# Inferring the Initial Condition for the BK Equation

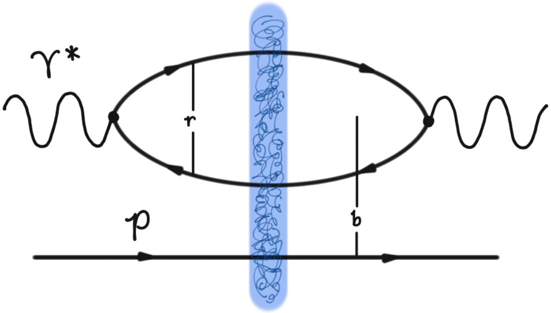
based on:
Phys.Rev.D 109 (2024) 054018 arXiv:2311.10491 with M.Karhunen and H.Mäntysaari; and,
(work in progress) with H.Hänninen and H.Mäntysaari

Speaker: Carlisle Casuga

Center of Excellence in Quark Matter, University of Jyväskylä

QCD Master Class 2024

# Deep Inelastic Scattering in the Dipole Picture



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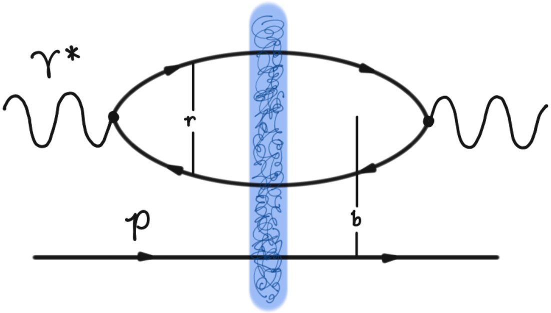


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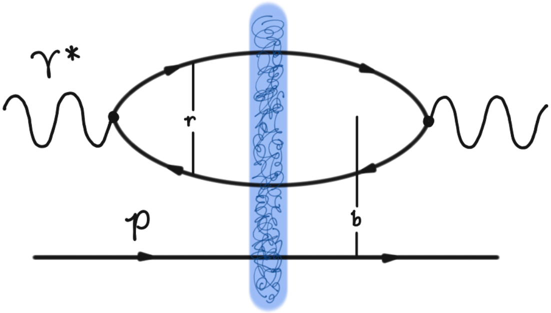


proton transverse area

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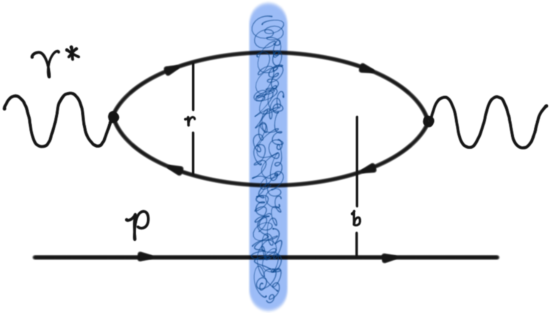


dipole-target scattering amplitude

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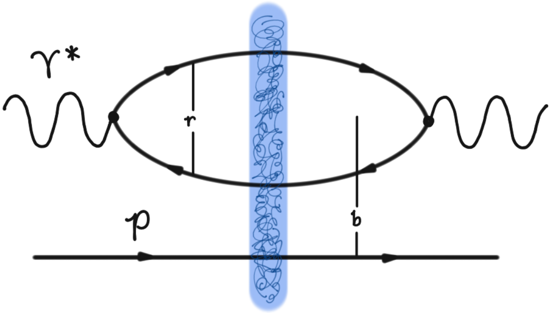


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$$\textbf{rcBK:}\, \mathcal{N}(r, x = x_0; Q_{s0}^2, \gamma, e_c) \xrightarrow{C^2} \mathcal{N}(r, x)$$

# Objectives

- Constrain model parameters, $[Q_{s0}^2, \gamma, e_c, C^2, \sigma_0/2]$ against combined HERA reduced cross section data ...

Some previous fits to HERA data:

- ✓ H.Mäntysaari, T. Lappi (2013): 1309.6963
- ✓ AAMQS Collaboration (2010) arXiv:1012.4408
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... this work: provides uncertainty for the BK initial condition!

