in the ω-plane

Contour for inverse Laplace - parallel to imaginary axis, right of all singularities

$$F(t) = \int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega t} f_\omega$$





Analytic solution of large-$\mathbf{N_c}$ equations

Rightmost singularity comes from the polarized anomalous dimension: $\Delta\gamma_{GG}(\omega) = \gamma_\omega^-$

$$\gamma_\omega^- = \frac{\omega}{2}\left[1 - \underline{\sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}}\right]$$

Branch point from the large square root

$$\alpha_h = \frac{4}{3^{1/3}}\sqrt{\mathrm{Re}\left[\left(-9 + i\sqrt{111}\right)^{1/3}\right]}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66074\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Large-$\mathrm{N_c}$ comparison to BER

Bartels, Ermolaev, and Ryskin (BER) IR evolution equations (Bartels, Ermolaev, Ryskin 9603204v1)

Polarized GG anomalous dimension

$$\Delta\gamma_{GG}^{\mathrm{BER}}(\omega) = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\,\frac{1-\frac{3\overline{\alpha}_s}{\omega^2}}{1-\frac{\overline{\alpha}_s}{\omega^2}}}\,\right] = \frac{4\overline{\alpha}_s}{\omega} + \frac{8\overline{\alpha}_s^2}{\omega^3} + \frac{56\overline{\alpha}_s^3}{\omega^5} + \frac{504\overline{\alpha}_s^4}{\omega^7} + \mathcal{O}(\alpha_s^5)$$

Compare to us

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}}\,\right] = \frac{4\overline{\alpha}_s}{\omega} + \frac{8\overline{\alpha}_s^2}{\omega^3} + \frac{56\overline{\alpha}_s^3}{\omega^5} + \frac{496\overline{\alpha}_s^4}{\omega^7} + \mathcal{O}(\alpha_s^5)$$

Large-N_c comparison to BER

Bartels, Ermolaev, and Ryskin (BER) IR evolution equations (Bartels, Ermolaev, Ryskin 9603204v1)

Small-*x* (pure-glue) intercept

$$\alpha_h^{\text{BER}} = \sqrt{\frac{17+\sqrt{97}}{2}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66394\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Compare to us

$$\alpha_h = \frac{4}{3^{1/3}}\sqrt{\text{Re}\left[\left(-9+i\sqrt{111}\right)^{1/3}\right]}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66074\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

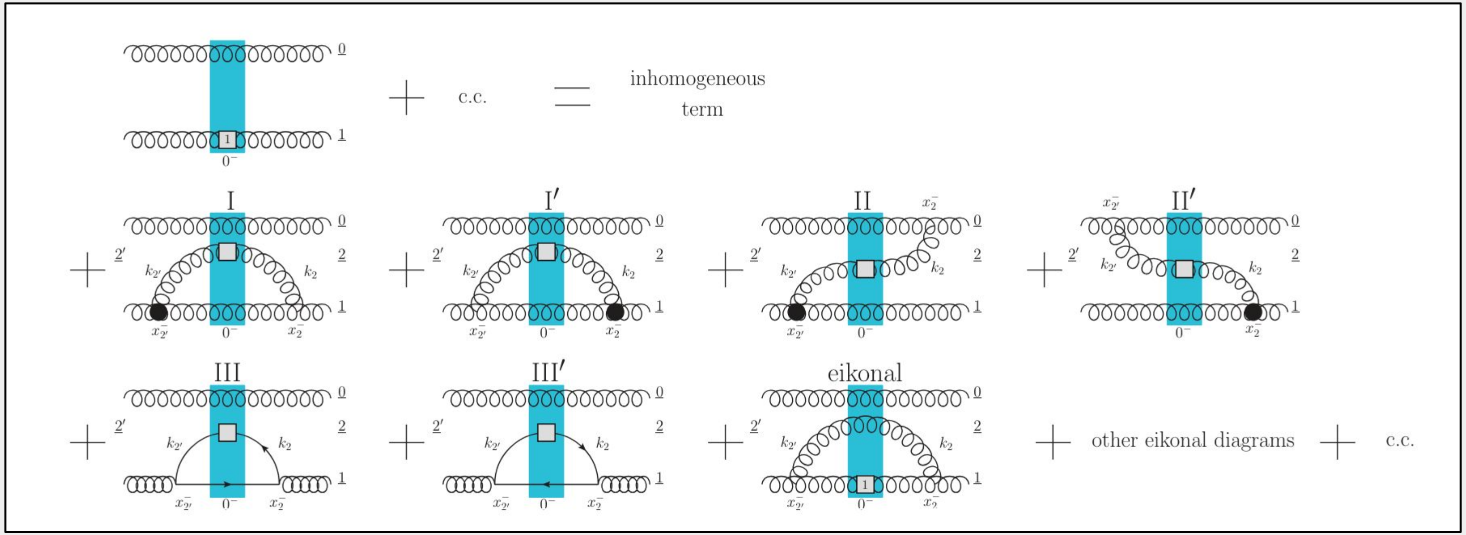
Why the (*very small*) disagreements with BER? No hard non-ladder gluons in IREE (?)

Kovchegov, Pitonyak, & Sievert 1610.06197v1
See also Boussarie, Hatta, Yuan 1904.02693v2

Helicity evolution at large-N_c&N_f

At large-N_c there is a simple relationship between fundamental and adjoint polarized dipole amplitudes, so no need for adjoint dipole's evolution there.