- Tool: Bayesian inference to extract posterior distribution.
- Account for correlated experimental uncertainties in HERA data.
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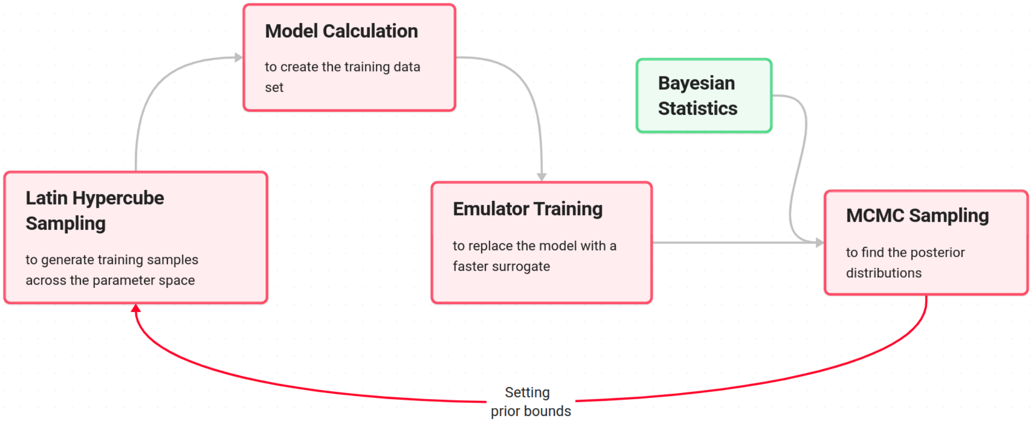
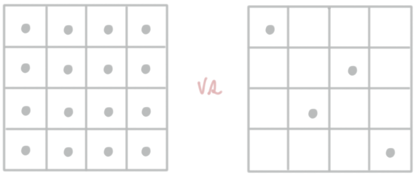
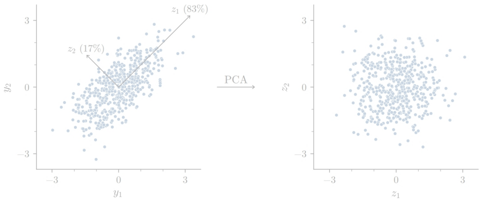
# Typical Bayesian Workflow



## Latin Hypercube Sampling




## Principal Component Analysis

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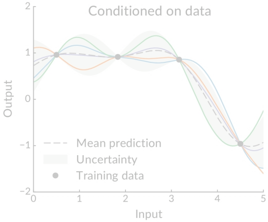
GPs learn the parameter dependence of the model!

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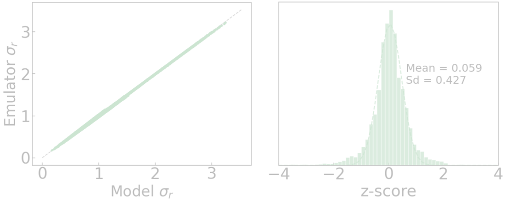

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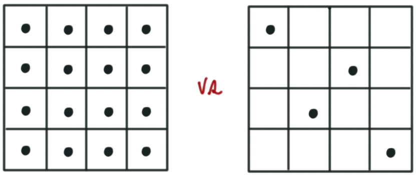


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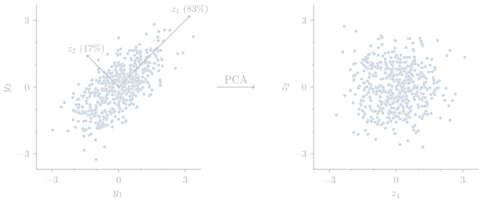


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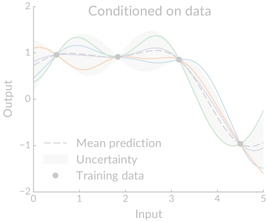


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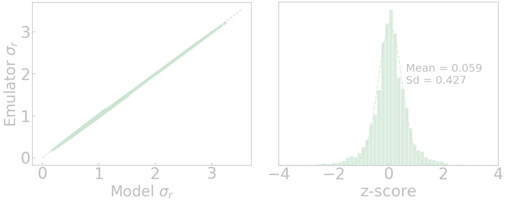

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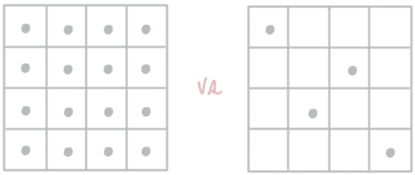


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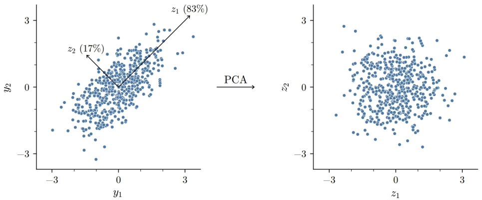


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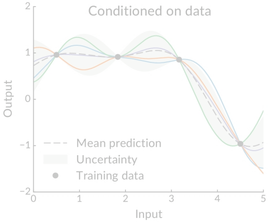


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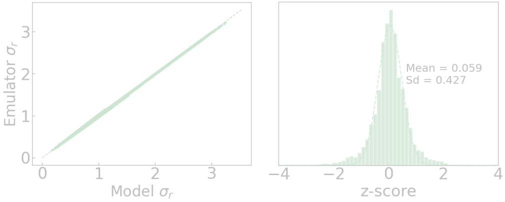

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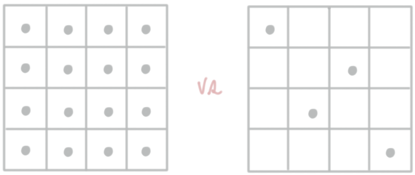


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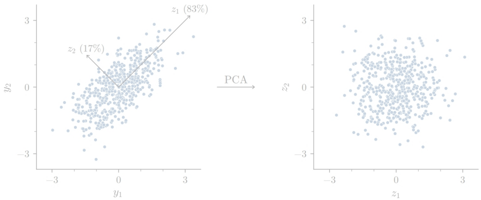


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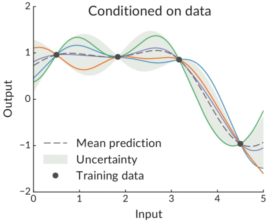


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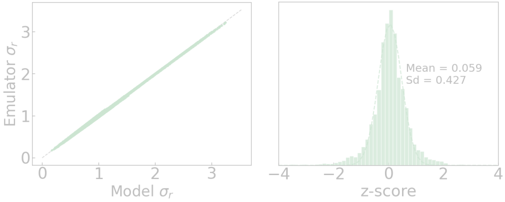

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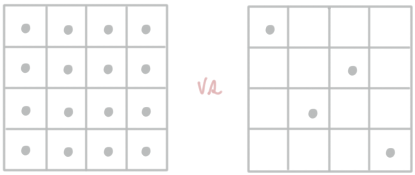


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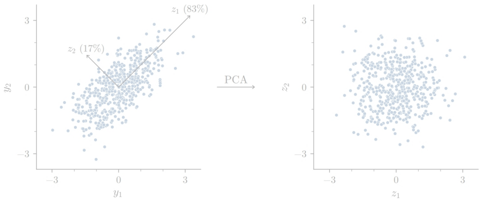


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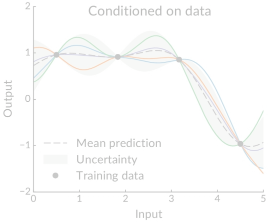


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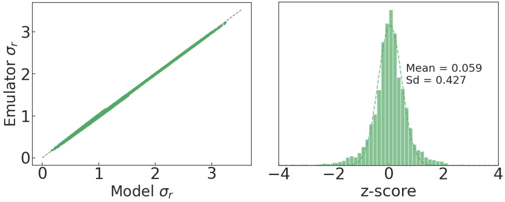
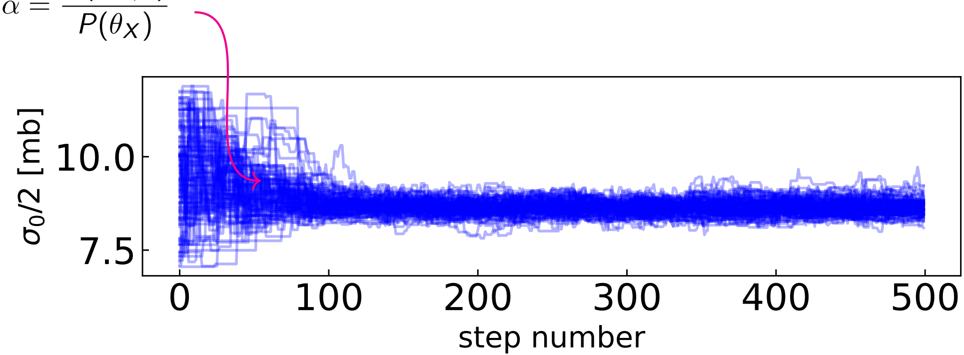