- Does not hold at large-N_c&N_f so we need to construct evolution of adjoint dipole



Cougoulic, Kovchegov, Tarasov, Tawabutr [2204.11898v3](https://arxiv.org/abs/2204.11898v3)

Helicity evolution at large-$N_c \& N_f$

$$Q(x_{10}^2, zs) = Q^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\widetilde{G}(x_{21}^2, z's) + 2\widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) + Q(x_{21}^2, z's) - \overline{\Gamma}(x_{10}^2, x_{21}^2, z's) + 2\Gamma_2(x_{10}^2, x_{21}^2, z's) + 2G_2(x_{21}^2, z's) \right] + \frac{\alpha_s N_c}{4\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{\min\{x_{10}^2 \frac{z}{z'}, \frac{1}{\Lambda^2}\}} \frac{dx_{21}^2}{x_{21}^2} \left[Q(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) \right]$$

$$\overline{\Gamma}(x_{10}^2, x_{21}^2, z's) = Q^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\widetilde{G}(x_{32}^2, z''s) + 2\widetilde{\Gamma}(x_{10}^2, x_{32}^2, z''s) + Q(x_{32}^2, z''s) - \overline{\Gamma}(x_{10}^2, x_{32}^2, z''s) + 2\Gamma_2(x_{10}^2, x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) \right] + \frac{\alpha_s N_c}{4\pi} \int\limits_{\Lambda^2/s}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{21}^2 \frac{z'}{z''}, \frac{1}{\Lambda^2}\}} \frac{dx_{32}^2}{x_{32}^2} \left[Q(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) \right]$$

$$\widetilde{G}(x_{10}^2, zs) = \widetilde{G}^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z} \frac{dz'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[3\widetilde{G}(x_{21}^2, z's) + \widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) + 2G_2(x_{21}^2, z's) + \left(2 - \frac{N_f}{2N_c}\right) \Gamma_2(x_{10}^2, x_{21}^2, z's) - \frac{N_f}{4N_c} \overline{\Gamma}(x_{10}^2, x_{21}^2, z's) \right] - \frac{\alpha_s N_f}{8\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{\max\{x_{10}^2, \frac{1}{z's}\}}^{\min\{x_{10}^2 \frac{z}{z'}, \frac{1}{\Lambda^2}\}} \frac{dx_{21}^2}{x_{21}^2} \left[Q(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) \right]$$

$$\widetilde{\Gamma}(x_{10}^2, x_{21}^2, z's) = \widetilde{G}^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z'} \frac{dz''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\}} \frac{dx_{32}^2}{x_{32}^2} \left[3\widetilde{G}(x_{32}^2, z''s) + \widetilde{\Gamma}(x_{10}^2, x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) + \left(2 - \frac{N_f}{2N_c}\right) \Gamma_2(x_{10}^2, x_{32}^2, z''s) - \frac{N_f}{4N_c} \overline{\Gamma}(x_{10}^2, x_{32}^2, z''s) \right] - \frac{\alpha_s N_f}{8\pi} \int\limits_{\Lambda^2/s}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\{x_{10}^2, \frac{1}{z''s}\}}^{\min\{x_{21}^2 \frac{z'}{z''}, \frac{1}{\Lambda^2}\}} \frac{dx_{32}^2}{x_{32}^2} \left[Q(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) \right]$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\Lambda^2/s}^{z} \frac{dz'}{z'} \int\limits_{\max\{x_{10}^2, \frac{1}{z's}\}}^{\min\{x_{10}^2 \frac{z}{z'}, \frac{1}{\Lambda^2}\}} \frac{dx_{21}^2}{x_{21}^2} \left[\widetilde{G}(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) \right]$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\Lambda^2/s}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\{x_{10}^2, \frac{1}{z''s}\}}^{\min\{x_{21}^2 \frac{z'}{z''}, \frac{1}{\Lambda^2}\}} \frac{dx_{32}^2}{x_{32}^2} \left[\widetilde{G}(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) \right]$$

Cougoulic, Kovchegov, Tarasov, Tawabutr
2204.11898v3
Adamiak, Kovchegov, Tawabutr
2306.01651

Helicity evolution at large-N_c&N_f

But the equations on the previous slide are not quite right! How do we know?

Solve the equations iteratively and extract polarized DGLAP splitting functions

→ Disagree with those calculated in $\overline{\text{MS}}$ scheme (and those of BER), with no scheme transformation between them.

Adamiak, Kovchegov, Tawabutr 2306.01651

Us	$\overline{\text{MS}}$ (small-x, large-N_c&N_f limit)
$\Delta P_{qq}(z) = \left(\frac{\alpha_s N_c}{4\pi}\right) + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \left(\frac{1}{2} - 4\frac{N_f}{N_c}\right) \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$	$\Delta \bar{P}_{qq}(z) = \left(\frac{\alpha_s N_c}{4\pi}\right) + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \left(\frac{1}{2} - 2\frac{N_f}{N_c}\right) \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$
$\Delta P_{qG}(z) = -\left(\frac{\alpha_s N_c}{4\pi}\right) \frac{2N_f}{N_c} - \left(\frac{\alpha_s N_c}{4\pi}\right)^2 13\frac{N_f}{N_c} \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$	$\Delta \bar{P}_{qG}(z) = -\left(\frac{\alpha_s N_c}{4\pi}\right) \frac{2N_f}{N_c} - \left(\frac{\alpha_s N_c}{4\pi}\right)^2 5\frac{N_f}{N_c} \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$
$\Delta P_{Gq}(z) = 2\left(\frac{\alpha_s N_c}{4\pi}\right) + 8\left(\frac{\alpha_s N_c}{4\pi}\right)^2 \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$	$\Delta \bar{P}_{Gq}(z) = 2\left(\frac{\alpha_s N_c}{4\pi}\right) + 5\left(\frac{\alpha_s N_c}{4\pi}\right)^2 \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$
$\Delta P_{GG}(z) = 8\left(\frac{\alpha_s N_c}{4\pi}\right) + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \left(16 - 0\frac{N_f}{N_c}\right) \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$	$\Delta \bar{P}_{GG}(z) = 8\left(\frac{\alpha_s N_c}{4\pi}\right) + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \left(16 - 2\frac{N_f}{N_c}\right) \ln^2 \frac{1}{z} + \mathcal{O}\left(\alpha_s^3\right)$

Modifying the large-$N_c\&N_f$ evolution

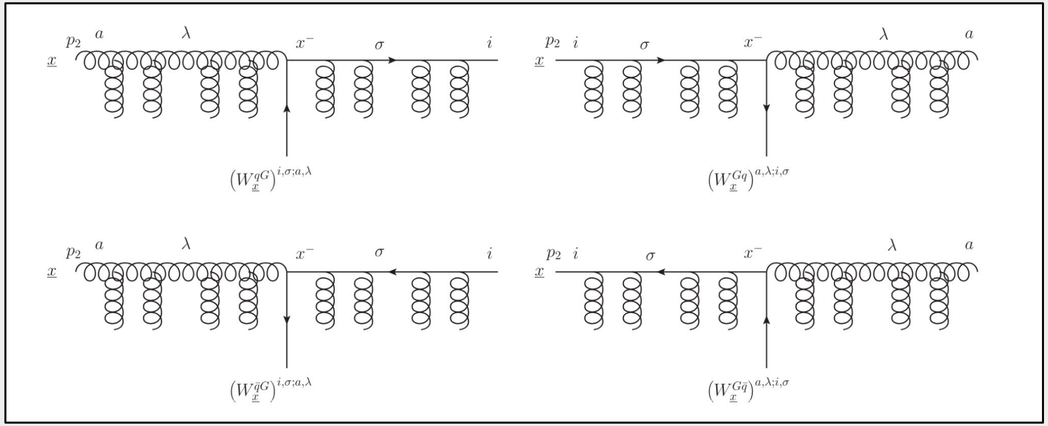
Our evolution contained only contributions where the interaction with the target does not change particle type, i.e.

$$q \to q\,,\ \bar{q} \to \bar{q}\,,\ G \to G$$

We must also include the interactions that do change particle type:

$$q \to G\,,\ \bar{q} \to G\,,\ G \to q\,,\ G \to \bar{q}$$

This is accomplished with a set of four new transition operators

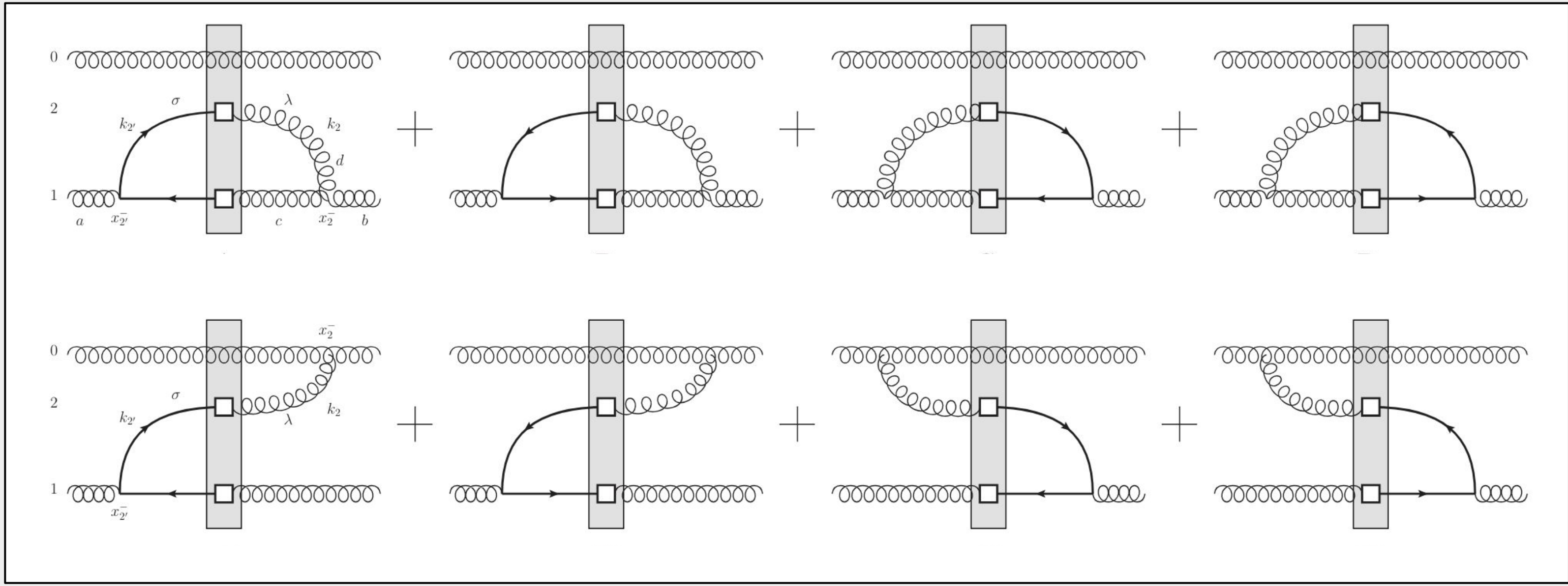


Borden, Kovchegov, Li
2406.11647
See also: Chirilli
2101.12744

Modifying the large-N_c&N_f evolution

New transition diagrams to include in the evolution

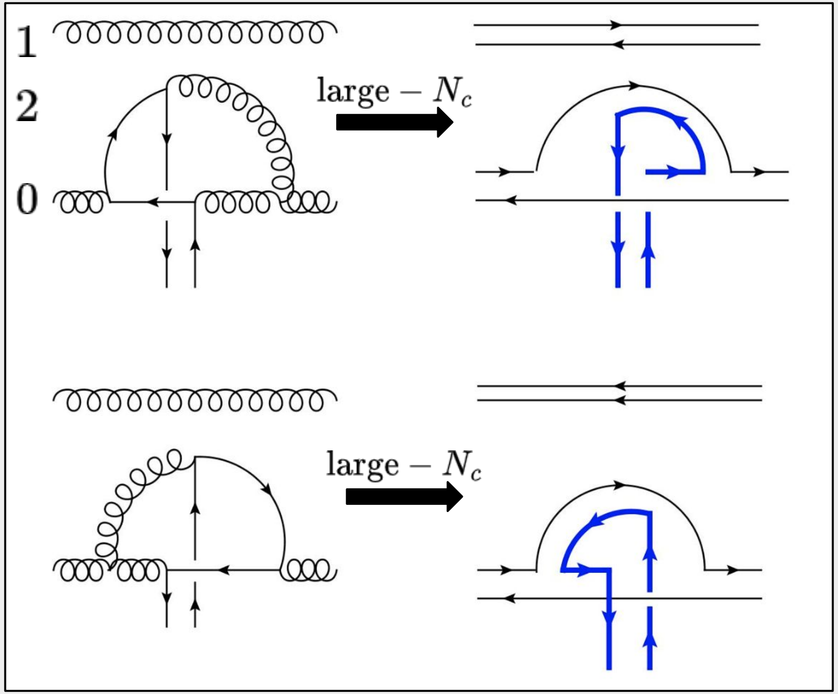
Starting with adjoint dipole ($\tilde{G}$) here. Contribution to fundamental dipole is N_c- suppressed.