Acceptance probability:

$$\alpha = \frac{P(\theta_{X+1})}{P(\theta_X)}$$



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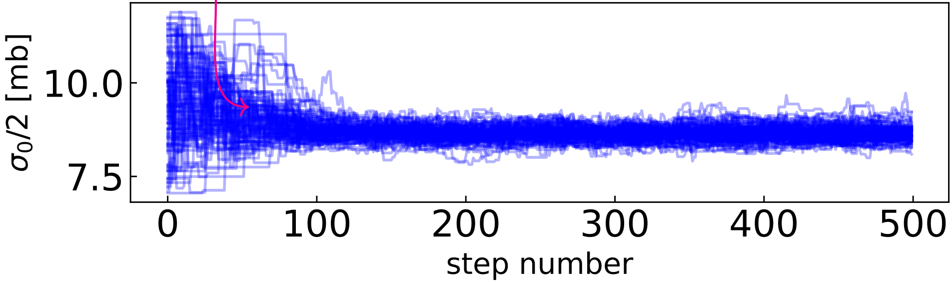
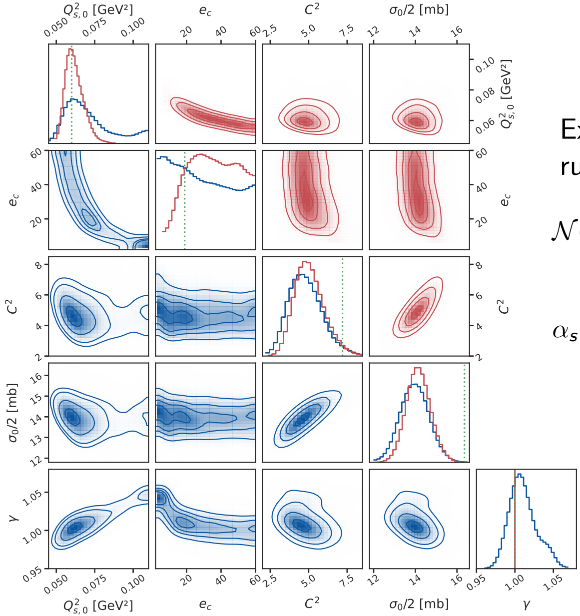
## Bayesian Statistics

$$P(\theta) = \text{posterior} = \text{likelihood} \times \text{prior}$$

- Likelihood: how well data matches the model at $\theta$
- Prior: bounds of the parameter space



# Posterior Distributions




C.C, M. Karhunen, H. Mäntysaari, 2311.10491

Explicit expressions for initial amplitude and running coupling:

$$\mathcal{N}(\mathbf{r}, x_0) = 1 - \exp\left[-\frac{(\mathbf{r}^2 Q^2_{s,0})^\gamma}{4} \ln\left(\frac{1}{|\mathbf{r}|\Lambda_{\mathrm{QCD}}} + e_c \cdot e\right)\right]$$

$$\alpha_s(\mathbf{r}) = \frac{12\pi}{(33 - N_f)\log\left(\frac{4C^2}{\mathbf{r}^2\Lambda^2_{QCD}}\right)}$$

— $[Q^2_{s0}, \gamma, e_c, C^2, \sigma_0/2]$
— $\gamma = 1$
— $\mathrm{MV}^e$ [H.M.& T.L. (2013)]

# Posterior Samples, Median and MAP

| 5 - parameter | $Q_{s0}^2[GeV^2]$ | $\gamma$ | $e_c$ | $C^2$ | $\sigma_0/2[\mathrm{mb}]$ | $\chi^2/\mathrm{dof}$ | $Q_s^2$ |
|---|---|---|---|---|---|---|---|
| median | 0.067 | 1.01 | 27.5 | 4.72 | 14.0 | 1.016 | 0.288 |
| MAP | 0.077 | 1.01 | 15.6 | 4.47 | 13.9 | 1.012 | 0.289 |