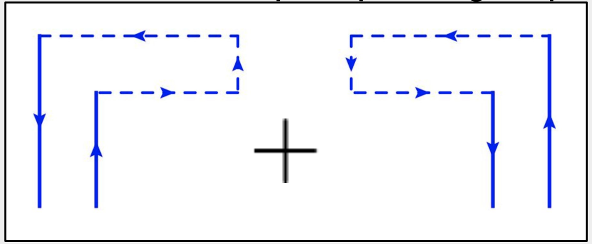
Here □ = one of the $q \to G$, $\bar{q} \to G$, $G \to q$, $G \to \bar{q}$ operators from the previous slide

Modifying the large-N_c&N_f evolution

New structure emerges at large-N_c

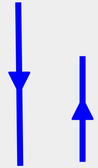


Sum of future- and past-pointing staples

$\tilde{Q} \propto$ 

$$\propto \int \mathrm{d}^2k_\perp e^{i\underline{k}\cdot\underline{x}_{10}}\, g_{1L}^S(x, k_T^2)$$

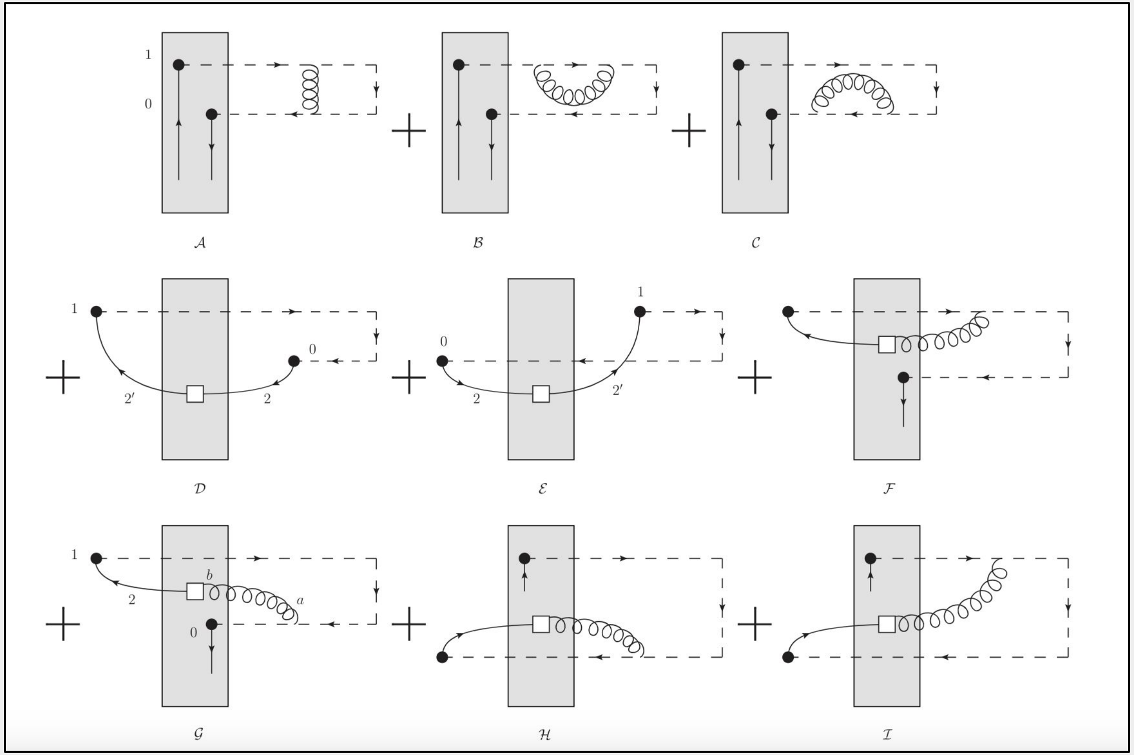
$$\propto \Delta\Sigma(x, Q^2)$$

 = quark/antiquark fields exchanged with target proton

 = fundamental Wilson line

Modifying the large-$N_c\&N_f$ evolution

Need to evolve our new structure (only future-pointing staple shown here, but past-pointing contribution needed as well)



Modifying the large-$N_c\&N_f$ evolution

Modification to evolution of adjoint dipole (neighbor modified similarly):

$$\delta\widetilde{G}(x_{10}^2, zs) = -\frac{\alpha_s N_f}{4\pi}\int\limits_{1/(sx_{10}^2)}^{z}\frac{dz'}{z'}\int\limits_{1/(z's)}^{x_{10}^2}\frac{dx_{21}^2}{x_{21}^2}\,\widetilde{Q}(x_{21}^2, z's)$$

Evolution of new object $\widetilde{Q}$:

$$\widetilde{Q}(x_{10}^2, zs) = \widetilde{Q}^{(0)}(x_{10}^2, zs) - \frac{\alpha_s N_c}{2\pi}\int\limits_{\Lambda^2/s}^{z}\frac{dz'}{z'}\int\limits_{\max\{x_{10}^2, \frac{1}{z's}\}}^{\min\{\frac{z}{z'}x_{10}^2, \frac{1}{\Lambda^2}\}}\frac{dx_{21}^2}{x_{21}^2}\left[Q(x_{21}^2, z's) + 2G_2(x_{21}^2, z's)\right]$$

<u>Can now solve the modified set of large-$N_c\&N_f$ equations to iteratively extract the polarized DGLAP splitting functions</u>

New predictions for large-$\mathrm{N_c}$&$\mathrm{N_f}$ polarized splitting functions