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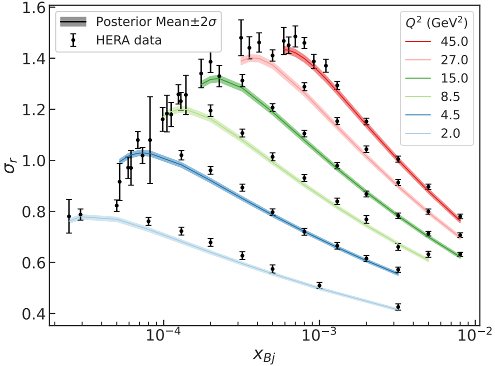


! Good agreement with HERA data

# Inclusive quark production

$$\mathrm{d}\sigma^{q+A\to q+X} = xq(x,\mathbf{k}^2)\tilde{S}_p(\mathbf{k})$$

$\tilde{S}_p(\mathbf{k}) \to$ 2DFT of Dipole amplitude

$$= \int \mathrm{d}^2\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\times [1 - \mathcal{N}(\mathbf{r}, x = x_0)]$$

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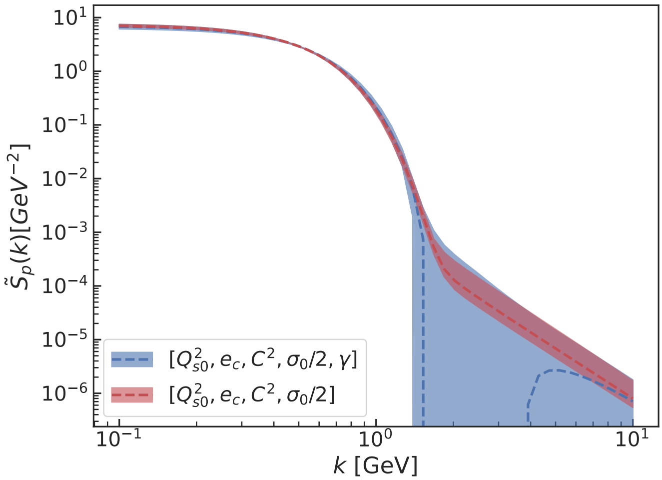
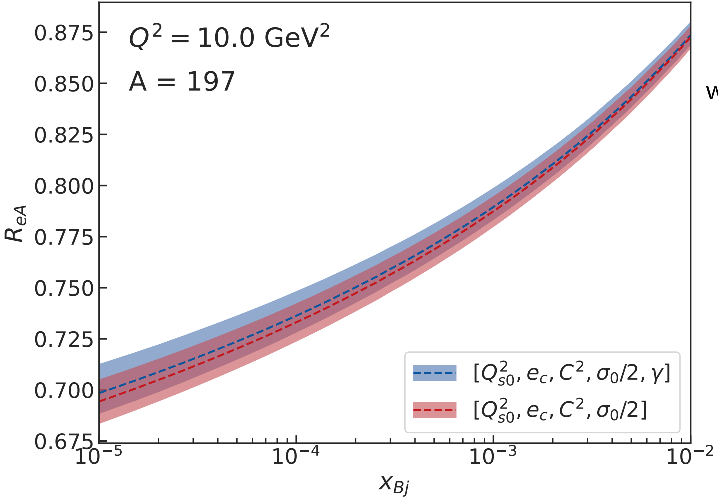


# Nuclear Modification factor at the EIC and beyond

C.C, M. Karhunen, H. Mäntysaari, 2311.10491




$$R_{eA} = \frac{F_{2,A}}{AF_{2,p}}$$

where $Q^2_{s,A} = Q^2_{s,p} \cdot \sigma_0/2 \cdot AT_A(b)$.

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! These measurements could provide further constraint for the initial condition

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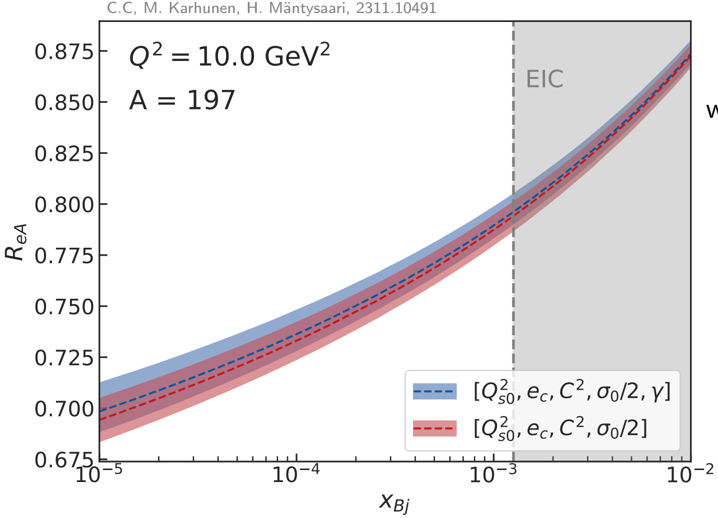