To three loops we predict:

$$\Delta P_{qq}(x) = \frac{\alpha_s N_c}{4\pi} + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \left(\frac{1}{2} - 2\frac{N_f}{N_c}\right) \ln^2\frac{1}{x} + \left(\frac{\alpha_s N_c}{4\pi}\right)^3 \frac{1}{12}\left(1 - 20\frac{N_f}{N_c}\right) \ln^4\frac{1}{x} + \mathcal{O}(\alpha_s^4)$$

$$\Delta P_{qG}(x) = -\left(\frac{\alpha_s N_c}{4\pi}\right)\frac{2N_f}{N_c} - \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \frac{5N_f}{N_c}\ln^2\frac{1}{x} - \left(\frac{\alpha_s N_c}{4\pi}\right)^3 \frac{1}{6}\frac{N_f}{N_c}\left(35 - 4\frac{N_f}{N_c}\right) \ln^4\frac{1}{x} + \mathcal{O}(\alpha_s^4)$$

$$\Delta P_{Gq}(x) = 2\left(\frac{\alpha_s N_c}{4\pi}\right) + 5\left(\frac{\alpha_s N_c}{4\pi}\right)^2 \ln^2\frac{1}{x} + \left(\frac{\alpha_s N_c}{4\pi}\right)^3 \frac{1}{6}\left(35 - 4\frac{N_f}{N_c}\right) \ln^4\frac{1}{x} + \mathcal{O}(\alpha_s^4)$$

$$\Delta P_{GG}(x) = 8\left(\frac{\alpha_s N_c}{4\pi}\right) + \left(\frac{\alpha_s N_c}{4\pi}\right)^2 \left(16 - 2\frac{N_f}{N_c}\right) \ln^2\frac{1}{x} + \left(\frac{\alpha_s N_c}{4\pi}\right)^3 \frac{1}{3}\left(56 - 11\frac{N_f}{N_c}\right) \ln^4\frac{1}{x} + \mathcal{O}(\alpha_s^4)$$

Complete agreement to three loops with predictions of BER
and with $\overline{\mathrm{MS}}$ after a scheme transformation

(Bartels, Ermolaev, Ryskin 9603204v1)
(Blümlein, Vogt 9606254)

New predictions for large-$N_c \& N_f$ polarized splitting functions

To four loops:

(Bartels, Ermolaev, Ryskin 9603204v1)
(Blümlein, Vogt 9606254

Us	BER
$\Delta P_{qq}^{(3)}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{720}\left(5 - 748\frac{N_f}{N_c} + 80\frac{N_f^2}{N_c^2}\right)\ln^6\frac{1}{x}$	$\Delta P_{qq}^{(3)(\mathrm{BER})}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{720}\left(5 - 764\frac{N_f}{N_c} + 80\frac{N_f^2}{N_c^2}\right)\ln^6\frac{1}{x}$
$\Delta P_{qG}^{(3)}(x) = -\left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{360}\frac{N_f}{N_c}\left(1213 - 224\frac{N_f}{N_c}\right)\ln^6\frac{1}{x}$	$\Delta P_{qG}^{(3)(\mathrm{BER})}(x) = -\left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{360}\frac{N_f}{N_c}\left(1229 - 224\frac{N_f}{N_c}\right)\ln^6\frac{1}{x}$
$\Delta P_{Gq}^{(3)}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{360}\left(1213 - 224\frac{N_f}{N_c}\right)\ln^6\frac{1}{x}$	$\Delta P_{Gq}^{(3)(\mathrm{BER})}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{360}\left(1229 - 224\frac{N_f}{N_c}\right)\ln^6\frac{1}{x}$
$\Delta P_{GG}^{(3)}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{180}\left(1984 - 549\frac{N_f}{N_c} + 20\frac{N_f^2}{N_c^2}\right)\ln^6\frac{1}{x}$	$\Delta P_{GG}^{(3)(\mathrm{BER})}(x) = \left(\frac{\alpha_s N_c}{4\pi}\right)^4 \frac{1}{180}\left(2016 - 557\frac{N_f}{N_c} + 20\frac{N_f^2}{N_c^2}\right)\ln^6\frac{1}{x}$

Minor disagreement at four loops, but consistent with disagreement already seen at large-N_c

Summary

- Small-x helicity evolution in s-channel/shockwave formalism - novel small-x evolution equations for polarized dipole amplitudes governing helicity PDFs and helicity TMDs.
- Equations solved analytically at large-N_c
  - Small-x asymptotics very close to predictions of BER, disagreement beginning at the third decimal place in the intercept.
  - Predicted polarized GG anomalous dimension agrees with finite order calculations (to 3 loops) but disagrees minorly with BER prediction beginning at 4 loops.
- Large-$N_c \& N_f$ evolution shown to *disagree* with finite order calculations, beginning at two loops.
  - Corrected by including contributions of shockwave transition operators.
  - New evolution eqns solved iteratively. Polarized DGLAP splitting functions extracted to four loops.
    - Full agreement to 3 loops with BER, and consistent with $\overline{\text{MS}}$ after a scheme transformation.
    - Small disagreement with predictions of BER beginning at four loops, consistent with large-N_c disagreement beginning at that same order.
    - **Positive outlook for phenomenological applications**
- New large-$N_c \& N_f$ equations should also be amenable to analytic solution.

Extra Slides

Sub-eikonal quark S-matrix (Extra Slides)

$$V_{\underline{x},\underline{y};\sigma',\sigma}\Big|_{\text{sub-eikonal}} \equiv \sigma\delta_{\sigma,\sigma'}\left[V_{\underline{x}}^{G[1]} + V_{\underline{x}}^{q[1]}\right]\delta^2\left(\underline{x}-\underline{y}\right) + \delta_{\sigma,\sigma'}\left[V_{\underline{x},\underline{y}}^{G[2]} + V_{\underline{x}}^{q[2]}\delta^2\left(\underline{x}-\underline{y}\right)\right]$$

$$V_{\underline{x}}^{G[1]} = \frac{igp_1^+}{s}\int\limits_{-\infty}^{\infty} \mathrm{d}x^- V_{\underline{x}}\left[\infty, x^-\right]F^{12}\left(x^-,\underline{x}\right)V_{\underline{x}}\left[x^-,-\infty\right]$$

$$V_{\underline{x}}^{q[1]} = \frac{g^2p_1^+}{2s}\int\limits_{-\infty}^{\infty}\mathrm{d}x_1^-\int\limits_{x_1^-}^{\infty}\mathrm{d}x_2^- V_{\underline{x}}\left[\infty,x_2^-\right]t^b\psi_\beta\left(x_2^-,\underline{x}\right)U_{\underline{x}}^{ba}\left[x_2^-,x_1^-\right]\left[\gamma^+\gamma^5\right]_{\alpha\beta}\bar{\psi}_\alpha\left(x_1^-,\underline{x}\right)t^aV_{\underline{x}}\left[x_1^-,-\infty\right]$$

$$V_{\underline{x},\underline{y}}^{G[2]} = -\frac{ip_1^+}{s}\int\limits_{-\infty}^{\infty}\mathrm{d}z^-\mathrm{d}^2z V_{\underline{x}}\left[\infty,z^-\right]\delta^2\left(\underline{x}-\underline{z}\right)\overleftarrow{D}^i\left(z^-,\underline{z}\right)D^i\left(z^-,\underline{z}\right)V_{\underline{y}}\left[z^-,-\infty\right]\delta^2\left(\underline{y}-\underline{z}\right)$$

$$V_{\underline{x}}^{q[2]} = -\frac{g^2p_1^+}{2s}\int\limits_{-\infty}^{\infty}\mathrm{d}x_1^-\int\limits_{x_1^-}^{\infty}\mathrm{d}x_2^- V_{\underline{x}}\left[\infty,x_2^-\right]t^b\psi_\beta\left(x_2^-,\underline{x}\right)U_{\underline{x}}^{ba}\left[x_2^-,x_1^-\right]\left[\gamma^+\right]_{\alpha\beta}\bar{\psi}_\alpha\left(x_1^-,\underline{x}\right)t^aV_{\underline{x}}\left[x_1^-,-\infty\right]$$

Sub-eikonal gluon S-matrix (Extra Slides)

$$\left(U_{\underline{x},\underline{y};\lambda',\lambda}\right)^{ba}\Big|_{\text{sub-eikonal}} \equiv \lambda\delta_{\lambda,\lambda'}\left(U_{\underline{x}}^{G[1]}+U_{\underline{x}}^{q[1]}\right)^{ba}\delta^2\left(\underline{x}-\underline{y}\right)+\delta_{\lambda,\lambda'}\left(U_{\underline{x},\underline{y}}^{G[2]}+U_{\underline{x}}^{q[2]}\delta^2\left(\underline{x}-\underline{y}\right)\right)^{ba}$$

$$\left(U_{\underline{x}}^{G[1]}\right)^{ba}=\frac{2ig\,p_1^+}{s}\int\limits_{-\infty}^{\infty}\mathrm{d}x^-\left(U_{\underline{x}}\left[\infty,x^-\right]\right)^{bb'}\left(\mathcal{F}^{12}\right)^{b'a'}\left(x^-,\underline{x}\right)\left(U_{\underline{x}}\left[x^-,-\infty\right]\right)^{a'a}$$

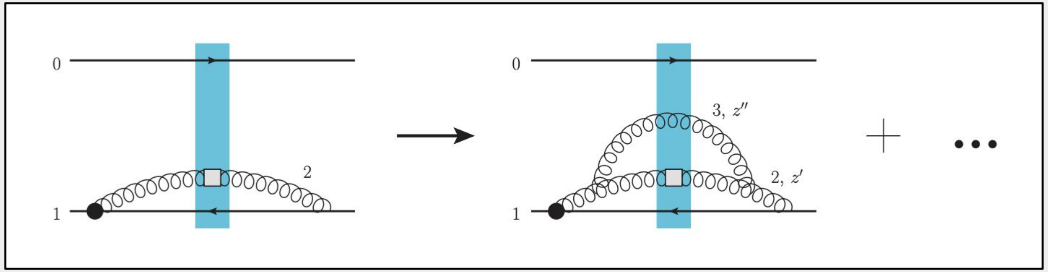
$$\left(U_{\underline{x}}^{q[1]}\right)^{ba}=\frac{g^2\,p_1^+}{2s}\int\limits_{-\infty}^{\infty}\mathrm{d}x_1^-\int\limits_{x_1^-}^{\infty}\mathrm{d}x_2^-\left(U_{\underline{x}}\left[\infty,x_2^-\right]\right)^{bb'}\bar{\psi}\left(x_2^-,\underline{x}\right)t^{b'}V_{\underline{x}}\left[x_2^-,x_1^-\right]\gamma^+\gamma^5t^{a'}\psi\left(x_1^-,\underline{x}\right)\left(U_{\underline{x}}\left[x_1^-,-\infty\right]\right)^{a'a}+\text{c.c.}$$

$$\left(U_{\underline{x},\underline{y}}^{G[2]}\right)^{ba}=-\frac{i\,p_1^+}{s}\int\limits_{-\infty}^{\infty}\mathrm{d}z^-\mathrm{d}^2z\left(U_{\underline{x}}\left[\infty,z^-\right]\right)^{bb'}\delta^2\left(\underline{x}-\underline{z}\right)\overleftarrow{\underline{\mathscr{D}}}^{b'c}\left(z^-,\underline{z}\right)\cdot\underline{\mathscr{D}}^{ca'}\left(z^-,\underline{z}\right)\left(U_{\underline{y}}\left[z^-,-\infty\right]\right)^{a'a}\delta^2\left(\underline{y}-\underline{z}\right)$$

$$\left(U_{\underline{x}}^{q[2]}\right)^{ba}=-\frac{g^2\,p_1^+}{2s}\int\limits_{-\infty}^{\infty}\mathrm{d}x_1^-\int\limits_{x_1^-}^{\infty}\mathrm{d}x_2^-\left(U_{\underline{x}}\left[\infty,x_2^-\right]\right)^{bb'}\bar{\psi}\left(x_2^-,\underline{x}\right)t^{b'}V_{\underline{x}}\left[x_2^-,x_1^-\right]\gamma^+t^{a'}\psi\left(x_1^-,\underline{x}\right)\left(U_{\underline{x}}\left[x_1^-,-\infty\right]\right)^{a'a}-\text{c.c.}$$

Neighbor Dipole Amplitudes (Extra Slides)

One step in evolution of neighbor dipole amplitude



(1) DLA lifetime ordering → $x_{21}^2 z' \gg x_{32}^2 z''$

(2) But also have IR cutoff for dipole 02 → $x_{32} \ll x_{20}$

When $x_{20}^2 > x_{21}^2 \frac{z'}{z''}$ → (1) is more constraining than (2)

<u>So for everything to be ordered properly, subsequent evolution in dipole 02 (here evolving to give dipole 32) 'knows' about dipole 21</u>