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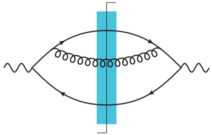
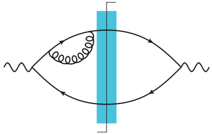
# Next-to-Leading Order Analysis

- **NLO Cross section**:

$$\sigma^{\rm NLO}_{\rm L,T} = \sigma^{\rm IC}_{\rm L,T} + \sigma^{\rm qg}_{\rm L,T} + \sigma^{\rm dip}_{\rm L,T}$$

1. Leading order term, amplitude evaluated at IC
2. Gluon term, a gluon is emitted into the final state
3. Dipole term, NLO $q\bar{q}$ contribution + virtual corrections





* Image from H. Hänninen et al. 2211.03504

- **NLO BK**:
  - Evolution equation: ResumBK [E. Iancu et. al., 1502.05642]
  - Running coupling prescription: Balistky + smallest dipole
  - Model parameters: $IC \to [Q^2_{s0}, \gamma, C^2, \sigma_0/2]$ and charm mass $\to m_c$

- **First analysis with simultaneous constraint** against HERA total reduced cross section + charm quark production

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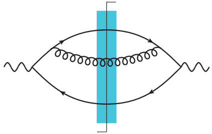

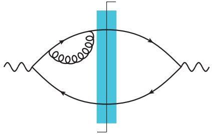

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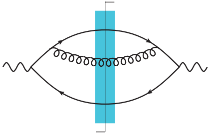

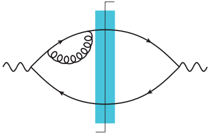

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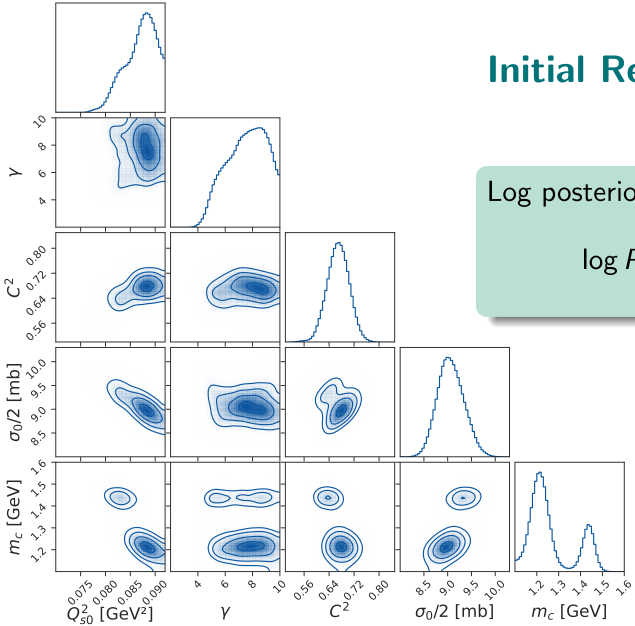


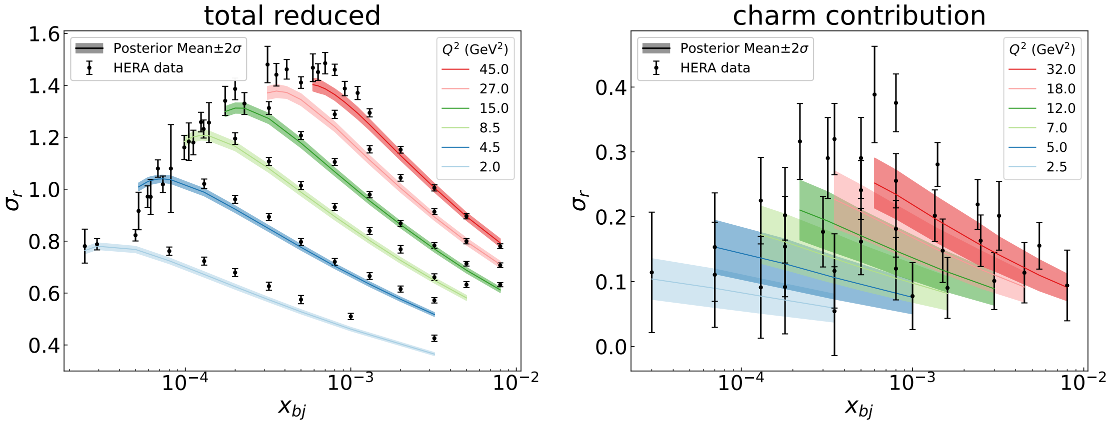

# Initial Results of NLO analysis

Log posterior for multiple constraints:

$$\log P(\theta|X) = \sum_{i=\mathrm{datasets}} \log P(\theta|X_\mathrm{i})$$

# Description of charm data

- $\chi^2$/d.o.f. over 100 samples: 1.70 (total reduced) and 1.39 (charm)



# Summary & Outlook

- Posterior distribution for model parameters of BK IC, a necessary input to LO and NLO CGC calculations

! First time:
  - Method for propagating the uncertainties BK IC
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- Other IC parametrizations and NLO BK (approximate) prescriptions
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# Back up Slides

# Parameters

- $\sigma_0/2$, half of normalization to the cross section, proton transverse area, $2\int \mathrm{d}^2 b \to \sigma_0$
- $Q^2_{s,0}$, **related** to the saturation scale at initial $x$. Previous fits used GBW parametrization:

$$\mathcal{N}(\mathbf{r}, x_0) = 1 - \exp\left[-\frac{(\mathbf{r}^2 Q^2_{s,0})^\gamma}{4}\right]$$

- $C^2$, connects the running coupling in $\mathbf{r}$ to its Fourier transform
- $\gamma$, anomalous dimension, controlling the steepness of the cross section related to its fall-off for small dipoles
- $e_c$, infrared cut-off in the MV model

# $F_2$ Structure Function for Nucleus

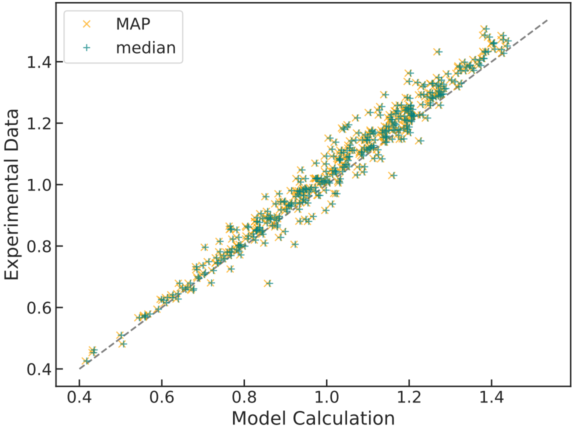
$$N_A(\mathbf{r}, \mathbf{b}, x = x_0) = 1 - \exp\left[-AT_A(\mathbf{b})\frac{\sigma_0}{2}\frac{\left(\mathbf{r}^2 Q_{\mathrm{s},0}^2\right)^\gamma}{4}\ln\left(\frac{1}{|\mathbf{r}|\Lambda_{\mathrm{QCD}}} + e_c \cdot e\right)\right]$$

where $T_A(\mathbf{b})$ is the transverse thickness function of the nucleus of mass number $A$, obtained through integrating the Woods-Saxon distribution

$$\rho_A(\mathbf{b}, z) = \frac{n}{1 + \exp\left[\frac{\sqrt{\mathbf{b}^2 + z^2} - R_A}{d}\right]}$$

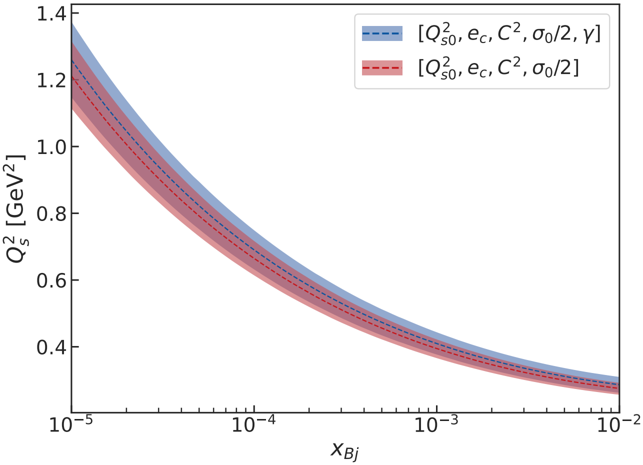
# Emulator vs Model

$$\frac{\chi^2}{\text{d.o.f}} = \frac{1}{N-p}\Delta\mathbf{y}(\boldsymbol{\theta})^T \Sigma_{\text{exp}}^{-1}\Delta\mathbf{y}(\boldsymbol{\theta}),$$