Full large-N_c solution (Extra Slides)

$$G_2(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zs x_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \left[e^{\omega \ln(z's x_{21}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) + e^{\omega \ln(z's x_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}^{(0)} \right]$$

$$G(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zs x_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{\omega\gamma}{2\overline{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) - 2G_{2\omega\gamma} \right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = \int \frac{d\omega}{2\pi i} e^{\omega \ln(z's x_{21}^2)} \left[\Gamma_\omega^+(x_{10}^2) e^{\delta_\omega^+ \ln\left(\frac{1}{x_{21}^2 \Lambda^2}\right)} + \Gamma_\omega^-(x_{10}^2) e^{\delta_\omega^- \ln\left(\frac{1}{x_{21}^2 \Lambda^2}\right)} \right]$$
$$+ \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(z's x_{21}^2) + \gamma \ln\left(\frac{1}{x_{21}^2 \Lambda^2}\right)} \left[\frac{\left(-\frac{3}{2}\omega\gamma + 4\overline{\alpha}_s\right) G_{2\omega\gamma} + \frac{3}{2}\omega\gamma G_{2\omega\gamma}^{(0)}}{\gamma^2 - \omega\gamma + \overline{\alpha}_s} \right]$$
$$- \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \left[2e^{\omega \ln(z's x_{21}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) + 2e^{\omega \ln(z's x_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}^{(0)} \right]$$

$$\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

Full large-N_c solution (Extra Slides)

$$G_{2\omega\gamma} = G^{(0)}_{2\omega\gamma} + \frac{\overline{\alpha}_s}{\omega\left(\gamma-\gamma_\omega^-\right)\left(\gamma-\gamma_\omega^+\right)}\left[2\left(\gamma-\delta_\omega^+\right)\left(G^{(0)}_{\delta_\omega^+\gamma} + 2G^{(0)}_{2\delta_\omega^+\gamma}\right) - 2\left(\gamma_\omega^+ - \delta_\omega^+\right)\left(G^{(0)}_{\delta_\omega^+\gamma_\omega^+} + 2G^{(0)}_{2\delta_\omega^+\gamma_\omega^+}\right) + 8\delta_\omega^-\left(G^{(0)}_{2\omega\gamma} - G^{(0)}_{2\omega\gamma_\omega^+}\right)\right]$$

$$G^{(0)}(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i} e^{\omega\ln\left(zsx_{10}^2\right)+\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G^{(0)}_{\omega\gamma}$$

$$G_2^{(0)}(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i} e^{\omega\ln\left(zsx_{10}^2\right)+\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G^{(0)}_{2\omega\gamma}$$

$$\Gamma_\omega^\pm(x_{10}^2) = \frac{e^{-\delta_\omega^\pm\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)}}{\bar\alpha_s\left(\delta_\omega^\pm - \delta_\omega^\mp\right)}\int\frac{d\gamma}{2\pi i}e^{\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)}\frac{\omega\delta_\omega^\pm}{2\left(\gamma-\delta_\omega^\pm\right)}\left[G_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\bar\alpha_s-\frac{8\bar\alpha_s}{\omega}\delta_\omega^\mp\right) - G^{(0)}_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\bar\alpha_s\right)\right]$$

$$\delta_\omega^\pm = \frac{\omega}{2}\left[1\pm\sqrt{1-\frac{4\overline{\alpha}_s}{\omega^2}}\right] \qquad \gamma_\omega^\pm = \frac{\omega}{2}\left[1\pm\sqrt{1-\frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1-\frac{4\overline{\alpha}_s}{\omega^2}}}\right]$$

Analytic solution of large-N_c equations (Extra Slides)

Can use the dipole amplitudes to obtain small-*x* large-N_c expressions for hTMDs and hPDFs (for arbitrary initial conditions)

Γ functions, not neighbor dipole amplitude

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{2N_c}{\alpha_s \pi^3}\frac{1}{k_T^2}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i} e^{\omega\ln\left(\frac{Q^2}{xk_T^2}\right)+\gamma\ln\left(\frac{k_T^2}{\Lambda^2}\right)} 2^{2\omega-2\gamma}\frac{\Gamma(\omega-\gamma+1)}{\Gamma(\gamma-\omega)} G_{2\omega\gamma}$$

$$g_{1L}^{S}(x,k_T^2) = -\frac{N_f}{\alpha_s 2\pi^3}\frac{1}{k_T^2}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i}\left[e^{\omega\ln\left(\frac{Q^2}{xk_T^2}\right)+\gamma\ln\left(\frac{k_T^2}{\Lambda^2}\right)} - e^{(\gamma-\omega)\ln\left(\frac{k_T^2}{\Lambda^2}\right)}\right] 2^{2\omega-2\gamma}\frac{\Gamma(\omega-\gamma+1)}{\Gamma(\gamma-\omega+1)}\gamma\left(G_{2\omega\gamma}-G_{2\omega\gamma}^{(0)}\right)$$

$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s\pi^2}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i}e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}G_{2\omega\gamma}$$

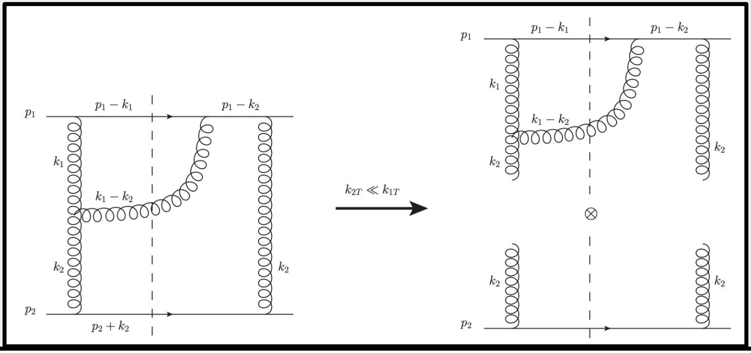
$$\Delta\Sigma(x,Q^2) = -\frac{N_f}{\alpha_s 2\pi^2}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i}\frac{\omega}{\omega-\gamma}\left(G_{2\omega\gamma}-G_{2\omega\gamma}^{(0)}\right)e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}$$

$$g_1(x,Q^2) = -\frac{1}{2}\sum_f Z_f^2\frac{1}{\alpha_s 2\pi^2}\int\frac{d\omega}{2\pi i}\int\frac{d\gamma}{2\pi i}\frac{\omega}{\omega-\gamma}\left(G_{2\omega\gamma}-G_{2\omega\gamma}^{(0)}\right)e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}$$

Disagreement with BER

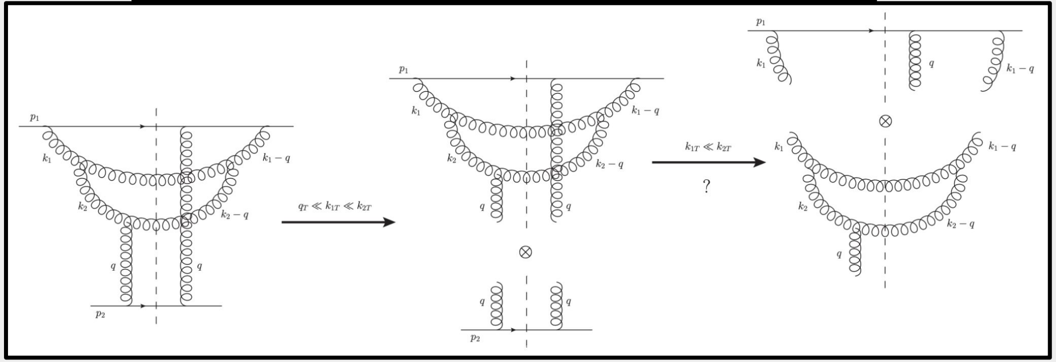
<u>No hard non-ladder gluons in IREE</u>

Two diagrams contained within our evolution



Ladder with rails $k_1 - k_2$ & k_2, (uncut) rung $p_1 - k_2$, and bremsstrahlung gluon k_1

Hard non-ladder gluon $k_1 - k_2$ accommodated at



3- and 5-point Green functions (BER IREE have only 4-point)

Problem at $\mathcal{O}(\alpha_s^4)$ (?)

Eigenvalues of anomalous dimension matrix (Extra Slides)

Can solve large-$N_c \& N_f$ equations analytically. The pole structure that emerges is

$$\frac{1}{\left(\gamma-\gamma_\omega^{--}\right)\left(\gamma-\gamma_\omega^{-+}\right)}(\dots)$$

These functions γ_ω^{--}, γ_ω^{-+} are the eigenvalues of the anomalous dimension matrix

Eigenvalue of matrix of anomalous dimensions

$$\begin{pmatrix}\Delta\gamma_{qq}(\omega) & \Delta\gamma_{qG}(\omega)\\ \Delta\gamma_{Gq}(\omega) & \Delta\gamma_{GG}(\omega)\end{pmatrix}$$

Finite order

Us

$$\left(\frac{\alpha_s}{4\pi}\right)\frac{1}{2}\left[9N_c+\sqrt{N_c\left(49N_c-16N_f\right)}\right]\frac{1}{\omega}$$

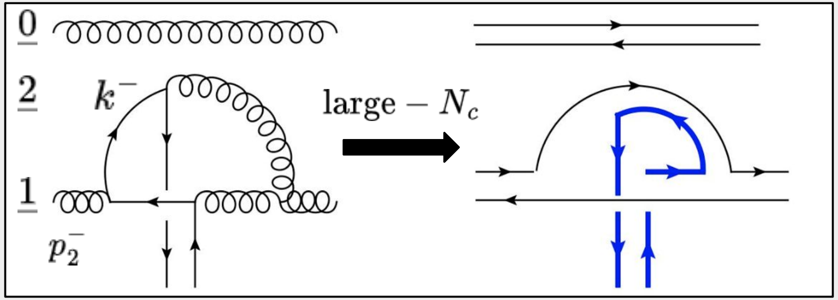
$$+\left(\frac{\alpha_s}{4\pi}\right)^2\frac{1}{2}\frac{N_c}{\left(49N_c-16N_f\right)}\left[\left(49N_c-16N_f\right)\left(33N_c-8N_f\right)+\sqrt{N_c\left(49N_c-16N_f\right)}\left(217N_c-\underset{112}{80}N_f\right)\right]\frac{1}{\omega^3}$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^3\frac{N_c}{\left(49N_c-16N_f\right)^2}\left[N_c\left(49N_c-16N_f\right)^2\left(225N_c-\underset{84}{64}N_f\right)\right.$$

$$\left.+\sqrt{N_c\left(49N_c-16N_f\right)}\left(76489N_c^3-\underset{68412}{60712}N_c^2N_f+\underset{16896}{15568}N_cN_f^2-\underset{1024}{1280}N_f^3\right)\right]\frac{1}{\omega^5}$$

$$+\mathcal{O}\left(\alpha_s^4\right)$$

Relations between $\widetilde{\mathbf{Q}}$ and $\mathbf{g_{1L}^S}$, $\Delta\Sigma$ (Extra Slides)



$$\widetilde{Q}_{12}(s) = \left\langle\!\!\left\langle \frac{g^2}{16\sqrt{k^- p_2^-}} \int_{-\infty}^{\infty} \mathrm{d}y^- \int_{-\infty}^{\infty} \mathrm{d}z^- \left[\bar{\psi}(y^-, \underline{x}_2) \left(\frac{1}{2}\gamma^+\gamma^5\right) \left(V_{\underline{2}}[y^-, \infty] V_{\underline{1}}[\infty, z^-] + V_{\underline{2}}[y^-, -\infty] V_{\underline{1}}[-\infty, z^-]\right) \psi(z^-, \underline{x}_1) + \text{c.c.} \right] \right\rangle\!\!\right\rangle (s)$$

$$\widetilde{Q}(x_{10}^2, zs) = \int \mathrm{d}^2 \left(\frac{x_0 + x_1}{2}\right) \widetilde{Q}_{10}(zs)$$

$$\widetilde{Q}(x_{10}^2, Q^2/x) = \alpha_s \pi^2 \int \mathrm{d}^2 k_\perp e^{i\underline{k}\cdot\underline{x}_{10}} g_{1L}^S(x, k_T^2)$$

$$\widetilde{Q}\left(x_{10}^2 = \frac{1}{Q^2}, s = \frac{Q^2}{x}\right) = \frac{\alpha_s \pi^2}{N_f} \Delta\Sigma(x, Q^2)$$