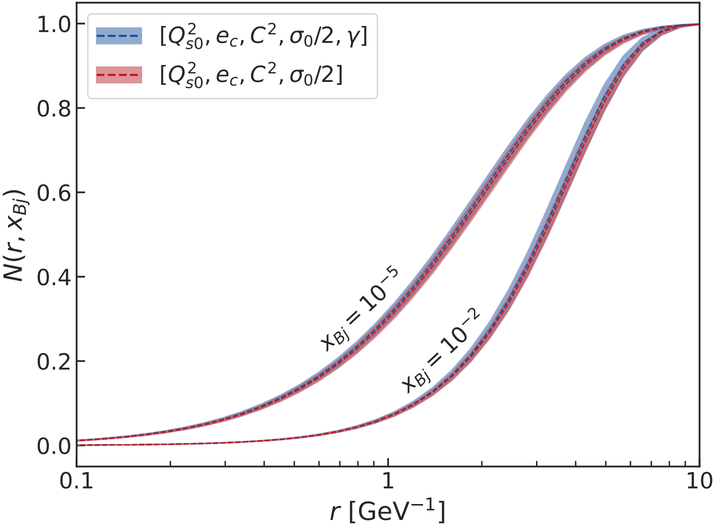


# Initial Saturation Scale

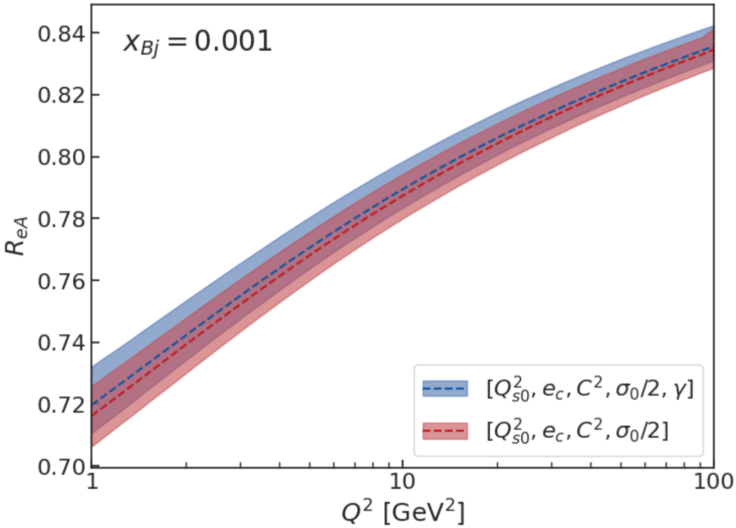
$$N(\mathbf{r}^2 = 2/Q_s^2) = 1 - e^{-1/2}$$



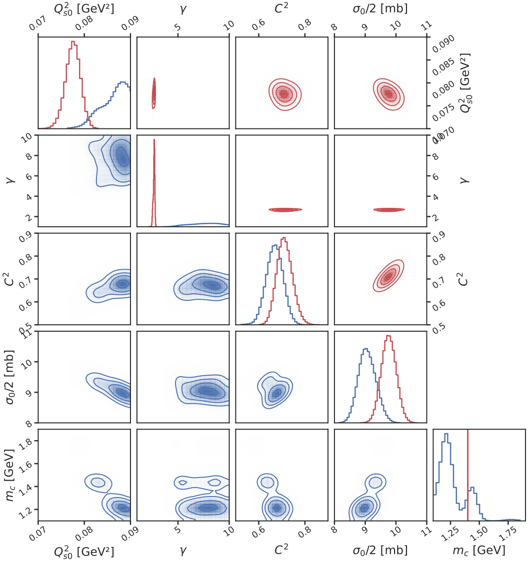
# Initial and Evolved $\mathcal{N}(r, y)$



# Nuclear Modification Ratio



# Comparison to 4-parameter set up (NLO)



# Total reduced cross section constraint only (NLO)